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The Polyakov Path Integral Over Bordered Surfaces

II. The Closed String Off-Shell Amplitudes

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Abstract. Following the general scheme of the convariant path integral quantization of gauge systems, two alternative formulations of the first quantized closed bosonic string in a position representation are presented. In both approaches the covariant path integral representations of the propagator and of the higher order off-shell amplitudes are constructed. For a wide class of gauges the explicit formulae for off-shell amplitudes are obtained. This paper is the continuation of our previous work where the corresponding problems in the open string case were considered [20].

1. Introduction

In the past few years, the elegant and self-contained S-matrix formulation of the interacting bosonic closed string was developed [1]. The basic ingredients of this formulation are the on-shell amplitudes defined by means of the Polyakov path integral over closed surfaces [2] with a prescribed topology and with vertex functionals [3] corresponding to ingoing and outgoing on-shell particle states. This is however thoroughly perturbative formulation and it is an important problem to derive an underlying theory the perturbative expansion of which we have. Despite numerous attempts this goal is not yet completely achieved. One possible way to go beyond the perturbative formulation is to understand whether the Polyakov on-shell amplitudes could be constructed from simpler pieces. If we adapt the ordinary field theoretical scheme of perturbation expansion these building pieces should be interpreted as Green functions (off-shell amplitudes) in a covariant second quantized string theory. The basic idea of the off-shell formulation proposed by Cohen, Moore, Nelson, and Polchinski [4] is that the off-shell amplitudes can be expressed by the Polyakov path integral over bordered surfaces without making use of string fields. Considering the simplest Green functions: the off-shell propagator and the off-shell three string vertex one can try to derive some information about an underlying string field theory [5].

The potential usefulness of the approach sketched above depends on whether the following two problems could be solved. The first one consists in constructing a well-defined path integral representation of off-shell amplitudes without referring to string field theory (much likely the on-shell amplitudes are unambiguously determined by the Polyakov path integral over closed surfaces [1, 6, 71]). The second problem is to derive sewing rules for off-shell amplitudes giving a way for constructing an arbitrary higher order on-shell amplitude from lower order "off-shell pieces."

The first approach to the first problem was proposed in the original paper [4]. The idea, based on Alvarez's work [8], was to consider the path integral over metrized surfaces connecting prescribed contours in the target space. The resulting functional contains an averaging over boundary reparametrizations which in contrast to other functional measures in the Polyakov theory remains undetermined and is an untractable formal symbol. The expression for the off-shell amplitudes is therefore well defined only for point-like string where this averaging decouples [9–11]. In several papers [12–17] it was pointed out that the off-shell amplitudes in string theory should be gauge dependent and the averaging over boundary reparametrizations seems to be spurious. At present the structure of the BRST extended off-shell closed string propagator is rather well understood. The derivation based on the proper time representation was proposed by Birmingham and Torre [12] and by Lee [13]. Another approach using the Batalin-Fradkin-Vilkoviski phase space path integral was presented by Karchev [14]. These results are closely related to the canonical operator (first) quantization, therefore a generalization to higher order amplitudes is not straightforward. On the other hand the considerations of [15, 16] are mainly of string field theory nature and the original geometrical interpretation of the Polyakov path integral over surfaces is lost. In particular the relation between boundary parametrization and a conformal structure on the world sheet remains unclear. This leads to problems in interpreting the off-shell amplitudes as functionals on boundary values of x- and ghosts variables [16, 17]. Despite difficulties with the definition of the off-shell amplitudes, important progress was recently made in the second problem of the off-shell approach [17–19]. In particular the problem of sewing at a fixed conformal structure was completely solved [19].

In this paper we address the first problem of the off-shell formulation – the construction of a path integral representation of closed string off-shell amplitudes. It is a continuation of our previous paper [20] where the corresponding problem in the open string case was considered.

In Sect. 2 the construction of the space of closed string wave functionals in the position representation is presented. The considerations of this section are based on the general scheme of the covariant path integral quantization of gauge systems with first class constraints described in [20, 21]. Similarly as in the open string case we consider two formulations determined by two different choices of the space of boundary conditions for closed string trajectories. In the first formulation this space is chosen as the space $\dot{\mathscr{C}}^d_M$ of closed oriented metrized contours with marked points, while in the second one as the quotient space $\bar{\mathscr{C}}^d_M = \dot{\mathscr{C}}^d_M/\mathbb{R}_+$. Special attention is paid to the gauge independent description of the residual gauge invariance and to the construction of ultralocal inner products in both formulations.

In Sect. 3 the path integral representation of the off-shell closed string propagator is derived and evaluated. It is shown that this representation requires a choice of an additional geometrical data concerning the global structure of the space of trajectories which can be interpreted as a choice of a gauge. For a wide class of gauges an explicit formula for the propagator is obtained. In the special gauges (called in this paper the fixed length and the constant curvature gauge) the known expression [4, 12–14] is reproduced. Let us note that due to the presence of the conformal Killing vector field on the cylinder the application of the F–P method is slightly more complicated than in the case of the rectangle and requires generalized (incomplete) gauges.

In Sect. 4 the path integral representations of the off-shell closed string amplitudes are derived. In the formulation based on the space $\dot{\mathscr{C}}^d_M$ and in the fixed length gauge they have a structure similar to that of the off-shell amplitudes considered in [17, 18]. Unlike in [17, 18] the boundary values of the x-variables are related in the present approach to a conformal structure on the world sheet via the reparametrization invariant boundary conditions. Moreover the final expression for amplitudes contains an averaging over boundary twists (this provides a natural way for incorporating an integration over relative twist in sewing rules). In the formulation based on the space $\dot{\mathscr{C}}^d_M$, in the constant curvature gauge the new expression for the off-shell amplitudes is obtained. This formulation was very much influenced by the work of D'Hoker and Phong [26].

Section 5 includes some comments about the first quantized string in the position representation and the discussion of some consequences of the present approach for the sewing problem.

2. The Space of Closed String Wave Functionals

Within the covariant path integral framework [20, 21] the first stage of quantization consists in the choice of a space of boundary conditions for trajectories of a system which determines a space of states. Guided by the open string case let us first consider the space $\mathscr{C}^d_S$ of all oriented closed metrized contours in the target space $\mathbb{R}^d$. It is defined by the following quotient construction. Let S denote some fixed model of 1-dim sphere, $\mathscr{M}_s$ – the space of all einbeins on S, and $\mathscr{E}^d_S$ – the space of all mappings $x: S \to \mathbb{R}^d$. The action of the group $\mathscr{D}_S$ of all orientation preserving diffeomorphisms of S on the space $\mathscr{M}_S \times \mathscr{E}^d_S$, defined by

$$\mathscr{M}_S \times \mathscr{E}^d_S \ni (e, \tilde{x}) \xrightarrow{\gamma \in \mathscr{D}_S} (\gamma^* e, \gamma^* \tilde{x}) \in \mathscr{M}_S \times \mathscr{E}^d_S$$

induces the principal fiber bundle structure:

$$\begin{array}{ccc} \mathscr{D}_S & \longrightarrow & \mathscr{M}_S \times \mathscr{E}^d_S \\ & & \Big\downarrow \Pi_{\dot{\mathscr{C}}} \\ & & \mathscr{C}^d_S \end{array} \quad . \tag{2.1}$$

Unlike in the open string case the bundle above is nontrivial and there are no tractable parametrizations of the base space $\mathscr{C}^d_S \approx \mathbb{R}_+ \times (\mathscr{E}^d_S/S^1)$. For this reason we will consider instead of $\mathscr{C}^d_S$ the space $\dot{\mathscr{C}}^d_S$ of closed oriented metrized contours with marked points. The space $\dot{\mathscr{C}}^d_S$ is defined as the base space of the principal fiber bundle:

$$\begin{array}{ccc} \mathscr{D}_S & \longrightarrow & \mathscr{M}_S \times S \times \mathscr{E}^d_S \\ & & \Big\downarrow \Pi_{\dot{\mathscr{C}}} \\ & & \dot{\mathscr{C}}^d_S \approx \mathbb{R}_+ \times \mathscr{E}^d_S , \end{array} \tag{2.2}$$

where the group action is determined by:

$$\mathcal{M}_S \times S \times \mathcal{E}_S^d \ni (e, s, \tilde{x}) \xrightarrow{\gamma \in \mathcal{D}_S} (\gamma^* e, \gamma^{-1}(s), \gamma^* \tilde{x}) \in \mathcal{M}_S \times S \times \mathcal{E}_S^d .$$

We define the space $\mathfrak{H}^{\text{off}}$ of the off-shell closed string wave functionals as the space of all functionals on $\dot{\mathcal{C}}_S^d$. Equivalently it can be defined as the space of $\mathcal{D}_S$-invariant functionals on $\mathcal{M}_S \times S \times \mathcal{E}_S^d$. In contrast to the bundle (2.1) the bundle (2.2) is trivial. In the sequel we will use a special class of global sections of (2.2) determined by the family of the gauge slices:

$$\varphi_{\hat{e},\hat{s}} = \{(e, s, \tilde{x}) \in \mathcal{M}_S \times S \times \mathcal{E}_S^d : e = \text{const} \times \hat{e},\ s = \hat{s}\} . \tag{2.3}$$

In these gauges a string wave functional $\Psi[\dot{c}] \in \mathfrak{H}^{\text{off}}$ can be regarded as a functional $\Psi_{\hat{e},\hat{s}}[\alpha, \tilde{x}] = \Psi[\Pi_{\mathcal{C}}(\alpha\hat{e}, \hat{s}, \tilde{x})]$ on $\mathbb{R}_+ \times \mathcal{E}_S^d$.

Let us now turn to the description of residual gauge transformations (i.e. transformations induced on the space of boundary conditions by gauge transformations on the space of string trajectories [20, 21]). In the space $\mathcal{M}_S \times S \times \mathcal{E}_S^d$ a large part of this transformation is described by the semidirect product $\mathcal{D}_S \odot \mathcal{W}_S$, where $\mathcal{W}_S$ denotes the additive group of all real valued functions on S with the following action on $\mathcal{M}_S \times S \times \mathcal{E}_S^d$:

$$\mathcal{M}_S \times S \times \mathcal{E}_S^d \ni (e, s, \tilde{x}) \xrightarrow{\varphi \in \mathcal{W}_S} (\exp(\varphi) e, s, \tilde{x}) \in \mathcal{M}_S \times S \times \mathcal{E}_S^d .$$

As a result of the extension of $\mathcal{M}_S \times \mathcal{E}_S^d$ to $\mathcal{M}_S \times S \times \mathcal{E}_S^d$ there exist additional residual gauge transformations connecting metrized contours with different marked points. It is convenient to describe these transformations in the following way. Let (S, e) be an oriented 1-dim manifold diffeomorphic to S^1 endowed with a Riemannian metric e^2. For every $\tau \in \mathbb{R}$ we define the diffeomorphism:

$$i[S, e|\tau] : S \ni s \to s + \tau \in S , \tag{2.4}$$

where $s + \tau$ denotes the shift of the point $s \in S$ by the Riemannian distance $|\tau|$ in the direction determined by the orientation of S for $\tau > 0$ and in the opposite direction for $\tau < 0$. Let $[\mathbb{R}, +\text{mod}\,2\pi]$ be the standard parametrization of $U(1)$, then the map:

$$i[S, e|\cdot] : \mathbb{R} \ni \theta \to i\left[S, e \,\middle|\, \frac{l\theta}{2\pi} \right] \in \mathcal{D}_S ,$$

where $l = \int_S e ds$, provides an isomorphism of $U(1)$ onto the group of the orientation preserving isometries of (S, e). We define the $U(1)$-action on $\mathcal{M}_S \times S \times \mathcal{E}_S^d$ by:

$$\mathcal{M}_S \times S \times \mathcal{E}_S^d \ni (e, s, \tilde{x}) \xrightarrow{\theta \in U(1)} \left(e, s + \frac{l\theta}{2\pi}, \tilde{x} \right) \in \mathcal{M}_S \times S \times \mathcal{E}_S^d .$$

The whole group of the residual gauge transformations in the space $\mathcal{M}_S \times S \times \mathcal{E}_S^d$ is then described by the semidirect product $(\mathcal{W}_S \odot (U(1) \times \mathcal{D}_S)$. In the space $\dot{\mathcal{C}}_S^d$ and in the 1-dimensional conformal gauges (2.3) this group reduces to the direct product $\mathbb{R}_+ \times \tilde{\mathcal{D}}_S(\hat{e}, \hat{s})$. $\mathbb{R}_+$ denotes the 1-dim group of constant rescalings of einbeins while $\tilde{\mathcal{D}}_S(\hat{e}, \hat{s})$ is the subgroup of $(\mathcal{W}_S \odot (U(1) \times \mathcal{D}_S)$ defined by:

$$\tilde{\mathcal{D}}_S(\hat{e}, \hat{s}) = \{(\varphi, u, \gamma) \in (\mathcal{W}_S \odot (U(1) \times \mathcal{D}_S) : \exp(\varphi)\gamma^* \hat{e} = \hat{e},\ u \circ \gamma^{-1}(\hat{s}) = \hat{s}\} .$$

In order to obtain a gauge independent description of residual gauge transformations in $\dot{\mathcal{C}}_S^d$ we introduce the space $\mathcal{M}_S \times S \times \tilde{\mathcal{D}}_S$, where $\tilde{\mathcal{D}}_S$ denotes another copy of the group $\mathscr{D}_S$. The action of the group $\mathbb{R}_+ \times \mathscr{D}_S$ on $\mathscr{M}_S \times S \times \tilde{\mathscr{D}}_S$ defined by:

$$\mathscr{M}_S \times S \times \tilde{\mathscr{D}}_S \ni (e, s, \tilde{\gamma}) \xrightarrow{(\lambda, \gamma) \in \mathbb{R}_+ \times \mathscr{D}_S} (\lambda \gamma^* e, \gamma^{-1}(s), \gamma^{-1} \tilde{\gamma} \gamma) \in \mathscr{M}_S \times S \times \tilde{\mathscr{D}}_S$$

induces the principal fiber bundle structure:

$$\begin{array}{ccc} \mathscr{D}_S & \longrightarrow & \mathscr{M}_S \times S \times \tilde{\mathscr{D}}_S \\ & & \Big\downarrow \Pi_{\mathscr{G}} \\ & & \mathscr{G}_S\,. \end{array} \tag{2.5}$$

Let us consider the following family of global gauge slices of the bundle (2.5):

$$\varphi_{\hat{e}, \hat{s}} = \{(e, s, \tilde{\gamma}) \in \mathscr{M}_S \times S \times \tilde{\mathscr{D}}_S : e = \hat{e},\ s = \hat{s}\}, \tag{2.6}$$

We define the group structure on $\mathscr{G}_S$ by:

$$\mathscr{G}_S \times \mathscr{G}_S \ni (\Gamma, \Gamma') \to \Gamma \Gamma' = \Pi_{\mathscr{G}}(\hat{e}, \hat{s}, \tilde{\gamma} \circ \tilde{\gamma}'), \tag{2.7}$$

where $\Gamma = \Pi_{\mathscr{G}}(\hat{e}, s, \tilde{\gamma})$ and $\Gamma' = \Pi_{\mathscr{G}}(\hat{e}, \hat{s}, \tilde{\gamma})$.
The action of $\mathscr{G}_S$ on $\dot{\mathscr{C}}_S^d$ is defined by:

$$\dot{\mathscr{C}}_S^d \ni \dot{c} \xrightarrow{\Gamma \in \mathscr{G}_S} \Gamma \dot{c} = \Pi_{\mathscr{C}}(\alpha \hat{e}, \hat{s}, \tilde{x} \circ \tilde{\gamma}) \in \dot{\mathscr{C}}_S^d, \tag{2.8}$$

where $\dot{c} = \Pi_{\mathscr{C}}(\alpha \hat{e}, \hat{s}, \tilde{x})$ and $\Gamma = \Pi_{\mathscr{G}}(\hat{e}, \hat{s}, \tilde{\gamma})$.

Both definitions (2.7), (2.8) are independent of the choice of $(\hat{e}, \hat{s})$. It follows from (2.7) that $\mathscr{C}_S$ is isomorphic to $\mathscr{D}_S$. We also have:

$$\dot{\mathscr{C}}_S^d / \mathbb{R}_+ \times \mathscr{G}_S \approx \mathscr{K}_S^d,$$

where $\mathscr{K}_S^d$ denotes the space of closed oriented (not metrized) contours in $\mathbb{R}^d$.

In the sequel we will use two subgroups of $\mathscr{G}_S$ defined as quotient spaces:

$$\mathscr{G}_S^0 = \varphi_S / \mathscr{D}_S, \qquad \mathscr{F}_S = \mathscr{U}_S / \mathscr{D}_S,$$

where

$$\varphi_S = \{(e, s, \tilde{\gamma}) \in \mathscr{M}_S \times S \times \tilde{\mathscr{D}}_S : \tilde{\gamma}^{-1}(s) = s\},$$
$$\mathscr{U}_S = \{(e, s, \tilde{\gamma}) \in \mathscr{M}_S \times S \times \tilde{\mathscr{D}}_S : \tilde{\gamma}^* e = e\}.$$

It follows from the definition that the subgroup $\mathscr{G}_S^0$ is isomorphic to the group $\mathscr{D}_S(s) = \{\gamma \in \mathscr{D}_S : \gamma(s) = s\}$. By means of (2.4) one can construct the following isomorphism of $U(1) = [\mathbb{R}, +\mathrm{mod}\, 2\pi]$ onto $\mathscr{I}_S$:

$$I : \mathbb{R} \ni \theta \to I_\theta = \Pi_{\mathscr{G}}\left(\hat{e}, \hat{s}, i\left[S, \hat{e}\left|\frac{\hat{1}\theta}{2\pi}\right]\right.\right) \in \mathscr{I}_S. \tag{2.9}$$

Let us recall that within the convariant functional approach the first class constraints linear in momenta are implemented in the quantum theory by means of the residual gauge invariance [20, 21]. The subspace $\mathfrak{H}_{ph}^{\mathrm{off}} \subset \mathfrak{H}^{\mathrm{off}}$ consisting of all $\mathbb{R}_+ \times \mathscr{G}_S$-invariant wave functionals can then be interpreted as the space of all string states annihilated by all linear in momenta constraints operators.

Our next task is to introduce an inner product in the space $\mathfrak{H}_{ph}^{\mathrm{off}}$. Following the geometrical approach to functional integration of invariant objects [6] we start with the $U(1)\odot\mathscr{D}_S$-invariant ultralocal Riemannian structure $G(\cdot\,|\,\cdot)$ on the space $\mathscr{M}_S\times S\times\mathscr{E}^d_S$:

$$G_{(e,s,\tilde{x})}(\delta e,\delta s,\delta\tilde{x}\,|\,\delta e',\delta s',\delta\tilde{x})$$
$$=\int\frac{\delta e\delta e'}{e}ds+e^2(s)\delta s\delta s'+\int e\delta\tilde{x}\delta\tilde{x}'ds\,, \tag{2.10}$$

where $(\delta e,\delta s,\delta\tilde{x})$, $(\delta e',\delta s',\delta\tilde{x}')\in\mathscr{T}_{(e,s,\tilde{x})}(\mathscr{M}_S\times S\times\mathscr{E}^d_S)$.

We define the inner product $(\cdot,\cdot)$ in $\mathfrak{H}^{\mathrm{off}}_{ph}$ by the following formal expression:

$$(\Psi,\Psi')=\int\limits_{\mathscr{M}_S\times S\times\mathscr{E}^d_S}\mathscr{D}(e,s,\tilde{x})\Big(\int\limits_{\mathscr{D}_S}\mathscr{D}\gamma\Big)^{-1}\Big(\int\limits_{U(1)}du\Big)^{-1}\Big(\int\limits_{\mathscr{W}_S}\mathscr{D}\varphi\Big)^{-1}\Psi[R\dot{c}]\Psi'[\dot{c}]\,, \tag{2.11}$$

where the functional measure $\mathscr{D}(e,s,\tilde{x})$ is related to the metric structure (2.10) and wave functionaly are regarded as $\mathscr{D}_S$-invariant functionals on $\mathscr{M}_S\times S\times\mathscr{E}^d_S$. The map $R:\dot{\mathscr{C}}^d_S\to\dot{\mathscr{C}}^d_S$ is defined by:

$$R:\dot{\mathscr{C}}^d_S\ni\dot{c}\to R\dot{c}=\Pi_{\mathscr{C}}(e,\ s,\ \tilde{x}\circ r[e,s])\in\dot{\mathscr{C}}^d_S\,,$$

where $\dot{c}=\Pi_{\dot{\mathscr{C}}}(e,s,\tilde{x})$ and $r[e,s]$ denotes the orientation reversing isometry of (S,e) uniquely determined by the condition $r[e,s](s)=s$.

In the 1-dim conformal gauges (2.3) the standard application of the F–P procedure with respect to the group $\mathscr{D}_S$ yields:

$$(\Psi,\Psi')=\int\limits_0^\infty\hat{l}d\alpha(\hat{l}\alpha)^{-1}\Big(\int\limits_{\mathscr{W}_S}\mathscr{D}\varphi\Big)^{-1}\int\limits_{\mathscr{E}^d_S}\mathscr{D}^{\alpha\hat{e}}\tilde{x}$$
$$\times\bar{\Psi}_{\hat{e},\hat{s}}[\alpha,\tilde{x}\circ r[\hat{e},\hat{s}]]\Psi'_{\hat{e},\hat{s}}[\alpha,\tilde{x}]\,, \tag{2.12}$$

where $\hat{l}=\int\hat{e}ds$ and the relation $\mathrm{Vol}(U(1))=\hat{l}\alpha$ was used. The functional measure $\mathscr{D}^{\alpha\hat{e}}\tilde{x}$ is related to the metric structure $E^{\alpha\hat{e}}(\cdot\,|\,\cdot)$ on $\mathscr{E}^d_S$:

$$E^{\alpha\hat{e}}(\delta\tilde{x}\,|\,\delta\tilde{x}')=\int\limits_S\alpha\hat{e}\delta\tilde{x}\delta\tilde{x}ds\,.$$

Note that the inner product (2.12) is independent of the choice of a gauge (2.3). It can be regarded as a path integral over $\dot{\mathscr{C}}^d_S$:

$$(\Psi,\Psi')=\int\limits_{\dot{\mathscr{C}}^d_S}\mathscr{D}\dot{c}\Big(\int\limits_{\mathscr{W}_S}\mathscr{D}\varphi\Big)^{-1}\bar{\Psi}[R\dot{c}]\Psi'[\dot{c}]\,. \tag{2.13}$$

The functional measure $\mathscr{D}\dot{c}$ in the formula above is related to the ultralocal Riemannian structure $C(\cdot\,|\,\cdot)$ on $\dot{\mathscr{C}}^d_S$ defined as a pull back of the following metric structure on $\varphi_{\hat{e},\hat{s}}$:

$$G^{\hat{e},\hat{s}}(\alpha\hat{e},\hat{s},\tilde{x})(\delta\alpha,\delta\tilde{x}\,|\,\delta\alpha',\delta\tilde{x}')=\frac{\delta\alpha\delta\alpha'}{\alpha^2}+E^{\alpha\hat{e}}\tilde{x}(\delta\tilde{x}\,|\,\delta\tilde{x}')$$

by the corresponding global section of (2.2). Let us note that the metric structure $C(\cdot\,|\,\cdot)$ is $\mathscr{I}_S$-invariant. It is however not $\mathbb{R}_+\times\mathscr{G}^0_S$-invariant and cannot be reduced by the F–P method to an ultralocal Riemannian structure on the space $\mathscr{K}^d_S\approx\dot{\mathscr{C}}^d_S/\mathbb{R}_+\times\mathscr{G}_S$.

As in the open string case there is an alternative formulation in which the space $\bar{\mathscr{C}}^d_S=\dot{\mathscr{C}}^d_S/\mathbb{R}_+$ plays the role of the space of boundary conditions for string trajectories. The space $\bar{\mathfrak{H}}^{\mathrm{off}}$ of the off-shell string states is defined as the space of all functionals on $\bar{\mathscr{C}}^d_S$ and the subspace $\bar{\mathfrak{H}}^{\mathrm{off}}_{ph}\subset\bar{\mathfrak{H}}^{\mathrm{off}}$ of the off-shell "physical" states consists of all $\mathscr{G}_S$-invariant functionals on $\dot{\bar{\mathscr{C}}}_S^d$. The 1-dim conformal gauges are in this case determined by the global gauge slices:

$$\mathscr{S}_{\hat{e},\hat{s}}=\{(e,s,\tilde{x})\in\mathscr{M}_S\times S\times\mathscr{E}_S^d: e=\hat{e},\ s=\hat{s}\} \tag{2.14}$$

of the principal fiber bundle:

$$\begin{array}{c}\mathbb{R}_+\times\mathscr{D}_S\longrightarrow\mathscr{M}_S\times S\times\mathscr{E}_S^d\\ \qquad\qquad\qquad\big\downarrow \Pi_{\dot{\bar{\mathscr{C}}}}\\ \qquad\qquad\qquad\dot{\bar{\mathscr{C}}}_S^d\approx\mathscr{E}_S^d.\end{array} \tag{2.15}$$

Since the metric structure $C(\cdot|\cdot)_{\dot{}}$ is not $\mathbb{R}_+$-invariant in order to determine an ultralocal metric structure on $\dot{\mathscr{C}}_S^d$ one should restrict himself to the subspace $\dot{\mathscr{C}}_S^d(\ell)\subset\dot{\mathscr{C}}_S^d$ $(\dot{\mathscr{C}}_S^d(\ell)\approx\dot{\bar{\mathscr{C}}}_S^d)$ of contours with fixed intrinsic length $\int e ds=\ell$. This restriction can be regarded as a part of the gauge fixing in the formulation based on the space $\mathfrak{H}^{\mathrm{off}}$. In this gauge and in the 1-dim conformal gauges (2.14) the inner product $(\cdot|\cdot)^{\ell}$ in $\mathfrak{H}_{ph}^{\mathrm{off}}$ is determined by:

$$(\Psi|\Psi')^{\ell}=\ell^{-1}\int_{\mathscr{E}_S^d}\mathscr{D}^{\hat{e}}\tilde{x}\left(\int_{\mathscr{W}_S^0}\mathscr{D}\varphi\right)^{-1}\bar{\Psi}_{\hat{e},\hat{s}}[\tilde{x}\circ r[\hat{e},\hat{s}]]\Psi_{\hat{e},\hat{s}}[\tilde{x}], \tag{2.16}$$

where $\ell=\int\hat{e}ds$ and $\mathscr{W}_S^0=\{\varphi\in\mathscr{W}_S:\int\exp(\varphi)\hat{e}ds=\ell\}$.

In the end of this section we will briefly describe the gauge invariant parametrizations of the spaces $\dot{\mathscr{C}}_S^d,\dot{\bar{\mathscr{C}}}_S^d$. Given $(e,s)\in\mathscr{M}_S\times S$ let us consider the 1-dim Laplace-Beltrami operator:

$$\mathscr{L}_e=-e^{-1}\frac{d}{ds}e^{-1}\frac{d}{ds}$$

acting on the space of scalar real valued functions on S. Let us denote by $\{\psi_n^{e,s}\}_{n=-\infty}^{+\infty}$ the complete basis of all normalized eigenfunctions of $\mathscr{L}_e$:

$$\begin{gathered}\mathscr{L}_e\psi_n^{e,s}=\lambda_n\psi_n^{e,s},\\ \psi_n^{e,s}\circ r[e,s]=\psi_n^{e,s},\qquad n\geqq 0,\\ \psi_n^{e,s}\circ r[e,s]=-\psi_n^{e,s},\qquad n<0,\\ \int e\psi_n^{e,s}\psi_m^{e,s}ds=\delta_{nm}.\end{gathered}$$

We define the map $\mathscr{P}$ with values in the space Λ^d of infinite sequences of modes in the following way:

$$\begin{gathered}\mathscr{P}:\mathscr{M}_S\times S\times\mathscr{E}_S^d\ni(e,s,\tilde{x})\to(\ell,\dot{c}_0^\mu,\dot{c}_{+1}^\mu,\dot{c}_{+2}^\mu,\dots)\in\Lambda^d,\\ \ell=\int e ds,\\ \dot{c}_n^\mu=\int e\psi_n^{e,s}\tilde{x}^\mu ds.\end{gathered} \tag{2.17}$$

It can be easily verified that $\mathscr{P}$ provides the gauge invariant parametrization of $\dot{\mathscr{C}}_S^d$. The corresponding parametrization of the space $\dot{\bar{\mathscr{C}}}_S^d$ has the following form:

$$\begin{gathered}\bar{\mathscr{P}}:\mathscr{M}_S\times S\times\mathscr{E}_S^d\ni(e,s,\tilde{x})\to(\dot{\bar{c}}_0^\mu,\dot{\bar{c}}_{+1}^\mu,\dot{\bar{c}}_{+2}^\mu,\dots)\in\bar{\Lambda}^d,\\ \dot{\bar{c}}_n^\mu=\ell^{-1/2}\dot{c}_n^\mu.\end{gathered} \tag{2.18}$$

3. The Off-Shell Closed String Propagator

Let us fix a model 2-dim cylinder M with two boundary components $\Sigma_i, \Sigma_f \subset \partial M$ called the initial and the final one respectively. We fix an orientation of M then Σ_i, Σ_f acquire the induced orientations. Let $\mathcal{M}_M$ denote the space of all Riemannian metrics on M and $\mathcal{E}^d_M$ – the space of all mappings $x: M \to \mathbb{R}^d$. The space $\bar{\mathcal{T}}_M(i,f)$ of all closed string trajectories starting at an initial class $\dot{\bar{c}} \in \dot{\mathcal{C}}^d_S$ of metrized contours with marked a point and ending at a final one $\dot{\bar{c}}_f \in \dot{\mathcal{C}}^d_S$ consists of elements $(g, \sigma_i, \sigma_f, x)$ of the space $\mathcal{M}_M \times \Sigma_i \times \Sigma_f \times \mathcal{E}^d_M$ fulfilling the conditions:

$$\Pi_{\dot{\mathcal{C}}}(\varrho_i^* e_i, \varrho_i^{-1}(\sigma_i), \varrho_i^* x_i) = \Pi_{\dot{\mathcal{C}}}(\hat{e}, \hat{s}, \tilde{x}_i) = \dot{\bar{c}}_i, \quad (i \to f), \tag{3.1}$$

where:

$$e_i^2 = I_i^* g, \quad x_i = x \circ I_i, \quad (i \to f),$$

and $I_i: \Sigma_i \to M$, $I_f: \Sigma_f \to M$ denote the inclusions of the initial and the final boundaries of M respectively. ϱ_i and ϱ_f in (3.1) are arbitrary diffeomorphisms:

$$\varrho_i: \Sigma_i \to S, \quad (i \to f). \tag{3.2}$$

A special remark is required concerning orientation. We fix the convention (consistent with our definition of the inner product $(\cdot|\cdot)$) in which both parametrizations (3.2) are orientation preserving diffeomorphisms. With this convention the conditions (3.1) are independent of the choice of ϱ_i, ϱ_f.

The requirement of the consistency of the F–P method imposes some additional conditions on the space of allowed trajectories (see [20] for discussion of this point in the open string case). First of all one should replace the space $\mathcal{M}_M$ by an integral submanifold $\mathcal{M}^\iota_M \subset \mathcal{M}_M$ of the distribution in $\mathcal{T}\mathcal{M}_M$ determined by the Alvarez (mixed) boundary conditions for metric variations [20]. Let M^D denote the double of M and $\iota: M^D \to M^D$ some (arbitrary chosen) involution on M^D, $\iota^2 = \mathrm{id}_{M^D}$. The subspace $\mathcal{M}^\iota$ consists of all metrics on M admitting smooth ι-symmetric extensions on M^D. In a similar way we define the subgroup $\mathcal{D}^\iota_M \subset \mathcal{D}_M$ of all orientation preserving connected to the identity diffeomorphisms f of M with the smooth ι-symmetric extension f^D on $M^D (\iota \circ f^D = f^D \circ \iota)$ and the additive group $\mathcal{W}^\iota_M$ of all real valued functions on M with the Neumann boundary conditions $n^a_\iota \partial_a \phi = 0$ (n_ι denotes the invariant direction of the involution ι). In addition we introduce the subgroup $\bar{\mathcal{W}}^\iota_M \subset \mathcal{W}^\iota_M$ consisting of all functions constant on ∂M and the subgroup $\mathcal{W}^{\iota 0}_M \subset \bar{\mathcal{W}}^\iota_M \subset \mathcal{W}^\iota_M$ of all functions vanishing on ∂M. We restrict the space of trajectories to the space:

$$\bar{\mathcal{T}}^\iota_M(\iota, f) = \bar{\mathcal{T}}_M(\iota, f) \cap \mathcal{M}^\iota_M \times \Sigma_\iota \times \Sigma_f \times \mathcal{E}^d_M.$$

In order to formulate an additional condition for allowed trajectories let us consider the action of the semidirect product $\mathcal{D}^\iota_M \odot \mathcal{W}^\iota_M$ on $\mathcal{M}^\iota_M \times \Sigma_\iota \times \Sigma_f$ defined by:

$$(g, \sigma_\iota, \sigma_f) \xrightarrow{(f,\phi) \in \mathcal{D}^\iota_M \odot \mathcal{W}^\iota_M} (\exp(\phi) f^* g, f^{-1}(\sigma_\iota), f^{-1}(\sigma_f)).$$

It induces the principal fiber bundle structure:

$$\begin{array}{ccc} \mathcal{D}^\iota_M \odot \mathcal{W}^\iota_M & \longrightarrow & \mathcal{M}^\iota_M \times \Sigma_\iota \times \Sigma_f \\ & & \Big\downarrow \Pi_{TS} \\ & & T^R_M \times S^1 \end{array} \tag{3.3}$$

over the cartesian product $T_M^R \times S^1$ of the relative Teichmüller space T_M^R of M [23, 24] and the 1-dim sphere S^1.

Let us consider the space $\mathfrak{B}_M$ of all $\mathscr{D}_M^{\imath} \odot \mathscr{W}_M^{\imath}$-subbundles of (3.3). For every $\mathscr{B}_M \in \mathfrak{B}_M$ and $\ell \in \mathbb{R}_+$ we define the $\mathscr{D}^{\imath} \odot \mathscr{W}_M^{\imath 0}$-subbundle of (3.3):

$$\mathscr{B}_M(\ell) = \left\{(g, \sigma_\imath, \sigma) \in \mathscr{B}_M : \int_{\Sigma} e_\imath d\sigma = \int_{\Sigma_f} e_f d\sigma = \ell\right\}. \tag{3.4}$$

Now we are ready to define the path integral representation of the closed string propagator in the formulation based on the space $\bar{\mathfrak{H}}^{\mathrm{off}}$. Let us recall that within this formulation in order to define an inner product in the space $\bar{\mathfrak{H}}_{ph}^{\mathrm{off}}$ one should fix a gauge $\dot{\mathscr{C}}_S^d(\ell) \approx \dot{\mathscr{C}}_S^d$. In this gauge, for every $\mathscr{B}_M \in \mathfrak{B}_M$ we define the string propagator by the following formal expression:

$$P[\dot{c}_\imath, \dot{c}_f \,|\, \mathscr{B}_M(\ell)] \equiv \int_{\bar{\mathscr{T}}_M^{\imath}(\imath, f \,|\, \mathscr{B}_M(\ell))} \mathscr{D}(g, \sigma_\imath, \sigma_f, x) \left(\int_{\mathscr{D}_M^{\imath}} \mathscr{D}f \times \int_{\mathscr{W}_M^{\imath 0}}\right)^{-1} e^{-S[g,x]}, \tag{3.5}$$

where:

$$\bar{\mathscr{T}}_M^{\imath}(\imath, f \,|\, \mathscr{B}_M(\ell)) = \bar{\mathscr{T}}_M^{\imath}(\imath, f) \cap \mathscr{B}_M(\ell) \times \mathscr{E}_M^d,$$

and $S[g, x]$ denotes the BDHP action for the bosonic string [22]. According to Polykov's ideas [2, 6], the functional measure $\mathscr{D}(g, \sigma_\imath, \sigma_f, x)$ in (3.5) is treated as an infinite dimensional volume form related to the ultralocal Riemannian structure $G(\cdot\,|\,\cdot)$ on $\mathscr{M}_M \times \Sigma_\imath \times \Sigma_f \times \mathscr{E}_M^d$ defined by:

$$\begin{aligned} &G_{(e, \sigma_\imath, \sigma_f, x)}(\delta e, \delta\sigma_\imath, \delta\sigma_f, \delta x \,|\, \delta e', \delta\sigma'_\imath, \delta\sigma'_f, \delta x') \\ &\quad = M_g(\delta g \,|\, \delta g') + e_\imath^2(\sigma_\imath)\delta\sigma_\imath\delta\sigma'_\imath + e_f^2(\sigma_f)\delta\sigma_f\delta\sigma'_f + E_x^g(\delta x \,|\, \delta x'), \end{aligned} \tag{3.6}$$

where $\delta g, \delta g' \in \mathscr{T}_g\mathscr{M}_M$, $\delta\sigma_\imath, \delta\sigma'_\imath \in T_\sigma\Sigma_\imath$, $\delta\sigma_f, \delta\sigma'_f \in T_{\varrho_f}\Sigma_f$, $\delta x, \delta x' \in \mathscr{T}_x\mathscr{E}_M^d \approx \mathscr{E}_M^d$ and

$$M_g(\delta g \,|\, \delta g') = \int_M \sqrt{g}\, d^2z\, g^{ac} g^{bd} \delta g_{ab} \delta g'_{cd}, \tag{3.7}$$

$$E_x^g(\delta x \,|\, \delta x') = \int_M \sqrt{g}\, d^2z\, \delta x^\mu \delta x'^\mu. \tag{3.8}$$

Similarly the functional measures $\mathscr{D}f, \mathscr{D}\phi$ are related to the Riemannian structures $H^g(\cdot\,|\,\cdot)$, $W^g(\cdot\,|\,\cdot)$ respectively:

$$H_{\imath d}^g(\delta f \,|\, \delta f') \equiv \int_M \sqrt{g}\, d^2z\, g_{ab} \delta f^a \delta f'^b, \qquad \delta f, \delta f' \in \mathscr{T}_{\imath d}\mathscr{D}_M^{\imath}, \tag{3.9}$$

$$W_\phi^g(\delta\phi \,|\, \delta\phi') \equiv \int_M \sqrt{g}\, d^2z\, \delta\phi\delta\phi', \qquad \delta\phi, \delta\phi' \in \mathscr{T}_\phi\mathscr{W}_M^{\imath 0} \approx \mathscr{W}_M^{\imath 0}. \tag{3.10}$$

The integration over the subspace of $\bar{\mathscr{T}}_M^{\imath}(\imath, f \,|\, \mathscr{B}_M(\ell))$ of a fixed

$$(g, \sigma_\imath, \sigma_f) \in \mathscr{B}_M(\ell)$$

is Gaussian and yields:

$$\begin{aligned} P[\dot{c}_\imath, \dot{c}_f \,|\, \mathscr{B}_M(\ell)] = &\int_{\mathscr{B}_M(\ell)} \mathscr{D}(g, \sigma_\imath, \sigma_f) \left(\int_{\mathscr{D}_M^{\imath}} \mathscr{D}f \times \int_{\mathscr{W}_M^{\imath 0}} \mathscr{D}\phi\right)^{-1} \\ &\times (\det{}_D \mathscr{L}_g)^{-\frac{d}{2}} e^{-S[g, x_\imath^f[g, \sigma_\imath, \sigma_f]]}, \end{aligned} \tag{3.11}$$

where $\det_D \mathscr{L}_g$ denotes the determinant of the Laplace-Beltrami operator $\mathscr{L}_g$ acting on the space of scalar real valued functions on M with the Dirichlet boundary conditions.

$x_\imath^f[g,\sigma_\imath,\sigma_f]$ in the formula (3.11) denotes the solution of the boundary value problem:

$$\mathscr{L}_g x_\imath^f[g,\sigma_\imath,\sigma_f]=0\,, \tag{3.12}$$
$$x_\imath^f[g,\sigma_\imath,\sigma_f]\circ I_\imath=\tilde{x}_\imath\circ\gamma_\imath[g,\sigma_\imath|\hat{e},\hat{s}]\,,\qquad(\imath\to f)\,,$$

where $\dot{c}_\imath=\Pi_{\dot{\mathscr{C}}}(\hat{e},\hat{s},\tilde{x}_\imath)$, $(\imath\to f)$.

The orientation preserving diffeomorphisms:

$$\gamma_\imath[g,\sigma_\imath|\hat{e},\hat{s}]:\Sigma_\imath\to S\,,\qquad(\imath\to f)\,,$$

are uniquely determined (for every $(g,\sigma_\imath,\sigma_f)\in\mathscr{M}_M^\imath\times\Sigma_\imath\times\Sigma_f$, $(\hat{e},\hat{s})\in\mathscr{M}_S\times S$) by the equations:

$$\mathrm{const}_\imath\times\gamma_\imath^*[g,\sigma_\imath|\hat{e},\hat{s}]\hat{e}=e_\imath=(I_\imath^*g)^{1/2}\,,\qquad(\imath\to f)\,,$$
$$\gamma_\imath[g,\sigma_\imath|\hat{e},\hat{s}]\,(\sigma_\imath)=\hat{s}\,,\qquad(\imath\to f)\,. \tag{3.13}$$

For every $(f,\phi)\in\mathscr{D}_M^\imath\odot\mathscr{W}_M^\imath$, $(\lambda,\gamma)\in\mathbb{R}_+\times\mathscr{D}_S$ we have the relations:

$$\gamma_\imath[\exp(\phi)f^*g,f^{-1}(\sigma_\imath)|\hat{e},\hat{s}]=\gamma_\imath[g,\sigma_\imath|\hat{e},\hat{s}]\circ f\circ I_\imath\,,\qquad(\imath\to f)\,, \tag{3.14}$$
$$\gamma_\imath[g,\sigma_\imath|\lambda\gamma^*\hat{e},\gamma^{-1}(\hat{s})]=\gamma^{-1}\circ\gamma_\imath[g,\sigma_\imath|\hat{e},\hat{s}]\,,\qquad(\imath\to f)\,. \tag{3.15}$$

It follows from (3.12–15) that the classical action $S[g,x_\imath^f[g,\sigma_\imath,\sigma_f]]$ is a $\mathscr{D}_M^\imath\odot\mathscr{W}_M^{\imath o}$-invariant functional with respect to the variables $(g,\sigma_\imath,\sigma_f)\in\mathscr{M}_M^\imath\times\Sigma_\imath\times\Sigma_f$ and is independent of the choice of parametrizations $(\hat{e},\hat{s},\tilde{x}_\imath)$, $(\hat{e},\hat{s},\tilde{x}_f)\in\mathscr{M}_S\times S\times\mathscr{E}_S^d$ of the initial $\dot{c}_\imath$ and of the final $\dot{c}_f$ contours. Therefore it can be regarded as a functional on the space $\mathscr{M}_M^\imath\times\Sigma_\imath\times\Sigma_f\times\dot{\mathscr{C}}_S^d\times\dot{\mathscr{C}}_S^d$:

$$W[g,\sigma_\imath,\sigma_f|\dot{c}_\imath,\dot{c}_f]=S[g,x_\imath^f[g,\sigma_\imath,\sigma_f]]\,. \tag{3.16}$$

Since $S[g,x_\imath^f[g,\sigma_\imath,\sigma_f]]$ is not invariant under conformal rescalings of metric by conformal factors $\phi\notin\mathscr{W}_M^\imath$, the functional (3.16) nontrivially depends on the choice of $\mathscr{B}_M\in\mathfrak{B}_M$. In order to describe more explicitly this dependence we introduce a suitable parametrization of the family $\mathfrak{B}_M$. We start with the construction of a special class of global sections of the bundle (3.3). (The statements below concerning the infinite dimensional geometry can be derived identifying the space $\mathscr{M}_M^\imath$ with the space of $\imath$-symmetric metrics on the double M^D [23, 24] and then using known results for closed surfaces [7].)

Let $\mathscr{M}_M^{\imath o}$ denote the space of all metrics $g\in\mathscr{M}_M^\imath$ with the zero scalar curvature. Let $\mathscr{F}_\Gamma^\imath(\ell)$ be the subspace of $\mathscr{M}_M^{\imath o}$ of metrics with a fixed Levi-Civita connection Γ and with fixed lengths of the initial and final boundaries equal to ℓ. $\mathscr{F}_\Gamma^\imath(\ell)$ is a global gauge slice of the principal fiber bundle:

$$\begin{array}{c}\mathscr{D}_M^\imath(\hat{\sigma}_\imath)\odot\mathscr{W}_M^\imath\longrightarrow\mathscr{M}_M^\imath\\ \qquad\qquad\qquad\qquad\Big\downarrow{\scriptstyle\Pi_T}\\ \qquad\qquad\qquad\qquad T_M^R\,,\end{array} \tag{3.17}$$

where $\hat{\sigma}_\imath$ is a fixed point of $\Sigma_\imath$ and $\mathscr{D}^\imath_M(\hat{\sigma}_\imath)=\{f\in\mathscr{D}^\imath_M : f(\hat{\sigma}_\imath)=\hat{\sigma}_\imath\}$. Using $\mathscr{F}^\imath_\Gamma(\ell)$ one can construct a global gauge slice $\mathscr{S}_\Gamma(\hat{\sigma}_\imath,\ell)$ of the bundle (3.3):

$$\mathscr{S}_\Gamma(\hat{\sigma}_\imath,\ell)=\mathscr{F}^\imath_\Gamma(\ell)\times\{\hat{\sigma}_\imath\}\times\Sigma_f\,. \tag{3.18}$$

We introduce the "reference" subbundle $\mathscr{B}^0_M\in\mathfrak{B}_M$ constructed from the gauge slice (3.18) by the $\mathscr{D}^\imath_M\odot\mathscr{W}^\imath_M$-action:

$$\mathscr{B}^0_M=\mathscr{D}^\imath_M\odot\mathscr{W}^\imath_M(\mathscr{F}^\imath_\Gamma(\ell))\times\Sigma_\imath\times\Sigma_f\,. \tag{3.19}$$

Let Ξ be a global section of a given subbundle $\mathscr{B}_M\in\mathfrak{B}_M$:

$$\Xi : T^R_S\times S^1\ni(t,s)\to(g^{t,s},\sigma^{t,s}_\imath,\sigma^{t,s}_f)\in\mathscr{B}_M\,.$$

There exists a section Ξ_0 of $\mathscr{B}^0_M$:

$$\Xi_0 : T^R_S\times S^1\ni(t,s)\to(g^{t,s}_0,\sigma^{t,s}_i,\sigma^{t,s}_f)\in\mathscr{B}^0_M\,,$$

and a smooth family of functions $\{\phi^{t,s}\}$ such that for every $(t,s)\in T^R_M\times S^1$:

$$g^{t,s}=\exp(\phi^{t,s})g^{t,s}_0\,. \tag{3.20}$$

Note that Ξ^0 is determined by Ξ up to a conformal factor $\phi^{t,s}_0\in\mathscr{W}^{\imath_0}_M$.

Let us consider the diffeomorphisms $\tilde{\gamma}^{t,s}_\imath[\hat{e},\hat{s}]$, $\tilde{\gamma}^{t,s}_f[\hat{e},\hat{s}]\in\mathscr{D}_S$ defined by:

$$\tilde{\gamma}^{t,s}_\imath[\hat{e},\hat{s}]=\gamma_\imath[g^{t,s},\sigma^{t,s}_\imath|\hat{e},\hat{s}]\circ\gamma^{-1}_\imath[g^{t,s}_0,\sigma^{t,s}_\imath|\hat{e},\hat{s}]\,,\qquad(\imath\to f)\,.$$

It follows from the definition and the relation (3.14) that $\tilde{\gamma}^{t,s}_\imath[\hat{e},\hat{s}]$, $\tilde{\gamma}^{t,s}_f[\hat{e},\hat{s}]$ are independent of the choice of section Ξ of $\mathscr{B}_M$ as well as of the choice of section $\bar{\Xi}$ of $\mathscr{B}^0_B$ fulfilling the relation (3.20). Moreover from (3.15) we have for every $(\lambda,\gamma)\in\mathbb{R}_+\times\mathscr{D}_S$:

$$\tilde{\gamma}^{t,s}_\imath[\lambda\gamma^*\hat{e},\gamma^{-1}(\hat{s})]=\gamma^{-1}\circ\tilde{\gamma}^{t,s}_\imath[\hat{e},\hat{s}]\circ\gamma\,,\qquad(\imath\to f)\,.$$

Therefore the function:

$$\delta[\mathscr{B}_M] : T^R_S\times S^1\ni(t,s)\to(\Gamma^{t,s}_\imath,\Gamma^{t,s}_f)\in\mathscr{G}^0_S\times\mathscr{G}^0_S$$

defined by the equations:

$$\Gamma^{t,s}_\imath=\Pi_{\mathscr{G}}(\hat{e},\hat{s},\tilde{\gamma}^{t,s}_\imath[\hat{e},\hat{s}])\,,\qquad(\imath\to f)\,, \tag{3.21}$$

is independent of the choice of $(\hat{e},\hat{s})\in\mathscr{M}_S\times S$ and provides an invariant characteristic of the subbundle $\mathscr{B}_M$. Proceeding in the opposite direction one can easily show that the construction above yields the $1-1$ correspondence between the space $\mathfrak{B}_M$ of all $\mathscr{D}^\imath_M\odot\mathscr{W}^\imath_M$-subbundles of (3.3) and the space $\mathfrak{b}$ of all functions $\delta : T^R_S\times S^1\to\mathscr{G}^0_S\times\mathscr{G}^0_S$.

Proceeding as in [20, the formulae (3, 34–42)] one can show that for every section Ξ of $\mathscr{B}_M\in\mathfrak{B}_M$:

$$\Xi : T^R_S\times S^1\ni(t,s)\to(g^{t,s},\sigma^{t,s}_\imath\sigma^{t,s}_f)\in\mathscr{B}_M\,.$$

and for every section $\hat{\Xi}$ of $\mathscr{B}^0_M$:

$$\hat{\Xi} : T^R_S\times S^1\ni(t,s)\to(\hat{g}^{t,s},\hat{\sigma}^{t,s}_\imath,\hat{\sigma}^{t,s}_f)\in\mathscr{B}^0_M\,,$$

the following relation holds:

$$W[g^{t,s},\sigma^{t,s}_\imath,\sigma^{t,s}_f|\dot{c}_\imath,\dot{c}_f]=W[\hat{g}^{t,s},\hat{\sigma}^{t,s}_\imath,\hat{\sigma}^{t,s}_f|\Gamma^{t,s}_\imath\dot{c}_\imath,\Gamma^{t,s}_f\dot{c}_f]\,, \tag{3.22}$$

where $(\Gamma^{t,s}_\imath,\Gamma^{t,s}_f)=\delta[\mathscr{B}_M](t,s)$.

Let us now turn to the path integral (3.11). For simplicity of presentation we will restrict ourselves to the subspace $\bar{\mathfrak{B}}_M \subset \mathfrak{B}_M$ corresponding to the subspace $\bar{\mathfrak{b}} \subset \mathfrak{b}$ consisting of all functions $\bar{\delta}$ constant with respect to the variable $s \in S^1$ $(\partial/\partial s\bar{\delta} = 0)$.

Let $\bar{\mathscr{B}}_M \in \bar{\mathfrak{B}}_M$. We start with the construction of an appropriate gauge slice of the principal fiber bundle:

$$\begin{array}{ccc} \mathscr{D}^{\iota}_M & \longrightarrow & \bar{\mathscr{B}}_M(\ell) \\ & & \big\downarrow \\ & & \bar{\mathscr{B}}_M(\ell)/\mathscr{D}^{\iota}_M \approx T^R_M \times S^1 \times \mathscr{W}^{\iota o}_M . \end{array} \tag{3.23}$$

Let us observe that for every $\Gamma, \hat{\sigma}_\iota$ the submanifold:

$$\mathscr{F}^{\iota}_{\Gamma}(\ell) \times \{\hat{\sigma}_\iota\} \times \Sigma_f \tag{3.24}$$

is a global gauge slice of the "reference" $\mathscr{D}^{\iota}_M \odot \mathscr{W}^{\iota o}_M$-subbundle $\mathscr{B}^0_M(\ell)$. With an appropriate parametrization $(t, \theta) \in \mathbb{R}_+ \times [-\pi, \pi]$ of $T^R_M \times S^1$ the gauge slice (3.24) determines the global section $\hat{\Xi}(\Gamma, \hat{\sigma}_\iota)$ of $\mathscr{B}^0_M(\ell)$:

$$\hat{\Xi}(\Gamma, \hat{\sigma}_\iota) : \mathbb{R}_+ \times [-\pi, \pi] \ni (t, \theta) \rightarrow \left(g^t_\Gamma, \hat{\sigma}_\iota, \hat{\sigma}_f + \frac{\ell\theta}{2\pi} \right) \in \mathscr{B}^0_M(\ell) .$$

It follows from the assumption $\bar{\mathscr{B}}_M \in \bar{\mathfrak{B}}_M$ that there exists a global section $\bar{\Xi}(\Gamma, \hat{\sigma}_\iota)$ of $\bar{\mathscr{B}}_M(\ell)$:

$$\bar{\Xi}(\Gamma, \hat{\sigma}_\iota) : \mathbb{R}_+ \times [-\pi, \pi] \ni (t, \theta) \rightarrow \left(\bar{g}^t, \hat{\sigma}_\iota, \hat{\sigma}_f + \frac{\ell\theta}{2\pi} \right) \beta \bar{\mathscr{B}}_M(\ell) .$$

such that:

$$\bar{g}^t = \exp(\phi^t) g^t_\Gamma , \tag{3.25}$$

where ϕ^t is independent of $s \in S^1$. From this section one can construct the global section $\bar{\Xi}^c(\Gamma, \hat{\sigma}_\iota)$ of the bundle (3.23) as follows:

$$\begin{aligned} \bar{\Xi}^c(\Gamma, \hat{\sigma}_\iota) : \mathbb{R}_+ \times [-\pi, \pi] \times \mathscr{W}^{\iota o}_M \ni (t, \theta, \phi) \\ \rightarrow \left(\exp(\phi) g^t_\Gamma, \hat{\sigma}_\iota, \hat{\sigma}_f + \frac{\ell\theta}{2\pi} \right) \in \mathscr{B}^0_M(\ell) . \end{aligned}$$

Let $\mathscr{S}^C(\Gamma, \hat{\sigma}_\iota)$ denote the global slice of (3.23) determined by $\bar{\Xi}^C(\Gamma, \hat{\sigma}_\iota)$. Since all metrics $g \in \mathscr{W}^{\iota}_M(\mathscr{F}^{\iota}_\Gamma(\ell))$ have the some conformal group $C_\Gamma \subset \mathscr{D}^{\iota}_M$ [7] the submanifold $\bar{\mathscr{S}}^C(\Gamma) = C_\Gamma(\mathscr{S}^C(\Gamma, \hat{\sigma}_\iota))$ is a C_Γ-reduction of the fiber bundle (3.23). We will use this reduction as a generalized (incomplete) gauge slice. Applying the geometrical formulation of the F–P method for generalized gauges developed in [7] to the path integral (3.11) with the gauge $\bar{\mathscr{S}}^C(\Gamma)$ one obtains:

$$\begin{aligned} P[\dot{c}_\iota, \dot{c}_f | \bar{\mathscr{B}}_M(\ell)] = \int_{\bar{\mathscr{S}}^C(\Gamma)} \mathscr{D}(g, \sigma_\iota, \sigma_f) \left(\int_{\mathscr{W}^{\iota 0}_M} \mathscr{D}\phi \right)^{-1} \left(\int_{C_\Gamma} d\omega^g \right)^{-1} \\ \times (\det{}'_A P^+_g P_g)^{1/2} (\det{}_D \mathscr{L}_g)^{-\frac{d}{2}} e^{-S[g, x^f_\iota[\sigma_\iota, \sigma_f]]} , \end{aligned} \tag{3.26}$$

where the measures $\mathscr{D}(g, \sigma_\iota, \sigma_f)$, $d\omega^g$ are related to the induced Riemannian structures on $\bar{\mathscr{S}}^C(\Gamma)$ and C_Γ respectively. The F–P operator $P^+_g P_g$ is defined by means of the conformal Lie derivative operator P_g and its adjoint P^+_g [6, 8]. The symbol $\det'_A$ for determinant means that the Alvarez's (mixed) boundary conditions [8] are used and the zero eigenvalue is omitted.

The generalized gauge slice $\mathscr{S}^C(\Gamma)$ can be parametrized by means of the section $\Xi^C(\Gamma, \hat{\sigma}_\imath)$ as follows:

$$\mathbb{R}_+ \times \mathscr{W}_M^{\imath 0} \times \Sigma_\imath \times \Sigma_f \ni (t, \phi, \sigma_\imath, \sigma_f) \to (\exp\{(\phi + \phi^t) \circ I_{\sigma_\imath}\} g_\Gamma^t, \sigma_\imath, I_{\sigma_\imath}^{-1}(\sigma_f)) \in \mathscr{B}_M^0(\ell), \tag{3.27}$$

where $I_{\sigma_\imath} \in C_\Gamma$ is uniquely determined by the condition $I_{\sigma_\imath}(\sigma_\imath) = \hat{\sigma}_\imath$. Changing variables in (3.26) by (3.27) and using the $\mathscr{D}_M^\imath$-invariance of the classical action we have:

$$P[\dot{\bar{c}}_\imath, \dot{\bar{c}}_f | \mathscr{B}_M(\ell)] = \int_0^\infty dt \int_{\Sigma_f} \hat{e}_f d\sigma_f \int_{\mathscr{W}_M^{\imath 0}} \tilde{\mathscr{D}}^t \phi \left(\int_{\mathscr{W}_M^{\imath 0}} \mathscr{D}^{\hat{g}} \phi \right)^{-1} \left(\int_{C_\Gamma} d\omega^{\hat{g}} \right)^{-1}$$
$$\times \ell M_{\hat{g}}(\delta\psi | \delta\chi) \left(\frac{\det'_A P_{\hat{g}}^+ P_{\hat{g}}}{H(P_{\hat{g}}^+)} \right)^{1/2} (\det_D \mathscr{L}_{\hat{g}})^{-\frac{d}{2}} e^{-S[\hat{g}, x_\imath^f[\hat{\sigma}_\imath, \sigma_f]]}, \tag{3.28}$$

where $\hat{g} = \exp(\phi + \phi^t) g_\Gamma^t$ and $(\hat{e}_f)^2 = I_f^* \hat{g} = \exp(\phi^t \circ I_f) I_f^* g_\Gamma^t$. $\delta\psi$ denotes an arbitrary element of the 1-dim space $\ker P_{\hat{g}}^+$ and:

$$(\delta\bar{\chi})_{ab} = \left(\frac{d}{dt} \hat{g}_{ab} - \frac{1}{2} \hat{g}_{ab} \hat{g}^{cd} \frac{d}{dt} \hat{g}_{cd} \right),$$

$$H(P_{\hat{g}}^+) = M_{\hat{g}}(\delta\psi | \delta\psi).$$

The functional measure $\tilde{\mathscr{D}}^t \phi$ in (3.28) is related to the nonconstant Riemannian structure $\tilde{W}^t(\cdot | \cdot)$ on $\mathscr{W}_M^{\imath 0}$:

$$\tilde{W}_\phi^t(\delta\phi | \delta\phi') = \int_M \exp(\phi + \phi^t) \sqrt{g_\Gamma^t} dz^2 \delta\phi \delta\phi'.$$

In order to extract the ϕ-dependence of the volume of the conformal group C_Γ, let us fix a conformal Killing vector field $\delta\varphi \in \ker P_{\hat{g}}$ and define:

$$H(P_{\hat{g}}) = H_{\imath d}^{\hat{g}}(\delta\varphi | \delta\varphi).$$

Note that for every $\hat{g} = \exp(\phi + \phi^t) g_\Gamma^t$ the 1-dim spaces $\ker P_{\hat{g}}$ are identical. The conformal Killing vector field $\delta\varphi \in \mathscr{T}_{\imath d} \mathscr{D}_M^\imath$ can be regarded as a right invariant vector field on C_Γ. Taking the 1-form $d\varphi$ dual to $\delta\varphi$ we have for every $\hat{g} = \exp(\phi + \phi^t) g_\Gamma^t$:

$$\int_{C_\Gamma} d\omega^{\hat{g}} = (H(P_{\hat{g}}))^{1/2} \int_{C_\Gamma} d\varphi, \tag{3.29}$$

where the integral on the right-hand side is independent on $\hat{g}$ and can be chosen to be equal 1.

Let us now consider the finite dimensional integral over Σ_f in (3.28). Changing variables (see Def. (2.4)):

$$[-\pi, \pi] \ni \theta \to \sigma_f = \hat{\sigma}_f + \frac{\ell\theta}{2\pi} \in \Sigma_f,$$

where $\hat{\sigma}_f$ is arbitrary chosen point of Σ_f and using the relation:

$$W[g, \sigma_\imath, \sigma_f + \tau | \dot{\bar{c}}_\imath, \dot{\bar{c}}_f] = W[g, \sigma_\imath, \sigma_f | \dot{\bar{c}}_\imath, I_\theta \dot{\bar{c}}_f]$$

we obtain:

$$\int_{\Sigma_f} \hat{e}_f d\sigma_f e^{-W[g,\hat{\sigma}_i,\sigma_f|\dot{c}_i,\dot{c}_f]} = \frac{\ell}{2\pi} \int_{-\pi}^{\pi} d\theta e^{-W[g,\hat{\sigma}_i,\hat{\sigma}_f|\dot{c}_i,I_\theta \dot{c}_f]}. \tag{3.30}$$

Inserting (3.29) into (3.28) and using the relations (3.22) and (3.30) we have:

$$P[\dot{c}_i,\dot{c}_f|\mathscr{B}_M(\ell)] = \frac{\ell^2}{2\pi} \int_0^\infty dt \int_{-\pi}^{\pi} d\theta M_{g_\Gamma^t}(\delta\psi|\delta\bar{\chi}) e^{-W[g_\Gamma^t,\hat{\sigma}_i,\hat{\sigma}_f|\Gamma_i^t\dot{c}_i,\Gamma_f^t I_\theta \dot{c}_f]}$$
$$\times \int_{\mathscr{W}_M^{i0}} \tilde{\mathscr{D}}^t\phi \left(\int_{\mathscr{W}_M^{i0}} \mathscr{D}^{\hat{g}}\phi\right)^{-1} \left(\frac{\det'_A P_{\hat{g}}^+ P_{\hat{g}}}{H(P_{\hat{g}}^+)H(P_{\hat{g}})}\right)^2 (\det_D \mathscr{L}_{\hat{g}})^{-\frac{d}{2}}.$$

As it was shown by Alvarez [8] in the critical dimension $d=26$ the conformal anomaly vanishes, therefore taking into account the formal relation:

$$\int_{\mathscr{W}_M^{i0}} \tilde{\mathscr{D}}^t\phi \left(\int_{\mathscr{W}_M^{i0}} \mathscr{D}^{\hat{g}}\phi\right)^{-1} = 1$$

we finally have:

$$P[\dot{c}_i,\dot{c}_f|\mathscr{B}_M(\ell)] = \frac{\ell^2}{2\pi} \int_0^\infty dt \int_{-\pi}^{\pi} d\theta M_{g_\Gamma^t}(\delta\psi|\delta\bar{\chi}) \left(\frac{\det'_A P_{g_\Gamma^t}^+ P_{g_\Gamma^t}}{H(P_{g_\Gamma^t}^+)H(P_{g_\Gamma^t})}\right)^{1/2}$$
$$\times (\det_D \mathscr{L}_{g_\Gamma^t})^{-13} e^{-W[g_\Gamma,\hat{\sigma}_i,\hat{\sigma}_f|\Gamma_i^t\dot{c}_i,\Gamma_f^t I_\theta \dot{c}_f]}. \tag{3.31}$$

Let us observe that the functional (3.31) is independent of the choice of a "reference" point $\hat{\sigma}_f \in \Sigma_f$. It is convenient to choose as $\hat{\sigma}_f$ the point $\hat{\hat{\sigma}}_i \in \Sigma_f$ of intersection of Σ_f with the geodesic line starting at $\hat{\sigma}_i$ and perpendicular to Σ_i (with respect to the metric g_Γ^t). One can show that $\hat{\hat{\sigma}}_i$ defined in this way is independent of $t \in T_M^R$ and of a flat Levi-Civita connection Γ. With this choice the functional:

$$W[t|\dot{c}_i,\dot{c}_f] = W[g_\Gamma^t,\hat{\sigma}_i,\hat{\hat{\sigma}}_i|\dot{c}_i,\dot{c}_f] \tag{3.32}$$

is independent of the choice of $\hat{\sigma}_i$ and Γ. This functional (with a slightly different interpretation) was first evaluated in [4]. It is however worthwhile for clarification of the present approach to repeat some steps of this derivation.

Let us fix a parametrization $(\sigma,\tau) \in [-\pi,\pi] \times [0,1]$ of M with the identification $[-\pi,\tau] = [\pi,\tau]$ and with the clockwise orientation on $[-\pi,\pi] \times [0,1] \subset \mathbb{R}^2$. In this parametrization the space $\mathscr{F}_0(\ell) \subset \mathscr{M}_M^{i0}$ of metrics with the vanishing Levi-Civita connection $\Gamma=0$ and with the lengths of the initial and of the final boundary component equal ℓ form the 1-parameter family:

$$\Xi_0 : T_M^R \approx \mathbb{R}_+ \ni t \to g_0^t = \frac{\ell^2}{4\pi^2} \begin{pmatrix} 1 & 0 \\ 0 & 4\pi^2 t^2 \end{pmatrix} \in \mathscr{F}_0(\ell) \subset \mathscr{B}^0(\ell). \tag{3.33}$$

We fix the point $\hat{\sigma}_i = (0,0) \in \Sigma_i$. It is easy to verify that $\hat{\hat{\sigma}}_i = (0,1)$. Let us now fix $(\hat{e},\hat{s}) \in \mathscr{M}_S \times S$ with $\int \hat{e} ds = \ell$. There exists the parametrization $\sigma \in [-\pi,\pi]$ of S in which $(\hat{e},\hat{s}) = \left(\frac{\ell}{2\pi}, 0\right)$. In this parametrization we have:

$$\mathscr{L}_{\hat{e}} = -\frac{4\pi^2}{\ell^2}\frac{d^2}{d\sigma^2},$$
$$\psi_0^{\hat{e},\hat{s}} = \ell^{-1/2},$$
$$\psi_n^{\hat{e},\hat{s}} = \sqrt{2}\ell^{-1/2}\cos(n\sigma),$$
$$\psi_{-n}^{\hat{e},\hat{s}} = \sqrt{2}\ell^{-1/2}\sin(n\sigma), \qquad n>0.$$

Let $\tilde{x}_i, \tilde{x}_f$ denote the representants of the contours $\dot{c}_i, \dot{c}_f$ in the gauge $\mathscr{S}_{\hat{e},\hat{s}}(\ell)$:

$$\Pi_{\dot{\mathscr{C}}}\left(\frac{\ell}{2\pi}, 0, \tilde{x}_i\right) = \dot{c}, \qquad (i \to f).$$

Proceeding to the gauge independent parametrization of $\dot{\mathscr{C}}_S^{26}$ (2.18) we have:

$$\dot{c}^\mu_{0i} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \tilde{x}^\mu_i(\sigma) d\sigma, \qquad (i \to f),$$

$$\dot{c}^\mu_{ni} = \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \tilde{x}^\mu_i(\sigma) \cos(n\sigma) d\sigma, \qquad (i \to f),$$

$$\dot{c}^\mu_{-ni} = \frac{\sqrt{2}}{2\pi} \int_{-\pi}^{\pi} \tilde{x}^\mu_i(\sigma) \sin(n\sigma) d\sigma, \qquad (i \to f),\ n > 0.$$

With our parametrizations of S and M the diffeomorphisms (3.13) are especially simple:

$$\gamma_i[g^t_0, \hat{\sigma}_i | \hat{e}, \hat{s}] : \Sigma_i = [-\pi, \pi] \times \{0\} \ni (\sigma, 0) \to -\sigma \in [-\pi, \pi] = S,$$

$$\gamma_f[g^t_0, \hat{\hat{\sigma}}_i | \hat{e}, \hat{s}] : \Sigma_f = [-\pi, \pi] \times \{1\} \ni (\sigma, 1) \to \sigma \in [-\pi, \pi] = S,$$

and the boundary value problem (3.12) takes the form:

$$\frac{4\pi^2}{\ell^2}\left(\frac{\partial^2}{\partial\sigma^2} + \frac{1}{4\pi^2 t^2}\frac{\partial^2}{\partial\tau^2}\right) x^f_i(\sigma, \tau) = 0,$$

$$x^f_i(\sigma, 0) = \dot{c}^\mu_{0i} + \sqrt{2} \sum_{n>0} (\dot{c}^\mu_{ni} \cos(n\sigma) - \dot{c}^\mu_{-ni} \sin(n\sigma)), \tag{3.34}$$

$$x^f_i(\sigma, 1) = \dot{c}^\mu_{0f} + \sqrt{2} \sum_{n>0} (\dot{c}^\mu_{nf} \cos(n\sigma) + \dot{c}^\mu_{-nf} \sin(n\sigma)).$$

Solving (3.34) for x^f_i we have:

$$\begin{aligned} x^f_i(\sigma, \tau) &= \dot{c}^\mu_{0f}\tau + \dot{c}^\mu_{0i}(1-\tau) \\ &+ \sqrt{2} \sum_{n>0} \{\dot{c}^\mu_{nf} \operatorname{sh}(2\pi n t\tau) + \dot{c}^\mu_{ni} \operatorname{sh}(2\pi n t(1-\tau))\} \frac{\cos(n\sigma)}{\operatorname{sh}(2\pi n t)} \\ &+ \sqrt{2} \sum_{n>0} \{\dot{c}^\mu_{-nf} \operatorname{sh}(2\pi n t\tau) + \dot{c}^\mu_{-ni} \operatorname{sh}(2\pi n t(1-\tau))\} \frac{\sin(n\sigma)}{\operatorname{sh}(2\pi n t)}. \end{aligned}$$

and

$$\begin{aligned} W[t | \dot{c}_i, \dot{c}_f] &= S[g^t_0, x^f_i[g^t_0, \hat{\sigma}_i, \hat{\hat{\sigma}}_i]] \\ &= \frac{1}{4\pi\alpha'}\left(\frac{(\dot{c}_{0i} - \dot{c}_{0f})^2}{t} + \sum_{n>0} \frac{2\pi n}{\operatorname{sh}(2\pi n t)}\left((\dot{c}^2_{ni} + \dot{c}^2_{nf}) \operatorname{ch}(2\pi n t) - \frac{|n|}{n} \dot{c}_{ni}\dot{c}_{nf}\right)\right). \end{aligned} \tag{3.35}$$

Evaluating determinants in the expression (3.31) as in [4] one obtains the final formula:

$$\begin{aligned} P[\dot{c}_i, \dot{c}_f | \mathscr{B}_M(\ell)] &= \ell^2 \int_0^\infty dt\, t^{-13} \exp(4\pi t) \prod_{n>0} \{1 - \exp(-4\pi n t)\}^{-24} \\ &\times \frac{1}{2\pi} \int_{\pi}^{\pi} d\theta e^{-W[t | \Gamma^t_i \dot{c}_i, \Gamma^t_f I_\theta \dot{c}_f]}. \end{aligned} \tag{3.36}$$

The formula above is the closed string counterpart of the expression for the open string propagator obtained in [20, 21]. Guided by the open string case we introduce the fixed length gauge consisting in the choice of the "reference" subbundle $\mathscr{B}^0_M(\ell)$. In this case $\Gamma^t_i = \Gamma^t_f = \mathrm{id}$ and the propagator:

$$P[\dot{c}_i, \dot{c}_f | \ell] = P[\dot{c}_i, \dot{c}_f | \mathscr{B}^0_M(\ell)] \tag{3.37}$$

takes the familiar form [4, 12–14].

The only new feature is the appearance of the averaging over boundary twist in (3.36). Let us introduce the projector $\Pi_{\mathscr{I}}$ on the subspace of $\mathscr{I}_S$-invariant states:

$$\Pi_{\mathscr{I}} : \mathfrak{H}^{\mathrm{off}} \ni \Psi[\dot{c}] \to \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \Psi[I_\theta \dot{c}] ,$$

and the operator $\tilde{P}_\ell$ defined on $\mathfrak{H}^{\mathrm{off}}$ by the integral kernel:

$$\tilde{P}[\dot{c}_i, \dot{c}_f | \ell] = \ell^2 \int_0^\infty dt t^{-13} \exp(4\pi t) \prod_{n>0} \{1 - \exp(-4\pi n t)\}^{-24} e^{-W[t|\dot{c}_i, I_\theta \dot{c}_f]} .$$

From the $\mathscr{I}_S$-invariance of the inner product $(\cdot,\cdot)^\ell$ (2.16) and from the properties of $W[t|\dot{c}_i, \dot{c}_f]$ we have:

$$P_\ell = \tilde{P}_\ell \circ \Pi_{\mathscr{I}} = \Pi_{\mathscr{I}} \circ \tilde{P}_\ell , \qquad [P_\ell, \Pi_{\mathscr{I}}] = [\tilde{P}_\ell, \Pi_{\mathscr{I}}] = 0 , \tag{3.38}$$

where P_ℓ denotes the operator corresponding to the integral kernel (3.37).

It follows from the formulae (3.38) that one can consistently replace P_ℓ by $\tilde{\mathscr{P}}_\ell$. In fact, since both operators lead (after the BRST extension [12]) to the same on-mass-shell condition on the subspace of $\mathscr{I}_S$-invariant states, the physical content of the quantum theory remains unchanged. The advantage of working with $\tilde{P}[\dot{c}_i, \dot{c}_f | \ell]$ is that its BRST extension is invertible on the whole (BRST extended) space of states.

The path integral representation of the functional $\tilde{P}[\dot{c}_i, \dot{c}_f | \ell]$ can be obtained taking in (3.5) the subspace of $\mathscr{T}^i_M(i, f | \mathscr{B}^0_M(\ell))$ consisting of all string trajectories with the zero relative twist. We say that $(g, \sigma_i, \sigma_f) \in \mathscr{M}^i_M \times \Sigma_i \times \Sigma_f$ has the zero relative twist iff for a zero scalar curvature metric $\hat{g} \in \mathscr{M}^{i0}_M$ related to the metric g by a conformal rescaling ($\hat{g} = \exp(\phi) g$) the points σ_i, σ_f can be connected by the geodesic line perpendicular to the boundaries Σ_i, Σ_f.

The quantum mechanical interpretation of the operator $\tilde{P}_\ell$ can be derived proceeding as in [4, 12]. As in the open string case [20] it can be regarded as the "body" of the operator inverse to the standard BRST extended closed string Hamiltonian.

As in the open string case there is an alternative formulation based on the space $\mathfrak{H}^{\mathrm{off}}_{ph}$ with the inner product $(\cdot,\cdot)$ (2.11). Within this formulation the propagator takes the following form:

$$P[\dot{c}_i, \dot{c}_f] = \delta(\ell_i - \ell_f) P[\dot{c}_i, \dot{c}_f | \ell_i] , \tag{3.39}$$

where $(\ell_i, \dot{c}_i)$, $(\ell_f, \dot{c}_f)$ are determined by the contours $\dot{c}_i, \dot{c}_f$ respectively. As was discussed in [20] there are some problems with the path integral representation of the functional $P[\dot{c}_i, \dot{c}_f]$. One can mimic the delta function structure of (3.39) defining the space $\mathscr{T}^i_M(i, f | \mathscr{B}^0_M)$ of allowable string trajectories as the space of all trajectories $(g, \sigma_i, \sigma_f, x) \in \mathscr{M}^i_M \times \Sigma_i \times \Sigma_f \times \mathscr{E}^{26}_M$ fulfilling the conditions:

$$\begin{gathered} \Pi_{\mathscr{C}}(\varrho_i^* e_i, \varrho_i^*) = \dot{c}_i , \qquad (i \to f) , \\ (g, \sigma_i, \sigma_f) \in \mathscr{B}^0_M \cap \{\mathscr{W}^{i0}_M(\mathscr{M}^{i0}_M) \times \Sigma_i \times \Sigma_f\} . \end{gathered} \tag{3.40}$$

Note that for $\dot{c}_i, \dot{c}_f$ with $\ell_i \neq \ell_f$ the space $\mathscr{T}^i_M(i, f\,|\,\mathscr{B}^0_M)$ is empty. This choice of the space of trajectories will be called hereafter the constant curvature gauge. As in the previous formulation the propagator $P[\dot{c}_i, \dot{c}_f]$ can be replaced by the propagator:

$$\tilde{P}[\dot{c}_i, \dot{c}_f] = \delta(\ell_i - \ell_f)\tilde{P}[\dot{\bar{c}}_i, \dot{\bar{c}}_f\,|\,\ell_i]\,, \tag{3.41}$$

which has an invertible BRST extension in $\mathfrak{H}^{\text{off}}_{\text{BRST}}$.

4. The Off-Shell Closed String Amplitudes

Let $M_{h,b}$ denote an oriented, bordered 2-dim surface with h-handles and b-boundary components Σ_i ($i = 1, 2, \ldots, b$) diffeomorphic to S^1 ($\partial M_{h,b} = \bigcup \Sigma_i$). We assume that the Euler characteristic of $M_{h,b}$ is negative $\chi(M_{h,b}) = 2 - 2h - b < 0$ (the disc and the cylinder cases are then excluded). For every $M_{h,b}$ we introduce the double $M^D_{h,b}$ with a fixed involution i. We define the spaces $\mathscr{M}^i_{h,b}, \mathscr{E}^d_{h,b}$, and the groups $\mathscr{D}^i_{h,b}, \mathscr{W}^i_{h,b}, \bar{\mathscr{W}}^i_{h,b}, \mathscr{W}^{i0}_{h,b}$ by the obvious generalization of conditions used in the previous section.

The action of the group $\mathscr{D}^i_{h,b} \odot \mathscr{W}^i_{h,b}$ on the space $\mathscr{M}^i_{h,b}$ induces the principal fiber bundle structure:

$$\begin{array}{ccc} \mathscr{D}^i_{h,b} \odot \mathscr{W}^i_{h,b} & \longrightarrow & \mathscr{M}^i_{h,b} \\ & & \Big\downarrow{\scriptstyle \Pi_{h,b}} \\ & & T^R_{h,b} \end{array} \tag{4.1}$$

where $T^R_{h,b}$ denotes the relative Teichmüller space of $M_{h,b}$ [23, 24]. Let us consider the action of the group $\mathscr{D}^i_{h,b} \odot \mathscr{W}^i_{h,b}$ on the space $\mathscr{M}^i_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b$ defined by:

$$\mathscr{M}^i_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b \ni (g, \sigma_1, \ldots, \sigma_b)$$
$$\xrightarrow{(f,\phi) \in \mathscr{D}^i_{h,b} \odot \mathscr{W}^i_{h,b}} (\exp(\phi) f^* g, f^{-1}(\sigma_1), \ldots, f^{-1}(\sigma_b)) \in \mathscr{M}^i_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b\,.$$

This action induces the principal fiber bundle structure:

$$\begin{array}{ccc} \mathscr{D}^i_{h,b} \odot \mathscr{W}^i_{h,b} & \longrightarrow & \mathscr{M}^i_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b \\ & & \Big\downarrow \\ & & T^R_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b \end{array} \tag{4.2}$$

Note that the bundle (4.2) is trivial. In fact for every global gauge slice $\mathscr{S}_{h,b}$ of (4.1) the submanifold $\mathscr{S}_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b$ is a global gauge slice of (4.2).

Let $\mathfrak{B}_{h,b}$ denote the space of all $\mathscr{D}^i_{h,b} \odot \bar{\mathscr{W}}^i_{h,b}$-subbundles of (4.2). Since the $\mathscr{D}^i_{h,b} \odot \mathscr{W}^i_{h,b}$-action on $\mathscr{M}^i_{h,b}$ is free (there are no conformal Killing vector fields on $M^i_{h,b}$) the problem of parametrization of $\mathfrak{B}_{h,b}$ simplifies. Let $\mathscr{A}^{\text{const}}_{n,b}$ denote the $\mathscr{D}^i_{h,b} \odot \bar{\mathscr{W}}^i_{h,b}$-subbundle of (4.1) obtained by the $\mathscr{W}^i_{h,b}$-action from the space $\mathscr{M}^{-1}_{h,b}$ of all metrics $g \in \mathscr{M}^i_{h,b}$ with the scalar curvature equal -1. Then the submanifold:

$$\mathscr{B}^0_{h,b} = \mathscr{A}^{\text{const}}_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b \tag{4.3}$$

is a $\mathscr{D}^i_{h,b} \odot \bar{\mathscr{W}}^i_{h,b}$-subbundle of (4.2). Using (4.3) as a "reference" subbundle and proceeding as in the previous section one can construct the $1-1$ correspondence between $\mathfrak{B}_{h,b}$ and the space $\mathrm{b}_{h,b}$ of functions:

$$\delta_{h,b}: T^R_{h,b} \times \Sigma_1 \times \ldots \times \Sigma_b \ni (t, \sigma) \to (\Gamma^{t,\sigma}_1, \ldots, \Gamma^{t,\sigma}_b) \in \overset{b}{\times}\, \mathscr{G}^0_S\,.$$

Within the formulation based on the space $\dot{\mathscr{C}}_S^{26}$ in the gauge $\dot{\mathscr{C}}_S^{26} \approx \dot{\mathscr{C}}_S^{26}(\ell)$ the h-loop b-states off-shell closed string amplitude is defined by:

$$\begin{aligned} &A_h[\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b}(\ell)] \\ &\quad = \int\limits_{\mathscr{T}_h^i(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b}(\ell))} \mathscr{D}(g, \sigma_1, \ldots, \sigma_b, x) \Bigg(\int\limits_{\mathscr{D}_{h,b}^i} \mathscr{D}f \times \int\limits_{\mathscr{W}_{h,b}^{i0}} \mathscr{D}\phi \Bigg)^{-1} e^{-S[g,x]}, \end{aligned} \tag{4.4}$$

where the space $\mathscr{T}_h^i(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b}(\ell))$ consists of all "trajectories"

$$(g, \sigma_1, \ldots, \sigma_b, x) \in \mathscr{M}_{h,b}^i \times \Sigma_1 \times \ldots \times \Sigma_b \times \mathscr{E}_{h,b}^{26}$$

fulfilling the conditions:

$$\begin{aligned} &(g, \sigma_1, \ldots, \sigma_b) \in \mathscr{B}_{h,b}^i(\ell) \times \Sigma_1 \times \ldots \times \Sigma_b, \\ &\Pi_{\dot{\mathscr{C}}}(\varrho_k^* e_k, \varrho_k^{-1}(\sigma_k), \varrho_k^* x_k) = \dot{c}_k, \qquad k = 1, \ldots, b; \end{aligned} \tag{4.5}$$

where: $e_k^2 = I_k^* g$, $x_k = I_k^* x$, $k = 1, \ldots, b$, and $\mathscr{B}_{h,b}^i(\ell)$ denotes the $\mathscr{D}_{h,b}^i \odot \mathscr{W}_{h,b}^{i0}$-subbundle of $\mathscr{B}_{h,b}^i$ determined by the conditions:

$$\int\limits_{\Sigma_k} e_k d\sigma_k = \ell, \qquad k = 1, \ldots, b. \tag{4.6}$$

(For every $k = 1, \ldots, b$, $I_k : \Sigma_k \to M_{h,b}$ denotes the inclusion of the boundary component Σ_k into $M_{h,b}$, and $\varrho_k : S \to \Sigma_k$ is an arbitrary orientation preserving diffeomorphism.)

The formal expression (4.4) fulfills the consistency requirement of the F–P method and can be evaluated (= defined) for any $\mathscr{B}_{h,b} \in \mathfrak{B}_{h,b}$. For the special subspace $\bar{\mathfrak{B}}_{h,b} \subset \mathfrak{B}_{h,b}$ of subbundles corresponding to the subspace $\bar{\mathfrak{b}}_{h,b} \subset \mathfrak{b}_{h,b}$ of functions independent of $(\sigma_1, \ldots, \sigma_b)$-variables one obtains the following result:

$$\begin{aligned} A_h[\dot{c}_1, \ldots, \dot{c}_b | \bar{\mathscr{B}}_{h,b}(\ell)] &= \frac{\ell^b}{(2\pi)^b} \int\limits_{-\pi}^{\pi} d\theta_1 \ldots \int\limits_{-\pi}^{\pi} d\theta_b \int\limits_{[T_{h,b}^R]} dt^{\sigma h - \sigma + 3b} t \\ &\times \det M_{\bar{g}^t}(\delta\psi_i | \delta\bar{\chi}_j) \left(\frac{\det_A P_{\bar{g}^t}^+ P_{\bar{g}^t}}{\det H(P_{\bar{g}^t}^+)} \right)^{1/2} (\det_D \mathscr{L}_{\bar{g}^t})^{-13} \\ &\times e^{-\bar{W}[\bar{g}^t, \hat{\sigma} | \bar{\Gamma}_1^t I_{\theta_1}, \ldots, \bar{\Gamma}_b^t I_{\theta_1} \dot{c}_b]}, \end{aligned} \tag{4.7}$$

where

$$T_{h,b}^R \ni t \to \bar{g}^t \in \mathscr{A}_{h,b}^{\mathrm{const}}(\ell) \subset \mathscr{M}_{h,b}^i \tag{4.8}$$

is a global section of the $\mathscr{D}_{h,b}^i \odot \mathscr{W}_{h,b}^{i0}$-subbundle $\mathscr{A}_{h,b}^{\mathrm{const}}(\ell) \subset \mathscr{A}_{h,b}^{\mathrm{const}}$ determined by the conditions (4.6). $[T_{h,b}^R]$ denotes a fundamental domain of the modular group in $T_{h,b}^R$.

For a given $\hat{\sigma} = (\hat{\sigma}_1, \ldots, \hat{\sigma}_b) \in \Sigma_1 \times \ldots \times \Sigma_b$ the functional $\bar{W}[\bar{g}^t, \bar{\sigma} | \dot{c}_1, \ldots, \dot{c}_b]$ is defined by:

$$\bar{W}[\bar{g}^t, \hat{\sigma} | \dot{c}_1, \ldots, \dot{c}_b] = S[\bar{g}^t, x_{cl}], \tag{4.9}$$

where x_{cl} is the solution of the boundary value problem:

$$\begin{aligned} &\mathscr{L}_{\bar{g}^t} x_{cl} = 0, \\ &(x_{cl})_k = \tilde{x}_k \circ \gamma_k[\bar{g}^t, \hat{\sigma}_k | \hat{e}, \hat{s}], \\ &\dot{c}_k = \Pi_{\dot{\mathscr{C}}}(\hat{e}, \hat{s}, \tilde{x}_k), \qquad k = 1, \ldots, b. \end{aligned} \tag{4.10}$$

It is easy to verify that the expression (4.7) is independent of the choice of a section $\bar{g}^t$ of the subbundle $\mathscr{A}_{h,b}^{\text{const}}(\ell)$ and of a point $\hat{\sigma} \in \Sigma_1 \times \ldots \times \Sigma_b$.

In the fixed length gauge the off-shell closed string amplitudes are defined by choosing for each topological type (h, b) the $\mathscr{D}_{h,b}^{\dot{\imath}} \odot \mathscr{W}_{h,b}^{\dot{\imath}}$-subbundle $\mathscr{B}_{h,b}^0$ (4.3):

$$A_h[\dot{\bar{c}}_1, \ldots, \dot{\bar{c}}_b | \ell] = A_h[\dot{\bar{c}}_1, \ldots, \dot{\bar{c}}_b | \mathscr{B}_{h,b}^0(\ell)]. \tag{4.11}$$

Note that due to the integration over twists of all boundary components in (4.7) the propagator $P[\dot{\bar{c}}_i, \dot{\bar{c}}_f | \ell]$ can be replaced by $\tilde{P}[\dot{\bar{c}}_i, \dot{\bar{c}}_f | \ell]$ also on the second quantized level.

Let us now proceed to the formulation based on the space $\dot{\mathscr{C}}_S^{26}$. In order to generalize the condition (3.40) to more complicated topologies we introduce the Frenchel-Nielsen coordinates of $T_{h,b}^R$:

$$T_{h,b}^R \ni t \to (L_1, \ldots, L_b, \theta_1, \ell_1, \ldots, \theta_{3h+b-3}, \ell_{3h+b-3}) \in \mathbb{R}_+^b \times (\mathbb{R} \times \mathbb{R}_+)^{3h+b-3}.$$

For a given pattern for gluing $2h-2+b$ parts to obtain the surface $M_{h,b}$, the coordinates $L_1, \ldots, L_b$ are lengths (with respect to the hyperbolic geometry on $M_{h,b}$) of the boundary components $\Sigma_1, \ldots, \Sigma_b$ while (θ_j, ℓ_j) $(j = 1, \ldots, 2h+b-3)$ are parameters of gluing [25]. (We use the convention where θ_j corresponds to the relative twist by the hyperbolic distance $\tau_j = (2\pi)^{-1} \ell_j \theta_j$, then the Dehn twists correspond to $\theta_j = 2\pi k$, $k \in \mathbb{Z}$).

For a given subbundle $\mathscr{B}_{h,b} \in \mathfrak{B}_{h,b}$ we define the space $\mathscr{T}_h^{\dot{\imath}}(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b})$ consisting of all string "trajectories"

$$(g, \sigma_1, \ldots, \sigma_b, x) \in \mathscr{M}_{h,b}^{\dot{\imath}} \times \Sigma_1 \times \ldots \times \Sigma_b \times \mathscr{E}_{h,b}^{26}$$

fulfilling the conditions:

$$\begin{aligned} L_k(\Pi_{h,b}(g)) &= \ell_k, \\ (g, \sigma_1, \ldots, \sigma_b) &\in \mathscr{B}_{h,b}, \\ \Pi_{\dot{\mathscr{C}}}(\varrho_k^* e_k, \varrho_k^{-1}(\sigma_k), \varrho_k^* x_k) &= \dot{c}_k, \qquad k = 1, \ldots, b. \end{aligned} \tag{4.12}$$

As was discussed in [20] in the open string case in order to construct conformally invariant (in $d = 26$) functional measure on $\mathscr{T}_h^{\dot{\imath}}(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b})$ it is necessary to introduce the larger space $\bar{\mathscr{T}}_h^{\dot{\imath}}(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b})$ determined by (4.12) with the first condition omitted. Furthermore we introduce the $\mathscr{D}_{h,b}^{\dot{\imath}} \odot \mathscr{W}_{h,b}^{\dot{\imath}0}$-invariant functional:

$$\prod_{k=1}^{b} \delta(\ell_k - L_k(\Pi_{h,b}(g))) \tag{4.13}$$

which can be regarded as a "characteristic functional" of the submanifold

$$\mathscr{T}_h^{\dot{\imath}}(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b}) \subset \bar{\mathscr{T}}_h^{\dot{\imath}}(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b}).$$

In the formulation based on the space $\dot{\mathscr{C}}_S$ the path integral representation of the h-loop, b-states off-shell closed string amplitude has the following form:

$$\begin{aligned} A_h[\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b}] = & \int\limits_{\bar{\mathscr{T}}_h^{\dot{\imath}}(\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}_{h,b}(\ell))} \mathscr{D}(g, \sigma_1, \ldots, \sigma_b, x) \prod_{k=1}^{b} \delta(\ell_k - L_k(\Pi_{h,b}(g))) \\ & \times \left(\int\limits_{\mathscr{D}_{h,b}^{\dot{\imath}}} \mathscr{D}f \times \int\limits_{\mathscr{W}_{h,b}^{\dot{\imath}0}} \mathscr{D}\phi \right)^{-1} e^{-S[g,x]}. \end{aligned} \tag{4.14}$$

As in the case of (4.4) the F–P procedure can be applied to evaluate (= to define) the expression above for any $\mathscr{B}_{h,b} \in \mathfrak{B}_{h,b}$. For $\bar{\mathscr{B}}_{h,b} \in \bar{\mathfrak{B}}_{h,b}$ proceeding along the standard lines [6, 7] and using the Frenchel-Nielsen coordinates one obtains the following result:

$$A_h[\dot{c}_1, \ldots, \dot{c}_b | \bar{\mathscr{B}}_{h,b}] = \frac{\ell_1}{2\pi} \int_{-\pi}^{\pi} d\theta \ldots \frac{\ell_b}{2\pi} \int_{-\pi}^{\pi} d\theta_b \int_{[\hat{T}^R_{h,b}]} d\hat{\omega}^{\mathrm{WP}}$$
$$\times (\det_A P^+_{g^t} P_{g^t})^{1/2} (\det_D \mathscr{L}_{g^t})^{-13} \mathrm{e}^{-W[g^t, \hat{\sigma} | \bar{\Gamma}^t_1 I_{\theta_1} \dot{c}_1, \ldots, \bar{\Gamma}^t_b I_{\theta_1} \dot{c}_b]} . \tag{4.15}$$

The symbol $d\hat{\omega}^{\mathrm{WP}}$ in (4.13) denotes the restricted Weil-Petersson volume form on $T^R_{h,b}$ defined by:

$$d\hat{\omega}^{\mathrm{WP}} = \prod_{j=1}^{3h-3+b} \ell_j d\ell_j \frac{d\theta_j}{2\pi},$$

and $[\hat{T}^R_{h,b}]$ is the submanifold of a fundamental domain $[T^R_{h,b}]$ determined by the equations $L_k(t) = \ell_k$, $k = 1, \ldots, b$.

The determinants in (4.15) are evaluated for a section

$$T^R_{h,b} \ni t \to g^t \in \mathscr{A}^{\mathrm{const}}_{h,b} \subset \mathscr{M}^i_{h,b} \tag{4.16}$$

of the subbundle $\mathscr{A}^{\mathrm{const}}_{h,b}$ with values in the space $\mathscr{M}^{i-1}_{h,b}$ of metrics with the scalar curvature equals -1. The functional $W[g^t, \hat{\sigma} | \dot{c}_1, \ldots, \dot{c}_b]$ is defined by replacing in Eq. (4.9, 10) the section $\bar{g}^t$ (4.8) by the section g^t (4.16) and substituting $\dot{\bar{c}}_k = \Pi(\dot{c}_k)$ (Π denotes the canonical projection $\Pi : \dot{\bar{\mathscr{C}}}^{26}_S \to \mathscr{C}^{26}_S$). These two sections can be connected by a transformation from $\bar{\mathscr{W}}^i_{h,b}$ so the functionals W and $\bar{W}$ coincide. They may have however different BRST extensions (the nonhomogeneous boundary conditions for ghost variables are not $\bar{\mathscr{W}}^i_{h,b}$-invariant.) Therefore we prefer to use different notations.

The closed string off-shell amplitudes in the constant curvature gauge are defined by:

$$A_h[\dot{c}_1, \ldots, \dot{c}_b | -1] = A_h[\dot{c}_1, \ldots, \dot{c}_b | \mathscr{B}^0_{h,b}] . \tag{4.17}$$

5. Conclusions

5.1. The First Quantized String

The considerations of Sect. 2.3 provide the first step of the covariant functional quantization of the closed bosonic string in the position representation. There are two formulations based on two different choices of the space of boundary conditions for closed string trajectories. They consist of the following three objects: the space of string wave functionals endowed with the ultralocal inner product, the family of off-shell propagators numerated by the family $\mathfrak{B}_M$ of $\mathscr{D}^i_M \odot \bar{\mathscr{W}}^i_M$-subbundles of (3.3) and the group of residual gauge transformations describing the subspace of "off-shell physical" states. As in the open string case [20] it turns out that the choice of the subbundle $\mathscr{B}^0_M$ constructed from the space of zero scalar curvature metrics by the $\bar{\mathscr{W}}^i_M$-action leads (in both formulations) to the propagator with a simple quantum mechanical interpretation. After the BRST extension [12–14] it can be related [4] by the quantum mechanical formula:

$$P = \int_0^\infty dt\, e^{-tH}$$

with the standard closed string Hamiltonian (the zero mode of the density of quadratic in momenta constraints).

It is an interesting question to find a quantum mechanical interpretation of the propagators corresponding to other choices of $\mathscr{B}_M \in \mathfrak{B}_M$. It seems that the freedom in the choice of $\bar{\mathscr{B}}_M \in \bar{\mathfrak{B}}_M$ corresponds to the freedom in the choice of a closed string Hamiltonian governing an (unphysical) evolution in the intrinsic time. In the present paper we adapt the point of view according to which the choice of $\mathscr{B}_M \in \mathfrak{B}_M$ is regarded as a part of the gauge fixing procedure. The general structure of gauge fixing in the first quantized theory is briefly summarized in Table 1. It is essentially the same as in the open string case and we refer to [20] for a more detailed discussion. Let us only note that the resulting formulation crucially depends on the first and the third stage of the gauge fixing. The choices made at the second stage have a technical character and do not influence the final expressions.

Within the covariant functional framework of quantization [20, 21] the first class constraints linear in momenta are implemented on the quantum level by the requirement of invariance of wave functionals under the residual gauge transformations. This requirement formulated in terms of infinitesimal transformations leads to a family of first order differential equations on string wave functionals. In order to determine the subspace of physical on-shell states it is necessary to add a second order differential equation corresponding to a quadratic in momenta constraint (all other quadratic constraints appear as integrability conditions for this extended system of equations). It could be done by taking the inverse of the string propagator, but in both formulations considered in Sect. 3 the propagators are not invertible. One possible way to overcome this difficulty consists in the construction of an invertible extension of the propagator. This is precisely what the BRST construction provides. Such an extension of the closed string propagator (in the fixed length gauge) was proposed in several papers [12–14]. It should be stressed however that a complete first quantized theory should include, besides a BRST extended invertible propagator, a BRST extended inner product as well as a BRST extension of the group of residual gauge transformations.

Table 1

		Space of boundary conditions	Space of string trajectories
I	$\bar{\mathfrak{H}}^{\mathrm{off}}$ / $\mathfrak{H}^{\mathrm{off}}$	S-fixed	M-fixed
II	$\bar{\mathfrak{H}}^{\mathrm{off}}$ / $\mathfrak{H}^{\mathrm{off}}$	(2.14) Conformal gauges (2.3)	Generalized conformal gauges $\bar{\mathscr{F}}^c(\Gamma)$
III	$\bar{\mathfrak{H}}^{\mathrm{off}}$	$\mathscr{G}_S, \dot{\mathscr{C}}_S^{26}(\ell)$	Fixed length gauge
	$\mathfrak{H}^{\mathrm{off}}$	$\mathscr{G}_S, \mathbb{R}_+$	Constant curvature gauge

5.2. The Second Quantized String

The path integral representations of the closed string off-shell amplitudes presented in Sect. 4 give a starting point for the off-shell formulation of interacting closed string theory. The basic idea of the present approach consists in the choice of $\mathcal{D}^i_{h,b}$-invariant boundary conditions (4.5), (4.12) for string "trajectories" (metrized surfaces). There are two advantages of the conditions (4.5), (4.12). First of all they lead (via the F–P method) to the well defined F–P determinant. Therefore to derive the path integral representations of off-shell amplitudes one can follow essentially the same line of reasoning as in the case of the Polykov path integral over closed surfaces [1, 2, 6, 7]. Secondly they allow to compare boundary values on different bordered surfaces without referring to any special parametrization of the corresponding boundary components. This gives a way to overcome the difficulty of previous approaches concerning the relation between boundary parametrizations (and therefore boundary values of the x-variables) and a conformal structure on the world sheet [17, 18].

Note that with the choice of $\mathcal{D}^i_{h,b}$-invariant boundary conditions the $\mathcal{W}^i_{h,b}$-invariance is broken to the $\mathcal{W}^{i0}_{h,b}$-invariance. In consequence the off-shell amplitudes are not invariant under the residual gauge transformations ($\mathcal{G}^0_S$ or $\mathbb{R}_+ \times \mathcal{G}^0_S$) and depend, for every topological type (h, b), on the choice of a $\mathcal{D}^i_{h,b} \odot \mathcal{W}^i_{h,b}$-subbundle $\mathcal{B}^*_{h,b}$ of the principal fiber bundle (4.2). (In different formulations the subbundle $\mathcal{B}^*_{h,b}$ is further reduced to different $\mathcal{D}^i_{h,b} - \mathcal{W}^{i0}_{h,b}$-subbundles, see conditions (4.5), (4.12).) This freedom in the definition of the off-shell amplitudes is in fact an expected feature of the off-shell formulation and can be understood as the freedom in the choice of a gauge. The gauge dependence of amplitudes is explicitly described for a wide class of gauges by the formulae (4.7), (4.15).

A complete discussion of sewing rules requires a BRST extension of the amplitudes constructed in Sect. 4. Therefore we restrict ourselves here only to a few remarks. In the formulation based on the space $\dot{\mathcal{C}}^{26}_S$ and in the constant curvature gauge the sewing rules are especially simple. It can be easily recognized from formula (4.15) that using the scalar product (2.12) for the sewing one obtains the correct Weil-Petersson measure. In fact the double integration over twists of common boundary components reduces, due to the $\mathcal{I}_S$-invariance of the inner product (2.12), to the single integration over relative twists. Moweover the constant curvature gauge has the sewing property [20] and the problem of sewing amplitudes can be reduced to the problem of sewing at a fixed conformal structure which was recently solved [19]. This pattern of the sewing amplitudes was first proposed by D'Hoker and Phong [26].

In the formulation based on the space $\dot{\bar{\mathcal{C}}}^{26}_S$ the expression for the off-shell amplitudes (4.7) contains the integration over Teichmüller parameters related to the lengths of boundary components. Therefore for every boundary component along which amplitudes are sewn we have one redundant integration. As it was argued in [17, 18] these superfluous integrations can be removed by inserting the inverse of the off-shell propagator "between" sewn amplitudes. Let us stress however that in the approach developed in [17, 18] the relation between boundary parametrizations and a conformal structure on $M_{h,b}$ is constructed by means of holomorphic quadratic differentials on $M_{h,b}$ and differs from that determined by the boundary conditions (4.5). The sewing rules proposed in [17, 18] require therefore a modification to be applied in the present formulation.

Both sewing patterns sketched above have (independent of details of a BRST extension) an important drawback: An infinite overcounting of the moduli space appears [17, 18]. Maybe it is an unavoidable feature of the Euclidean off-shell formulation. The comparison with the Mandelstam light-cone formalism [27, 28] suggests however that a solution of this problem exists. It seems that in this problem a better understanding of the relation between the covariant off-shell formulations in the Euclidean and in the Minkowski spaces is required.

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