Commun. Math. Phys. 143, 27–42 (1991)

Communications in Mathematical Physics

Instability in Superspace

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Received August 10, 1990; in revised form April 15, 1991

Abstract. Using Morse's theory of reconstructions we define the space of all the universes – the Superspace. On the Superspace we investigate the geometry of the DeWitt metric. It is shown that the geodesic flow corresponding to the DeWitt metric is exponentially instable. The dynamical system described by the Einstein equations of evolution (Einstein dynamics) has the same type of instability also, if 1) the Universe is inflationary in some local domain, 2) in some local domain the Universe does not change its volume, but changes the conformal geometry very quickly as compared with the conformal potential. So, the Einstein dynamics is unstable on the Superspace, therefore the following quantum theory considered on the minisuperspace (a submanifold of the Superspace with a finite dimension) says nothing about the "real" quantum theory on the Superspace, and in the Superspace the semiclassical approximation is close to the quantum approximation only during a short time.

1. Introduction

In a number of recent papers different approaches to quantizing gravitation are considered. However, in all these papers the methods presented are applied solely to minisuperspace models and quantized in a semiclassical approximation. Here a question arises: what connection is there between these toy models and the "real" quantum theory? In this paper we try to answer this question.

For quantization of the classical theory first of all a corresponding space is needed, which must include possible states of variables describing the theory. In cosmology that space must include the set of possible Universes [1]. While considering the quantum cosmology the global properties of that space are used [2]. Therefore we consider several properties of the Superspace. For this purpose a natural definition of the Superspace is given and the geometry of the metric given by the ADM Hamiltonian of gravitational and material fields is investigated. The geodesic flow stability and the Einstein dynamics in the Superspace has been investigated. In fact, this work is a continuation of the approach of [3].

2. World, Universe, Superspace

The set of d-dimensional Universes will be described as follows. We assume, that the Universe is closed (compact and without boundary).

By $\mathfrak{M}^{d+1}$ we denote the set of all $d+1$-dimensional, smooth (hereafter "smooth" means from class C^r, $r>2$), oriented, compact manifolds ($d>1$),

$$\mathfrak{M}^{d+1}=\{M^{d+1}\}=\{\text{all } d+1\text{-dimensional smooth, oriented, compact manifolds}\}.$$

c-world (i.e. spacetime with material fields, see Appendix A) will mean the following triad:

$$(M^{d+1}, g(M), \Phi(M)),$$

where $M^{d+1}\in\mathfrak{M}^{d+1}$, and $g(M)$ is a smooth Riemannian metric on M^{d+1}, $\Phi(M)$ is a smooth scalar field. (We consider scalar fields for simplicity; the following definitions can be evidently generalized for material fields of any type.) Denote the set of c-worlds by W^{d+1},

$$W^{d+1}=\{w\}=\{(M^{d+1}, g(M), \Phi(M))\}\equiv\{(M_w, g_w, \Phi_w)\}.$$

Let us consider the set of smooth functions on M^{d+1} without singular critical points (Morse's function) [4]. We denote that set by $\mathfrak{F}(M^{d+1})$,

$$f\in\mathfrak{F}(M^{d+1})$$

if

$$f\in C^r(M^{d+1}),$$

$$f: M^{d+1}\to S^1\equiv[0,2\pi]/\{0,2\pi\},$$

$$\partial M^{d+1}=f^{-1}[\partial f(M^{d+1})].$$

For every $c\in S^1$, $f\in\mathfrak{F}(M^{d+1})$ we denote

$$f_c[M^{d+1}]=\{x\,|\,x\in M^{d+1},\ f(x)=c\},$$

$$Y\{f_c[M^{d+1}]\}=\{x\,|\,x\in f_c[M^{d+1}],\ df|_{f_c[M^{d+1}]}(x)=0\}.$$

The compactness of M^{d+1} and non-singularity of critical points lead, for every $c\in S^1$, $f\in\mathfrak{F}(M^{d+1})$, to the set $Y\{f_c[M^{d+1}]\}$ which contains a finite number of points.

For given $w\in W^{d+1}$, $f\in\mathfrak{F}(M_w)$, and $c\in S^1$ we have the following triad:

$$u(w,f,c)=(f_c[M_w],\ g,\ \phi),$$

where g is the metric induced on $f[M_w]$ by g_w, $\phi=\Phi_w|_{f_c[M_w]}$. C-Universes (i.e. space with material fields) are members of the set $\mathfrak{U}^d$,

$$\mathfrak{U}^d=\bigcup_{w\in W^{d+1}}\ \bigcup_{f\in\mathfrak{F}(M_w)}\ \bigcup_{c\in S^1} u(w,f,c),$$

$$\mathfrak{U}^d=\{u\}=\{T_u, g_u, \phi_u\}.$$

According to Morse's theory [4, 5] (here we do not distinguish between the embedded T_u and the abstract T_u manifolds)

$$\mathfrak{U}^d=\Sigma^d\cup\Omega^d\cup\{\emptyset\},$$

where Σ^d is the set of all d-dimensional smooth, oriented, closed manifolds with a smooth Riemannian metric and a smooth scalar field on them. We shall consider the empty set $\emptyset$ as a trivial manifold.

At Morse's reconstructions Ω^d are critical with a given metric and scalar field. Thus, if we consider the c-world w and $f \in \mathfrak{F}(M_w^{d+1})$,

$$u(w, f, c_1) \in \Sigma^d \cup \{\emptyset\}, \qquad u(w, f, c_2) \in \Sigma^d \cup \{\emptyset\},$$

and if there exists a unique c such as $c_1 < c < c_2$, $Y\{f_c[M_w]\} \neq \emptyset$, then

$$u(w, f, c) \equiv \omega \in \Omega^d.$$

So the manifold T_w is defined by two manifolds belonging to $\Sigma^d \cup \{\emptyset\}$ and indexes $\lambda_1, \ldots, \lambda_k$ of critical function from $\mathfrak{F}(M^{d+1})$ [5], then we introduce T_w as follows:

$$T_w = T_{\sigma_1}(\lambda_1 - 1, \ldots, \lambda_k - 1) T_{\sigma_2}, \qquad 0 \leqq \lambda_i \leqq d+1,\ 1 \leqq i \leqq k,$$

where k is the number of point belonging to $Y\{f_c[M_w]\}$. In the case of $k=1$ Morse's $n = \lambda - 1$ reconstructions mean contraction of the sphere S^n embedded in T_{σ_1} into a point, and then expansion into the sphere S^{d-n-1}.

For example, if $d=2$, $k=1$, $n=1$, we have

$$S^2 \to S^2(1)\{S^2 + S^2\} \to S^2 + S^2.$$

It is clear that $T_{\sigma_1}(n) T_{\sigma_2} = T_{\sigma_2}(d-n-1) T_{\sigma_1}$ [4, 5]. If $n = -1$, then

$$\emptyset \to \emptyset(-1) S^d \to S^d,$$

a sphere is born from nothing, and vice versa

$$S^d \to S^d(d) \emptyset \to \emptyset,$$

the sphere vanishes.

$k > 1$ is the unification of the reconstructions of the $k=1$ cases. The spaces ω from Ω^d have singular points ($Y\{f_c[M^{d+1}]\}$), though the c-worlds including those spaces are smooth.

In order to construct the set $\mathfrak{U}^d$ as a space there is needed a topology, a system of open sets. We solve this problem as follows: first we define the set of "smooth" curves $\mathfrak{U}^d$.

A mapping like

$$\lambda : [0, 1] \to \mathfrak{U}^d$$

is needed a "smooth" one in $\mathfrak{U}^d$, if

$$\exists w \in W^{d+1}, \qquad \exists f \in \mathfrak{F}(M_w)$$

so, that

$$\lambda_{(w,f)}(\tau) = u(w, f, \tau), \qquad \tau \in [0, 1] \subset S^1.$$

$\mathfrak{O}$ is the strongest topology on $\mathfrak{U}^d$, for which every curve from the set of "smooth" curves

$$\bigcup_{w \in W^{d+1}} \bigcup_{f \in \mathfrak{F}(M_w)} \lambda_{(w,f)}(\cdot)$$

is continuous on $[0, 1]$ (cf. [6]).

By c-Superspace we mean an $\mathfrak{U}^d$ with topology $\mathfrak{D}(\mathfrak{U}^d, \mathfrak{D})$. Then we denote that space by $\mathfrak{U}^d$ again.

Linear connectivity of $\mathfrak{U}^d$ depends on the Ω_d^{SO}-boardism group [4]. If $\Omega_d^{SO} \neq 0$ ($\Omega_0^{SO} = Z$, $\Omega_1^{SO} = \Omega_2^{SO} = \Omega_3^{SO} = 0$, $\Omega_4^{SO} = Z$), then $\mathfrak{U}^d$ is not connected linearly and the number of non-connected pieces of $\mathfrak{U}^d$ depends on Ω_d^{SO}. Connectivity of $\mathfrak{U}^d$ also depends on the type of material fields as well: if ϕ is not a usual scalar field (i.e. having real values), but has values from the non-trivial group G, then $\mathfrak{U}^d$ will not be connected even if $\Omega_d^{SO} = 0$ [7].

The space of c-Universes with given manifold T we denote by $\mathfrak{U}_T^d$,

$$\mathfrak{U}_T^d = \{u \in \mathfrak{U}^d, \ T_u = T\}.$$

Notice that, if $T \in \Sigma^d$, then $C\mathfrak{U}_T^d$ (closing of the subspace $\mathfrak{U}_T^d$ with respect to $\mathfrak{D}$ topology) includes points from Ω^d. It is clear that the $\mathfrak{U}^d$ is not complete and if $T \in \Omega^d$, then the topology induced on the $\mathfrak{U}_T^d$-subspaces is discrete.

By c-Superspace some authors mean $\mathfrak{U}_T^d$ for some fixed $T \in \Sigma^d$. We call such subspace T-superspaces.

3. Geometry of Superspace

It is clear that there is no Banach structure on $\mathfrak{U}^d$, i.e. $\mathfrak{U}^d$ is not a manifold (it follows from discreteness of the topology induced on Ω^d). But such structures exists on any $\mathfrak{U}_M^d$, for $M \in \Sigma^d$.

Let us fix any $M \in \Sigma^d$ and consider $\mathfrak{U}_M^d$. In this case $\mathfrak{U}_M^d$ is the space of all smooth Riemannian metrics and smooth scalar fields on M. It is known [8, 9] that there exists smooth Banach structure on such spaces. If $S_2(M)$ is the space of symmetric 2-covariant tensor on M and $S(M)$ is the space of function then the tangent bundle of $\mathfrak{U}_M^d$ is [10]

$$T\mathfrak{U}_M^d \approx \mathfrak{U}_M^d \times [S_2(M) \oplus S(M)],$$

where $\oplus$ is the Withney sum.

Let us denote the set of symmetric 2-contravariant tensor densities (scalar densities) by $S_d^2(M)\,(S_d(M))$ [10, 11]. Let

$$T^*\mathfrak{U}_M^d \approx \mathfrak{U}_M^d \times [S_d^2(M) \oplus S_d(M)].$$

If $(k, \chi) \in T\mathfrak{U}_M^d$, $(\pi, p) \in T^*\mathfrak{U}_M^d$, then

$$\langle(\pi, p), (k, \chi)\rangle = \int_M \pi \cdot k + p \cdot \chi.$$

Now we introduce a metric on $\mathfrak{U}_M^d$ such that the kinematical part of the Hamiltonian given by ADM formalism could be expressed by that metric (we take $N = 1$, $N_i = 0$). So, we have the following metric on $\mathfrak{U}_M^d$ [10, 12, 13]

$$\mathfrak{S}[g, \phi](k, \chi; h, \vartheta) = \int_M d\mu(g)(-\operatorname{tr}(k)\operatorname{tr}(h) + \operatorname{tr}(k \times h) + \chi\vartheta),$$

where

$$(g, \phi) \in \mathfrak{U}_M^d, \ (k, \chi), (h, \vartheta) \in T\mathfrak{U}_M^d,$$

$$d\mu(g) = (\det g)^{1/2} dx^1 \wedge \ldots \wedge dx^d,$$

$$\operatorname{tr}(k) = g^{ab}k_{ab}, \qquad (k \times h)_{ab} \equiv k_{ac}g^{cd}h_{db}.$$

The metric $\mathfrak{G}$ has inverse metric $\mathfrak{G}^{-1}$ and

$$\mathfrak{G}^{-1}[g,\phi](\pi,p;\,\varrho,q)=\int_M d\mu(g)\left(-\frac{1}{d-1}\operatorname{tr}(\pi')\operatorname{tr}(\varrho')+\operatorname{tr}(\pi'\times\varrho')+p'q'\right),$$

where

$$(\pi,p),\ (\varrho,q)\in T^*\mathfrak{U}^d_M,$$
$$\pi=\pi' d\mu(g),\qquad \varrho=\varrho' d\mu(g),$$
$$p=p' d\mu(g),\qquad q=q' d\mu(g).$$

By means of this metric we can map $T\mathfrak{U}^d_M$ on $T^*\mathfrak{U}^d_M$ and vice versa – $T^*\mathfrak{U}^d_M$ on $T\mathfrak{U}^d_M$. These mappings are of the following form [10]:

$$\mathfrak{G}^b: T\mathfrak{U}^d_M\to T^*\mathfrak{U}^d_M,$$
$$\mathfrak{G}^b[g,\phi](k,\chi)=d\mu(g)(-\operatorname{tr}(k)g^{-1}+k^{-1},\chi)=d\mu(g)(-\operatorname{tr}(k)g^{ab}+k^{ab},\chi),$$
$$\mathfrak{G}^{\#}: T^*\mathfrak{U}^d_M\to T\mathfrak{U}^d_M,$$
$$\begin{aligned}\mathfrak{G}^{\#}[g,\phi](\pi,p)&=(-1/(d-1)\operatorname{tr}(\pi')g+\pi'^b,p')\\&=(-1/(d-1)\operatorname{tr}(\pi')g_{ab}+\pi'_{ab},p').\end{aligned}$$

Notice that every k-tensor from $S_2(M)$ can be introduced by the following form [14]

$$k=k^{TT}+k^L+k^{\mathrm{tr}},$$

where

$$\operatorname{tr}(k^{TT})=0,\qquad \nabla\cdot k^{TT}=0\ ((\nabla\cdot k^{TT})_b\equiv\nabla^a k^{TT}_{ab}),$$

there exists a vector field Z on M such that

$$k^L=\mathfrak{L}_Z g-(2/d)g\nabla\cdot Z,\qquad k^L_{ab}=\nabla_a Z_b+\nabla_b Z_a-(2/d)g_{ab}\nabla_c\cdot Z^c,$$
$$k^{\mathrm{tr}}=\frac{\operatorname{tr}(k)}{d}g,\qquad k^{\mathrm{tr}}_{ab}=\frac{\operatorname{tr}(k)}{d}g_{ab}.$$

Hence $T_{(g,\phi)}\mathfrak{U}^d_M$ can be introduced as a sum of perpendicular subspaces

$$T_{(g,\phi)}\mathfrak{U}^d_M=T^{TT}_{(g,\phi)}\mathfrak{U}^d_M\oplus T^L_{(g,\phi)}\mathfrak{U}^d_M\oplus T^{\mathrm{tr}}_{(g,\phi)}\mathfrak{U}^d_M\oplus T^{\phi}_{(g,\phi)}\mathfrak{U}^d_M,\tag{3.1}$$

where

$$T^{TT}_{(g,\phi)}\mathfrak{U}^d_M=\{(k,0),\ k=k^{TT}\},$$
$$T^{L}_{(g,\phi)}\mathfrak{U}^d_M=\{(k,0),\ k=k^{L}\},$$
$$T^{\mathrm{tr}}_{(g,\phi)}\mathfrak{U}^d_M=\{(k,0),\ k=k^{\mathrm{tr}}\},$$
$$T^{\phi}_{(g,\phi)}\mathfrak{U}^d_M=\{(0,\chi)\}.$$

Perpendicularity of these subspaces is evident

$$\begin{aligned}&\mathfrak{G}[g,\phi](\xi^{TT}+\xi^L+\xi^{\mathrm{tr}}+\xi^\phi,\eta^{TT}+\eta^L+\eta^{\mathrm{tr}}+\eta^\phi)\\&=\mathfrak{G}[g,\phi](\xi^{TT},\eta^{TT})+\mathfrak{G}[g,\phi](\xi^L,\eta^L)\\&\quad+\mathfrak{G}[g,\phi](\xi^{\mathrm{tr}},\eta^{\mathrm{tr}})+\mathfrak{G}[g,\phi](\xi^\phi,\eta^\phi).\end{aligned}$$

In the above 4 spaces the vectors belonging to $T^{\mathrm{tr}}_{(g,\phi)}\mathfrak{U}^d_M$ have negative length ("timelike" vectors) and the others have positive length ("spacelike" vectors). Note that vectors tangent to the orbit, passes through the point (g,ϕ) and is diffeomorphic to (g,ϕ) (see Appendix A), have the form [6, 10, 11]:

$$(\mathfrak{L}_Z g, \mathfrak{L}_Z \phi) = (\nabla_a Z_b + \nabla_b Z_a, \nabla_c \phi Z^c),$$

and

$$T_{(g,\phi)}\text{orbit} \subset T^L_{(g,\phi)}\mathfrak{U}^d_M \oplus T^{\mathrm{tr}}_{(g,\phi)}\mathfrak{U}^d_M \oplus T^{\phi}_{(g,\phi)}\mathfrak{U}^d_M\,. \tag{3.2}$$

As on a manifold of finite dimension, on $\mathfrak{U}^d_M$ there exists a Levi-Civita connection, i.e. a Riemannian one without torsion [15]. In that case, as it is shown in Appendix B (cf. [3, 12, 16]),

$$\begin{aligned}\Gamma[g,\phi](k,\chi;\,\omega,\varphi;\,h,\vartheta) = \tfrac{1}{4}\int_M d\mu(g)\,\{&-\mathrm{tr}(k)\,\mathrm{tr}(\omega)\,\mathrm{tr}(h)\\ &+3\,\mathrm{tr}(k)\,\mathrm{tr}(\omega\times h)+\mathrm{tr}(\omega)\,\mathrm{tr}(k\times h)+\mathrm{tr}(h)\,\mathrm{tr}(k\times\omega)\\ &-4\,\mathrm{tr}(k\times\omega\times h)+\mathrm{tr}(h)\chi\varphi+\mathrm{tr}(\omega)\chi\vartheta-\mathrm{tr}(k)\varphi\vartheta\}\,,\end{aligned}$$

$$\begin{aligned}&\mathfrak{Riem}[g,\phi](\omega,\varphi;\,k,\chi;\,l,\sigma;\,h,\vartheta)\\ &\quad=\int_M d\mu(g)\,\{(1/4)\,\mathrm{tr}[(k\times\omega-\omega\times k)\times(h\times l-l\times h)]\\ &\qquad+\kappa^2[\mathrm{tr}(h\times\omega)\,\mathrm{tr}(k)\,\mathrm{tr}(l)-\mathrm{tr}(l\times\omega)\,\mathrm{tr}(k)\,\mathrm{tr}(h)+\mathrm{tr}(k\times l)\,\mathrm{tr}(\omega)\,\mathrm{tr}(h)\\ &\qquad-\mathrm{tr}(k\times h)\,\mathrm{tr}(\omega)\,\mathrm{tr}(l)+d\{\mathrm{tr}(\omega\times l)\,\mathrm{tr}(k\times h)-\mathrm{tr}(h\times\omega)\,\mathrm{tr}(k\times l)\}\\ &\qquad+\mathrm{tr}(k)\,\mathrm{tr}(l)\vartheta\varphi-\mathrm{tr}(k)\,\mathrm{tr}(h)\varphi\sigma+\mathrm{tr}(\omega)\,\mathrm{tr}(h)\chi\sigma-\mathrm{tr}(\omega)\,\mathrm{tr}(l)\chi\vartheta\\ &\qquad+d\{\mathrm{tr}(\omega\times l)\chi\vartheta-\mathrm{tr}(\omega\times h)\chi\sigma+\mathrm{tr}(k\times h)\varphi\sigma-\mathrm{tr}(k\times l)\varphi\vartheta\}]\}\,,\end{aligned}$$

or [see Eq. (C.4)]

$$\begin{aligned}&\mathfrak{Riem}[g,\phi](\omega,\varphi;\,k,\chi;\,l,\sigma;\,h,\vartheta)\\ &\quad=\int_M d\mu(g)\,\{(1/4)\,\mathrm{tr}[(\bar{k}\times\bar{\omega}-\bar{\omega}\times\bar{k})\times(\bar{h}\times\bar{l}-\bar{l}\times\bar{h})]\\ &\qquad+\kappa^2\{\mathrm{tr}(\bar{\omega}\times\bar{l})\,\mathrm{tr}(\bar{k}\times\bar{h})-\mathrm{tr}(\bar{h}\times\bar{\omega})\,\mathrm{tr}(\bar{k}\times\bar{l})\\ &\qquad+\mathrm{tr}(\bar{\omega}\times\bar{l})\chi\vartheta-\mathrm{tr}(\bar{\omega}\times\bar{h})\chi\sigma+\mathrm{tr}(\bar{k}\times\bar{h})\varphi\sigma-\mathrm{tr}(\bar{k}\times\bar{l})\varphi\vartheta\}\}\,,\end{aligned}$$

where

$$\kappa^2=\frac{d}{16(d-1)},\qquad \omega=\bar{\omega}+\frac{\mathrm{tr}(\omega)}{d}g,\qquad \bar{\omega}\in T^{TT}_{(g,\phi)}\mathfrak{U}^d_M\oplus T^L_{(g,\phi)}\mathfrak{U}^d_M\,.$$

Note that Ricci ($\mathfrak{Ric}$) and scalar ($\mathfrak{R}$) tensors not always exist on infinite dimensional manifolds. Appendix C [Eqs. (C.2), (C.3), (C.5), (C.6)] implies

$$\mathfrak{Ric}[g,\phi](k,\chi;\,k,\chi)\simeq(d-6)(+\infty)\,,$$

$$\mathfrak{R}[g,\phi]\simeq(d-6)(+\infty)^2\,.$$

Therefore, if $d=6$ then

$$\mathfrak{Ric}[g,\phi](k,\chi;\,k,\chi)=0\,,$$

$$\mathfrak{R}[g,\phi]=0\,.$$

Hence, it is meaningless to write $\mathfrak{R}$ in the Wheeler-DeWitt equation, so far as $\mathfrak{R}$ is 0 or $\pm\infty$ [2].

And if we consider m scalar fields, then Eqs. (C.9), (C.10),

$$\mathfrak{Ric}[g,\phi](k,\chi;\, k,\chi) \simeq (d^2-7d+2m+4)(+\infty),$$

$$\mathfrak{R}[g,\phi] \simeq (d^2-7d+2m+4)(+\infty)^2 .$$

These relations show that in case of small dimensions and few scalar fields in c-Superspace there exists a certain instability. The geodesic flow of such c-Superspace is more unstable than in case of larger dimensions or many fields (cf. [17]).

4. Dynamics in Superspace

Let us consider some M-superspace and investigate the geodesic flow given by the $\mathfrak{G}$-metric on it. That geodesic flow coincides with the dynamical systems defined by the following Hamiltonian [9],

$$\mathscr{H}[g,\phi;\,\pi,p] = (1/2)\cdot \mathfrak{G}^{-1}[g,\phi](\pi,p;\,\pi,p).$$

Let us consider now the stability of this geodesic flow (cf. [19]) corresponding to a velocity-dominated Universe [18]. We describe the projection of two neighbouring geodesics on the submanifold of the homogeneous conformal metrics c-HC, (cf. [3, 12, 20])

$$c\text{-HC} = \{(g,\phi)\in \mathfrak{U}^d_M,\ (M,g,\phi)\text{-homogeneous},\ \xi(x)=\text{const},\ \phi(x)=\text{const}\},$$

where

$$\xi(x) = \frac{(\det g(x))^{1/4}}{\kappa},$$

and therefore,

$$L(s) = \xi_s L_{\text{HC}}(s).$$

As shown in [3], there exists an exponential instability on c-HC, i.e.

$$L_{\text{HC}}(\bar{s}) \simeq \exp(\lambda \bar{s}),$$

where $\bar{s}$ is an affine parameter for the geodesics projected on c-HC [12]

$$\frac{d\bar{s}}{ds} = \frac{\alpha}{\xi_s^2},$$

and [3], (see Appendix D),

$$\lambda = \max_i \{(-\omega_i)^{1/2},\ \omega_i<0\}.$$

Now we evaluate λ. Equation (C.7) implies

$$\sum_{i=1}^{n} K_c(v_i,u) = \text{Ric}_c(u,u) = -\frac{d}{4},$$

where v_i and u are vectors from c-HC and are orthogonal to each other, n is the dimension of c-HC $\frac{d(d+1)}{2}-1-1$ (vector u), i.e. $n=\frac{d(d+1)}{2}-2$, therefore

$$\langle K_c(u)\rangle \equiv \frac{1}{n}\sum_{i=1}^{n} K_c(v_i, u) = -\frac{d}{4n} = -\frac{d}{4\left(\frac{d(d+1)}{2}-2\right)}.$$

Following [3],

$$\lambda > 0 \quad \text{and} \quad \lambda^2 > \frac{d}{4\left(\frac{d(d+1)}{2}-2\right)}. \tag{4.1}$$

$$L(s) \simeq \xi_s \exp(\lambda \bar{s}),$$

and [12] implies

$$L(s)=\begin{cases} (s-s_2)^{q_+}(s-s_1)^{q_-} & \text{“timelike” geodesic} \\ \sqrt{2c}(s-s_0)^{q_+} & \text{“null” geodesic} \\ (s-s_1)^{q_+}(s_2-s)^{q_-} & \text{“spacelike” geodesic}, \end{cases}$$

where $c^2=\kappa^2(\alpha^2+p^2)$, $p=\xi_s^2(d\phi/ds)|_0$, $s_1=s_0-c$, $s_2=s_0+c$, $ds_0/ds=0$, $q_\pm=\frac{1}{2}\left(1\pm\frac{\lambda\alpha}{c}\right)$. Which implies that, for $s\to+\infty$,

$$L(s) \propto \begin{cases} s & \text{“timelike” geodesic} \\ \sqrt{2c}s^{q_+} & \text{“null” geodesic}, \end{cases}$$

where $q_+<\frac{1}{2}(1+\lambda/\kappa)$. So, “timelike” and “null” geodesics have weak instability (for these geodesics the Lyapunov characteristic number is 0).

In the “spacelike” case, in a finite interval of “time” $\Delta s=s_2-s_1=2c$, $L(s)$ becomes $+\infty$. This means that “spacelike” geodesic is very unstable (for these geodesics the Lyapunov characteristic number is $+\infty$).

Consider now the dynamics with the following Hamiltonian in M-superspace [10–13]

$$\mathscr{H}[g,\phi,\pi,p]=(1/2)\cdot\mathfrak{G}^{-1}[g,\phi](\pi,p,\pi,p)+V[g,\phi],$$

where

$$V[g,\phi]=\int_M d\mu(g)\{-R(g)+(1/2)\|d\phi\|^2+F(\phi)\},$$

$$\|d\phi\|^2\equiv g^{ab}\phi_{|a}\phi_{|b}.$$

The dynamics given by this Hamiltonian corresponds to Einstein's equations, if the constraint equations [11] are added,

$$\mathscr{H}[g,\phi;\pi,p]=0,$$

$$\mathfrak{P}_b[g,\phi;\pi,p]\equiv-2\pi^a_{b|a}+p\phi_{|b}=0.$$

These equations constrain only initial conditions, but not dynamics. The equations corresponding to the Hamiltonian are [10, 11]

$$\nabla_X X=-\mathfrak{G}^{\#}(dV)\equiv-\mathrm{grad}(V),$$

where ∇ is a covariant derivative on the M-superspace (see Appendix B),

$$X=\left(\frac{dg}{ds},\frac{d\phi}{ds}\right)\in T\mathfrak{U}_M^d,$$

and

$$\forall Z=(\omega,\chi)\in T\mathfrak{U}_M^d$$

we have

$$\begin{aligned}(dV,Z)=ZV&=\int_M d^dx\left\{\frac{dV}{dg_{ab}}\omega_{ab}+\frac{dV}{d\phi}\chi\right\}\\&=\int_M d\mu(g)\,\{(\mathrm{Ric}^{ab}(g)-\frac{1}{2}\phi^{|a}\phi^{|b})\omega_{ab}\\&\quad+\frac{\mathrm{tr}(\omega)}{2}\{-R(g)+(1/2)\,\|d\phi\|^2+F(\phi)\}+(F'(\phi)+\varDelta\phi)\chi\},\end{aligned}$$

where

$$\varDelta\phi=-g^{ab}\phi_{|ab},\qquad F'(\phi)=\frac{dF(\phi)}{d\phi}.$$

Then,

$$\frac{dV}{dg_{ab}}=(\det g)^{1/2}\{(\mathrm{Ric}^{ab}(g)-\frac{1}{2}\phi^{|a}\phi^{|b})+(1/2)\{-R(g)+(1/2)\,\|d\phi^2\|+F(\phi)\}g^{ab},$$

$$\frac{dV}{d\phi}=(\det g)^{1/2}(F'(\phi)+\varDelta\phi).$$

Therefore,

$$\langle d\xi,\nabla_XX\rangle=-\langle d\xi,\mathrm{grad}(V)\rangle=\frac{-2\kappa^2\xi}{d}\{(d-2)\left[-R(g)+\frac{1}{2}\,\|d\phi\|^2\right]+d\cdot F(\Phi)\},\tag{4.2}$$

$$\langle d\eta^A,\nabla_XX\rangle=-\langle d\eta^A,\mathrm{grad}(V)\rangle=\frac{dg_{ab}}{d\eta_A}\{\mathrm{Ric}^{ab}(g)-\frac{1}{2}\phi^{|a}\phi^{|b}\},\tag{4.3}$$

$$\langle d\phi,\nabla_XX\rangle=-\langle d\phi,\mathrm{grad}(V)\rangle=-(F'(\phi)+\varDelta\phi).\tag{4.4}$$

Here the following expressions have been used [3, 12]:

$$\frac{\partial g_{ab}}{\partial\xi}=\frac{4}{d}\xi^{-1}g_{ab},\qquad \mathrm{tr}\left(\frac{\partial g_{ab}}{\partial\eta^A}g^{ab}\right)=0,$$

where (cf. [3, 12]) $\xi(x)$; $\eta^A(x)$, $A=1,\dots,\frac{d(d+1)}{2}-1$; $\phi(x)$-coordinates are chosen such that

$$\frac{\partial}{\partial\xi(x)}\in T^{\mathrm{tr}}\mathfrak{U}_M^d,\qquad \frac{\partial}{\partial\eta^A(x)}\in T^{TT}\mathfrak{U}_M^d\oplus T^L\mathfrak{U}_M^d,\qquad \frac{\partial}{\partial\phi(x)}\in T^\phi\mathfrak{U}_M^d.$$

It is known that this system of equations has inflationary solutions,

$$\xi\propto\xi_0\exp\{(d/2)Hs\},\qquad \eta^A\propto\eta_0^A=\mathrm{const}^A,$$

$$\phi\propto\phi_0=\mathrm{const},\qquad F'(\phi_0)\propto 0,$$

$$\bar{s}(s)\propto\bar{s}(0),$$

hence,

$$L(s)=\xi_s L_{\mathrm{HC}}(\bar{s}) \propto \mathrm{const}\exp\{(d/2)Hs\},$$

i.e. inflationary solutions are unstable with respect to conformal perturbations ($L_{\mathrm{HC}}(0)>0$).

In general, when

$$|\langle d\eta^A, \nabla_X X\rangle| \gg |\langle d\eta^A, \mathrm{grad}(V)\rangle|,$$

i.e. conformal geometry changes very quickly as compared with the conformal potential, then

$$L(s)=\xi_s \exp\left(\lambda\alpha \int^s [d\tau/(\xi_\tau^2)]\right),$$

or

$$\frac{dL(s)}{ds}=\left(\frac{d\ln\xi_s}{ds}+\frac{\alpha\lambda}{\xi_s^2}\right)L(s),$$

and L increases exponentially, if

$$\frac{d\ln\xi_s}{ds}+\frac{\alpha\lambda}{\xi_s^2} \propto \mathrm{const}$$

(e.g. $\xi \propto \mathrm{const}$).

So, L increases exponentially, if

I. $\xi_s \propto \exp((d/2)Hs)$, local volume of the Universe increases exponentially (inflationary Universe).

II. $\frac{d\xi_s}{ds} \propto 0$, the Universe changes local conformal metrics very quickly as compared with the conformal potential, leaving the local volume unchanged (conformal Universe).

5. Conclusions

So it is clear that in any M-superspace, therefore in the Superspace, the Einstein dynamics and the geodesic flow are unstable. The instability is exponential, if
1) the gravitational and material fields are changed very quickly as compared with the potential (velocity dominated Universe [18]), the case of geodesic flow. Notice that with small dimensions d and few scalar fields m the geodesic flow is more unstable, than in case of larger dimensions or many scalar fields;
2) the Universe is inflationary in some local domain, (inflationary Universe);
3) the Universe does not change its volume in some local domain, but changes the conformal geometry, (conformal Universe).

In such cases the instability of dynamics implies that:
a) the quantized system on a submanifold of finite dimension (e.g. the minisuperspace) tells us very little about the "real, complete" quantized system, because according to the Heisenberg uncertainty principle there are always virtual perturbations along other frozen directions, and these perturbations are unstable,
b) in the Superspace (moreover, in the minisuperspace) the semiclassical approximation is close to the quantum approximation only during a short time [21, 22],

$t_{\inf} \propto \frac{2}{d} H^{-1}$ for inflationary Universe ($H \gg 1$), and $t_{\text{conf}} \propto \left(\frac{\alpha\lambda}{\xi^2}\right)^{-1}$ for "conformal Universe" $\left(\frac{\alpha}{\xi^2} \gg 1\right.$ and (4.1) implies that $\left. t_{\text{conf}} < \sqrt{2(d+1)} \left(\frac{\alpha}{\xi^2}\right)^{-1}\right)$.

In further investigations, where the complete Hamiltonian will be considered [23] without preliminary condition, we shall give final answers to these questions (see the Introduction).

Appendix A

Let Q be a smooth manifold and $E(Q)$ a vector bundle over Q with a projection $\pi: E(Q) \to Q$. Denote the group of (orientation-preserving) diffeomorphisms of Q by $\text{Diff}(Q)$. Consider now an equivalence relation on $E(Q)$. We say that $f_1 \in E(Q)$ is equivalent to $f_2 \in E(Q)$, $f_1 \sim f_2$, if $\exists \imath \in \text{Diff}(Q)$ such that $\imath_*(f_1) = f_2$.

Let $\mathfrak{E}(Q)$ be a space of all the equivalence classes

$$\mathfrak{E}(Q) = \frac{E(Q)}{\text{Diff}(Q)},$$

and

$$\text{orbit}(f) = \{\psi \in E(Q), \psi \sim f\}.$$

In that case we name the members of $E(Q)$ "c-objects" (c comes from the word "coordinate") and the members of $\mathfrak{E}(Q)$ we name "objects." So, defining some "c-object" (c-world, c-Universe, c-Superspace) we shall have an "object" (world, Universe, Superspace). In this paper the symbol "c" has only this meaning.

Appendix B

If f is a function defined on $\mathfrak{U}_M^d$, then by $Df(x)$ we denote the derivative of the function f and by $Df \cdot \omega(x)$ the value of the derivative on $\omega \in T_x \mathfrak{U}_M^d$ [5]. Notice that the following formulae are true [here $D_g k \cdot h = 0 = D_g l \cdot h$ and $k, l, h \in S_2(M)$],

$$D_g[d\mu(g)] \cdot h = \frac{\text{tr}(h)}{2} d\mu(g),$$

$$D_g g^{ab} \cdot h = -h^{ab},$$

$$D_g(\text{tr}(k)) \cdot h = -\text{tr}(k \times h),$$

$$D_g(\text{tr}(k \times l)) \cdot h = -2\,\text{tr}(k \times l \times h),$$

$$\text{tr}(k \times l \times h) = \text{tr}(l \times k \times h).$$

We have a Levi-Civita connection, therefore [15]

$$\begin{aligned} \Gamma[g,\phi](k,\chi;\, \omega,\varphi;\, h,\vartheta) = (1/2)\{ & D\mathfrak{G}[g,\phi] \cdot (h,\vartheta)(k,\chi;\, \omega,\varphi) \\ & + D\mathfrak{G}[g,\phi] \cdot (\omega,\varphi)(k,\chi;\, h,\vartheta) - D\mathfrak{G}[g,\phi] \cdot (k,\chi)(\omega,\varphi;\, h,\vartheta)\}, \end{aligned}$$

where

$$D_g k \cdot l = D_g h \cdot l = D_g \omega \cdot l = 0\,,$$
$$D_\phi \chi \cdot \sigma = D_\phi \vartheta \cdot \sigma = D_\phi \varphi \cdot \sigma = 0$$
$$\forall (l, \sigma) \in T\mathfrak{U}^d_M\,,$$

$$\begin{aligned} D\mathfrak{G}[g, \phi] \cdot (h, \vartheta)(k, \chi;\, \omega, \varphi) &= D_g \mathfrak{G}[g, \phi] \cdot h(k, \chi;\, \omega, \varphi) + D_\phi \mathfrak{G}[g, \phi] \cdot \vartheta(k, \chi;\, \omega, \varphi) \\ &= D_g \mathfrak{G}[g, \phi] \cdot h(k, \chi;\, \omega, \varphi) \\ &= D_g \left\{ \int_M d\mu(g) (-\operatorname{tr}(k) \operatorname{tr}(\omega) + \operatorname{tr}(k \times \omega) + \chi\varphi \right\} \cdot h \\ &= \int_M d\mu(g) \, [(1/2) \operatorname{tr}(h) \operatorname{tr}(k \times \omega) + \operatorname{tr}(\omega) \operatorname{tr}(h \times k) + \operatorname{tr}(k) \operatorname{tr}(h \times \omega) \\ &\quad - (1/2) \operatorname{tr}(k) \operatorname{tr}(\omega) \operatorname{tr}(h) + (1/2) \operatorname{tr}(h) \chi\varphi - 2 \operatorname{tr}(h \times k \times \omega)]\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \Gamma[g, \phi](k, \chi;\, \omega, \varphi;\, h, \vartheta) = \tfrac{1}{4} \int_M d\mu(g) \, \{ &- \operatorname{tr}(k) \operatorname{tr}(\omega) \operatorname{tr}(h) \\ &+ 3 \operatorname{tr}(k) \operatorname{tr}(\omega \times h) + \operatorname{tr}(\omega) \operatorname{tr}(k \times h) + \operatorname{tr}(h) \operatorname{tr}(k \times \omega) \\ &- 4 \operatorname{tr}(k \times \omega \times h) + \operatorname{tr}(h) \chi\varphi + \operatorname{tr}(\omega) \chi\vartheta - \operatorname{tr}(k) \varphi\vartheta \}\,. \end{aligned}$$

For any vector fields $X, Y, Z, U \in T\mathfrak{U}^d_M$ the covariant derivative and the $\mathfrak{Riem}$ tensor are defined as

$$\nabla_Y X \equiv \nabla X \cdot Y = DX \cdot Y + \mathfrak{G}^{\#}[g, \phi](\Gamma[g, \phi](\cdot\,;X;Y)),$$
$$\mathfrak{Riem}[g, \phi](X, Y)Z = \{[\nabla_X, \nabla_Y] - \nabla_{[X, Y]}\}Z\,,$$
$$\mathfrak{Riem}[g, \phi](U, Z, X, Y) = \mathfrak{G}[g, \phi](U, \mathfrak{Riem}[g, \phi](X, Y)Z)\,.$$

Then, one can see that

$$\begin{aligned} &\mathfrak{Riem}[g, \phi](\omega, \varphi;\, k, \chi;\, l, \sigma;\, h, \vartheta) \\ &\quad = (1/2)\{D^2\mathfrak{G}[g, \phi] \cdot (l, \sigma;\, k, \chi)(\omega, \varphi;\, h, \vartheta) - D^2\mathfrak{G}[g, \phi] \cdot (h, \vartheta;\, k, \chi)(\omega, \varphi;\, l, \sigma) \\ &\qquad + D^2\mathfrak{G}[g, \phi] \cdot (h, \vartheta;\, \omega, \varphi)(k, \chi;\, l, \sigma) - D^2\mathfrak{G}[g, \phi] \cdot (l, \sigma;\, \omega, \varphi)(k, \chi;\, h, \vartheta)\} \\ &\qquad + \mathfrak{G}^{-1}[g, \phi](\Gamma[g, \phi](\cdot\,;\, k, \chi;\, l, \sigma),\, \Gamma[g, \phi](\cdot\,;\, \omega, \varphi;\, h, \vartheta)) \\ &\qquad - \mathfrak{G}^{-1}[g, \phi](\Gamma[g, \phi](\cdot\,;\, k, \chi;\, h, \vartheta),\, \Gamma[g, \phi](\cdot\,;\, \omega, \varphi;\, l, \sigma)), \end{aligned}$$

where

$$\begin{aligned} D^2\mathfrak{G}[g, \phi] \cdot (l, \sigma;\, k, \chi)(\omega, \varphi;\, h, \vartheta) &= D[D\mathfrak{G}[g, \phi] \cdot (l, \sigma)] \cdot (k, \chi)(\omega, \varphi;\, h, \vartheta) \\ = (1/4) \int_M d\mu(g) \, \{ &\operatorname{tr}(l) \operatorname{tr}(k) \operatorname{tr}(\omega \times h) + 2 \operatorname{tr}(h) \operatorname{tr}(k) \operatorname{tr}(l \times \omega) \\ &+ 2 \operatorname{tr}(\omega) \operatorname{tr}(k) \operatorname{tr}(l \times h) + 2 \operatorname{tr}(l) \operatorname{tr}(h) \operatorname{tr}(k \times \omega) + 2 \operatorname{tr}(l) \operatorname{tr}(\omega) \operatorname{tr}(k \times h) \\ &+ 2 \operatorname{tr}(\omega) \operatorname{tr}(h) \operatorname{tr}(k \times l) - 4 \operatorname{tr}(k) \operatorname{tr}(l \times \omega \times h) - 4 \operatorname{tr}(l) \operatorname{tr}(k \times \omega \times h) \\ &- 8 \operatorname{tr}(h) \operatorname{tr}(k \times \omega \times l) - 8 \operatorname{tr}(\omega) \operatorname{tr}(k \times h \times l) - 4 \operatorname{tr}(\omega \times l) \operatorname{tr}(k \times h) \\ &- 4 \operatorname{tr}(h \times l) \operatorname{tr}(k \times \omega) - 2 \operatorname{tr}(\omega \times h) \operatorname{tr}(k \times l) \\ &+ 8 \operatorname{tr}(k \times l \times \omega \times h) + 8 \operatorname{tr}(l \times k \times \omega \times h) \\ &+ 8 \operatorname{tr}(l \times \omega \times k \times h) - \operatorname{tr}(k) \operatorname{tr}(l) \operatorname{tr}(h) \operatorname{tr}(\omega) + (\operatorname{tr}(k) \operatorname{tr}(l) - 2 \operatorname{tr}(k \times l)) \varphi\vartheta \}\,, \end{aligned}$$

$$\mathfrak{G}^{-1}[g,\phi](\Gamma[g,\phi](\cdot;\,k,\chi;\,l,\sigma),\,\Gamma[g,\phi](\cdot;\,\omega,\,\varphi;\,h,\vartheta))$$
$$=(1/16)\int_M d\mu(g)\{-d/(d-1)\,\mathrm{tr}(k)\,\mathrm{tr}(l)\,\mathrm{tr}(h)\,\mathrm{tr}(\omega)$$
$$+4[\mathrm{tr}(h\times\omega\times l\times k)+\mathrm{tr}(h\times\omega\times k\times l)+\mathrm{tr}(h\times l\times k\times\omega)+\mathrm{tr}(h\times k\times l\times\omega)]$$
$$+(3d-2)/(d-1)\,[\mathrm{tr}(k)\,\mathrm{tr}(l)\,\mathrm{tr}(h\times\omega)+\mathrm{tr}(h)\,\mathrm{tr}(\omega)\,\mathrm{tr}(k\times l)]$$
$$+\mathrm{tr}(h)\,\mathrm{tr}(k)\,\mathrm{tr}(\omega\times l)+\mathrm{tr}(h)\,\mathrm{tr}(l)\,\mathrm{tr}(\omega\times k)+\mathrm{tr}(\omega)\,\mathrm{tr}(k)\,\mathrm{tr}(h\times l)$$
$$+\mathrm{tr}(\omega)\,\mathrm{tr}(l)\,\mathrm{tr}(h\times k)-4[\mathrm{tr}(h)\,\mathrm{tr}(\omega\times k\times l)+\mathrm{tr}(l)\,(k\times h\times\omega)$$
$$+\mathrm{tr}(\omega)\,\mathrm{tr}(k\times l\times h)+\mathrm{tr}(k)\,\mathrm{tr}(h\times\omega\times l)]-(9d-8)/(d-1)\,\mathrm{tr}(h\times\omega)\,\mathrm{tr}(k\times l)$$
$$+(3d-4)/(d-1)\,(\mathrm{tr}(h\times\omega)\chi\sigma+\mathrm{tr}(k\times l)\vartheta\varphi)-(d-2)/(d-1)\,(\mathrm{tr}(h)\,\mathrm{tr}(\omega)\chi\sigma$$
$$+\mathrm{tr}(k)\,\mathrm{tr}(l)\vartheta\varphi)+(\mathrm{tr}(h)\varphi+\mathrm{tr}(\omega)\vartheta)\,(\mathrm{tr}(l)\chi+\mathrm{tr}(k)\sigma)\,-d/(d-1)\chi\vartheta\varphi\sigma\}\,.$$

Therefore,

$$\mathfrak{Riem}[g,\phi]\,(\omega,\varphi;\,k,\chi;\,l,\sigma;\,h,\vartheta)$$
$$=\int_M d\mu(g)\,\{(1/4)\,\mathrm{tr}[(k\times\omega-\omega\times k)\times(h\times l-l\times h)]$$
$$+\kappa^2[\mathrm{tr}(h\times\omega)\,\mathrm{tr}(k)\,\mathrm{tr}(l)-\mathrm{tr}(l\times\omega)\,\mathrm{tr}(k)\,\mathrm{tr}(h)+\mathrm{tr}(k\times l)\,\mathrm{tr}(\omega)\,\mathrm{tr}(h)$$
$$-\mathrm{tr}(k\times h)\,\mathrm{tr}(\omega)\,\mathrm{tr}(l)+d\{\mathrm{tr}(\omega\times l)\,\mathrm{tr}(k\times h)-\mathrm{tr}(h\times\omega)\,\mathrm{tr}(k\times l)\}$$
$$+\mathrm{tr}(k)\,\mathrm{tr}(l)\vartheta\varphi-\mathrm{tr}(k)\,\mathrm{tr}(h)\varphi\sigma+\mathrm{tr}(\omega)\,\mathrm{tr}(h)\chi\sigma-\mathrm{tr}(\omega)\,\mathrm{tr}(l)\chi\vartheta$$
$$+d\{\mathrm{tr}(\omega\times l)\chi\vartheta-\mathrm{tr}(\omega\times h)\chi\sigma+\mathrm{tr}(k\times h)\varphi\sigma-\mathrm{tr}(k\times l)\vartheta\varphi\}]\}\,.$$

Appendix C

We can consider the metric $\mathfrak{G}$ on $\mathfrak{U}_M^d$ as follows (cf. [12]):

$$\mathfrak{G}_{\text{Superspace}}=\int_M d^dx\,G^x_{\text{Homogeneous}}=\lim_{N\to\infty}\left\{\frac{v(M)}{N}\sum_{i=1}^N G^i_{\text{Homogeneous}}\right\},$$

where

$$v(M)\equiv\int_M d^dx\,,$$

$$G^x_{\text{Homogeneous}}=-d\xi^2(x)+\kappa^2\xi^2(x)\,(G^x_{\text{Conformal}\,\phi})$$
$$=-d\xi^2(x)+\kappa^2\xi^2(x)\,(G^x_{\text{Conformal}}+d\phi^2(x)),$$

then,

$$\Gamma_s=\int_M d^dx\,\Gamma^x_H=\lim_{N\to\infty}\left\{\frac{v(M)}{N}\sum_{i=1}^N \Gamma^i_H\right\},$$

$$\mathfrak{Riem}_s=\int_M d^dx\,\mathrm{Riem}^x_H=\lim_{N\to\infty}\left\{\frac{v(M)}{N}\sum_{i=1}^N \mathrm{Riem}^i_H\right\},\tag{C.1}$$

$$\mathfrak{Ric}_s=\int_M d^dx\,\mathrm{Ric}^x_H=\lim_{N\to\infty}\left\{1\cdot\sum_{i=1}^N \mathrm{Ric}^i_H\right\},\tag{C.2}$$

$$\mathfrak{R}_s=\int_M d^dx\,R^x_H=\lim_{N\to\infty}\left\{\left(\frac{v(M)}{N}\right)^{-1}\sum_{i=1}^N R^i_H\right\}.\tag{C.3}$$

Using these relations we compute $\mathfrak{Riem}_s$ by Riem_H^x. Notice that

$$G_H^x = -d\xi^2(x) + \kappa^2\xi^2(x)G_{c\phi}^x = -d\xi^2(x) + \kappa^2\xi^2(x)(G_c^x + d\phi^2(x))$$
$$= -d\xi^2 + \kappa^2\xi^2(G_c + d\phi^2),$$
$$G_H = -d\xi^2 + f(\xi)G_{c\phi},$$

i.e. G_H is torsional multiplication of $-d\xi^2$ and $G_{c\phi}$ [24]. If ∇^H is a covariant derivative of G_H, ∇^ξ of $-d\xi^2$, $\nabla^{c\phi}$ of $G_{c\phi}$, then [24].

$$\nabla_X^H Y = \nabla_{X_\xi}^\xi Y_\xi + \nabla_{X_{c\phi}}^{c\phi} Y_{c\phi} + \tfrac{1}{2}[X_\xi(\varphi)Y_{c\phi} + Y_\xi(\varphi)X_{c\phi} - G_{c\phi}(X_{c\phi}, Y_{c\phi})\,\mathrm{grad}_\xi(\varphi)],$$

where $(X_\xi, X_{c\phi})$ the natural projection of $X, \psi = \ln[f(\xi)] = \ln(\kappa^2\xi^2)$,

$$\begin{aligned}
&\mathrm{Riem}_H(X,Y)Z = \mathrm{Riem}_\xi(X_\xi, Y_\xi)Z_\xi + \mathrm{Riem}_{c\phi}(X_{c\phi}, Y_{c\phi})Z_{c\phi}\\
&+\tfrac{1}{2}\{h_\psi(X_\xi, Z_\xi)Y_{c\phi} - h_\psi(Y_\xi, Z_\xi)X_{c\phi} + G_{c\phi}(X_{c\phi}, Z_{c\phi})H_\psi(Y_\xi)\\
&\quad - G_{c\phi}(Y_{c\phi}, Z_{c\phi})H_\psi(X_\xi)\} + \tfrac{1}{4}[\{X_\xi(\psi)Z_\xi(\psi) + G_{c\phi}(X_{c\phi}, Z_{c\phi})\,\|d\psi\|^2\}Y_{c\phi}\\
&\quad - \{Y_\xi(\psi)Z_\xi(\psi) + G_{c\phi}(Y_{c\phi}, Z_{c\phi})\,\|d\psi\|^2\}X_{c\phi}\\
&\quad + \{Y_\xi(\psi)G_{c\phi}(X_{c\phi}, Z_{c\phi}) - X_\xi(\psi)G_{c\phi}(Y_{c\phi}, Z_{c\phi})\}\,\mathrm{grad}_\xi(\psi)],
\end{aligned}$$

where

$$h_\psi(X_\xi, Y_\xi) = \frac{d^2\psi}{d\xi^2}X_\xi Y_\xi, \quad \|d\psi\|^2 = -\left(\frac{d\psi}{d\xi}\right)^2, \quad \mathrm{grad}_\xi(\psi) = -\frac{d\psi}{d\xi},$$
$$H_\psi(X_\xi) = -\frac{d^2\psi}{d\xi^2}X_\xi, \quad \mathrm{Riem}_\xi = 0,$$

therefore

$$\begin{aligned}
\mathrm{Riem}_H(X,Y)Z &= \mathrm{Riem}_{c\phi}(X_{c\phi}, Y_{c\phi})Z_{c\phi}\\
&- \kappa^2\{G_{c\phi}(X_{c\phi}, Z_{c\phi})Y_{c\phi} - G_{c\phi}(Y_{c\phi}, Z_{c\phi})X_{c\phi}\}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{Riem}_H(U,Z,X,Y) &= \kappa^2\xi^2[\mathrm{Riem}_{c\phi}(U_{c\phi}, Z_{c\phi}, X_{c\phi}, Y_{c\phi})\\
&- \kappa^2\{G_{c\phi}(X_{c\phi}, Z_{c\phi})G_{c\phi}(Y_{c\phi}, U_{c\phi}) - G_{c\phi}(Y_{c\phi}, Z_{c\phi})G_{c\phi}(X_{c\phi}, U_{c\phi})\}].
\end{aligned}$$

If $U=(k,\chi)$, then $U_{c\phi} = (\bar{k}, \chi)$ [here $Y=(\omega, \vartheta)$] and

$$G_{c\phi}(U,Y) = G_{c\phi}(U_{c\phi}, Y_{c\phi}) = \mathrm{tr}(\bar{k}\times\bar{\omega}) + \chi\vartheta.$$

Hence it follows from Eqs. (C.1–C.3) and [3], that

$$\begin{aligned}
&\mathfrak{Riem}[g,\phi](\omega,\varphi;k,\chi;l,\sigma;h,\vartheta)\\
&= \int_M d\mu(g)\{1/4)\,\mathrm{tr}[(\bar{k}\times\bar{\omega} - \bar{\omega}\times\bar{k})\times(\bar{h}\times\bar{l} - \bar{l}\times\bar{h})]\\
&+ \kappa^2\{\mathrm{tr}(\bar{\omega}\times\bar{l})\,\mathrm{tr}(\bar{k}\times\bar{h}) - \mathrm{tr}(\bar{h}\times\bar{\omega})\,\mathrm{tr}(\bar{k}\times\bar{l})\\
&+ \mathrm{tr}(\bar{\omega}\times\bar{l})\chi\vartheta - \mathrm{tr}(\bar{\omega}\times\bar{h})\chi\sigma + \mathrm{tr}(\bar{k}\times\bar{h})\varphi\sigma - \mathrm{tr}(\bar{k}\times\bar{l})\varphi\vartheta\}\},
\end{aligned} \tag{C.4}$$

and

$$\mathrm{Ric}_H = \frac{d(d-6)}{32}G_{c\phi}, \tag{C.5}$$

$$R_H = \frac{d(d-6)}{32\kappa^2\xi^2}\,\frac{d(d+1)}{2}. \tag{C.6}$$

From [3] we have

$$\mathrm{Ric}_c = -(d/4)G_c\,, \tag{C.7}$$

$$R_c = -(d/4)\left\{\frac{d(d-1)}{2} - 1\right\}. \tag{C.8}$$

In the general case, if there are m scalar fields,

$$\mathrm{Ric}_H = \frac{d}{32(d-1)}\{d^2 - 7d + 2m + 4\}G_{c\phi}\,, \tag{C.9}$$

$$R_H = \frac{d}{32\kappa^2\xi^2(d-1)}\{d^2 - 7d + 2m + 4\}\cdot(d(d+1)/2 - 1 + m)\,. \tag{C.10}$$

Appendix D

Notice, that in HC the instability of geodesic flow does not immediately follow from [3], i.e. from instability on c-HC, because only the existence of the 2-surface is shown, on which the two-dimensional curvature $K = \mathrm{const} < 0$. But in the general case (not only homogeneous Universe) that direction can be from spaces tangent to the orbit (we mean conformal orbit). Hence, we say that two c-Universes are going exponentially away from each other, but in fact this movement is along the orbit. It means that departing c-conformal Universes (conformal metrics) can be very close Universes (conformal geometries), while the c-Universes can differ much.

Now let us show that there is a vector orthogonal to the velocity of geodesics, which is not tangent to the orbit, and on the 2-surface extended on that vector and velocity of geodesics, $\mathfrak{K} = \mathrm{const} < 0$.

Assume that k is a vector tangent to the geodesic. Let us take any β vector field on M and consider

$$\mathfrak{B}_{ab} = \beta_a\beta_b - \frac{\beta_c\beta^c}{d}g_{ab}\,,$$

where

$$\mathfrak{G}[g,\phi](\mathfrak{B},\mathfrak{B}) = 1\,, \qquad \beta^a_{|a}\beta^b + (1-2/d)\beta^a\beta^b_{|a} = 0\,.$$

It is clear that

$$\mathfrak{K}(k,\mathfrak{B}) \equiv \mathfrak{Riem}(k,\mathfrak{B},k,\mathfrak{B}) = \int_M d\mu(g)\,\{(\beta^a k_{ab}\beta^b)^2 - (\beta_c\beta^c)(\beta^a k^c_a k_{cb}\beta^b)\} \leqq 0\,.$$

Choose β so that

$$\beta^a k_{ab}\beta^b = 0\,.$$

There exists such β, because $\mathrm{tr}(k) = 0$, $\mathrm{tr}(k \times k) \neq 0$, which means that (the smallest $\mathfrak{B}$, $\mathfrak{K}(k,\mathfrak{B}) = \mathrm{const} < 0$. As $\mathrm{tr}(\mathfrak{B}) = 0$ and $\mathfrak{B}^a_{b|a} = 0$, then $\mathfrak{B} \in T^{TT}_{(g,\phi)}\mathfrak{U}^d_M$ and we have [see Eqs. (3.1), (3.2)] $\forall (g,\phi) \in \mathfrak{U}^d_M$, $\mathfrak{B} \notin T_{(g,\phi)}$orbit, on which the statement made above is based.

Acknowledgements. The author is grateful to V. G. Gurzadyan, V. V. Harutyunyan, R. T. Jantzen, S. G. Matinyan, R. L. Mkrtchyan, and A. G. Sedrakyan for useful discussions and help. We wish to thank Prof. R. Ruffini and the Dipartimento di Fisica of the University of Rome for their hospitality.

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Communicated by S.-T. Yau