

Area-Preserving Diffeomorphisms and Supermembrane Lorentz Invariance

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Abstract. The Lie algebra of area-preserving diffeomorphisms on closed membranes of arbitrary topology is investigated. On the basis of a harmonic decomposition we define the structure constants as well as two other tensors which appear in the supermembrane Lorentz generators. We derive certain identities between these tensors and analyze their validity when the area-preserving diffeomorphisms are approximated by $SU(N)$. One of the additional tensors can then be identified with the invariant symmetric three-index tensor of $SU(N)$, while the second has no obvious analog. We prove that the Lorentz generators are classically conserved in the light-cone gauge for arbitrary membrane topology, as a consequence of these tensor identities. This formulation allows a systematic study of the violations of Lorentz invariance in the $SU(N)$ approximation.

1. Introduction

The recent interest in (super)membrane theory has been motivated as much by their mathematical structure as by their possible interest for the unification of fundamental interactions (recent developments are reviewed in [1]). Of particular importance is the group of area-preserving (symplectic) diffeomorphisms which naturally appears in membrane theory as a residual symmetry in the light-cone gauge [2, 3] (the corresponding symmetry in string theory consists only of the “length-preserving” diffeomorphism $\sigma \rightarrow \sigma + \text{constant}$, and is thus rather trivial). Little is known about these infinite-dimensional groups in general (for an early reference in the physics literature, see [4]), and one of the possible benefits of membrane theories could well be an improved understanding of their mathematical structure. A remarkable result, already obtained some time ago, is that, for spherical membranes, the structure constants of the group of area-preserving diffeomorphisms can be obtained as the $N \rightarrow \infty$ limit of the $SU(N)$ structure constants [2, 3]. More recently, and partly motivated by the discovery of supermembranes [5], several papers have appeared dealing with area-preserving diffeomorphisms on the sphere and the torus [6–12], which have led to a certain amount of progress.

For instance, it was realized that (the simple subgroup of) the area-preserving diffeomorphisms on the torus can also be obtained from $SU(\infty)$ ([10, 12, 19] and Appendix A of this paper); this result, however, does not necessarily imply that the area-preserving diffeomorphisms for spherical and toroidal membranes are the “same.” As for topologically more complicated membranes, results are scarce due to the lack of explicit representations for the associated Lie algebras. Another complication is that the Lie algebra associated with the group of area-preserving diffeomorphisms is no longer simple for nonzero genus (it is simple for the sphere [13]). This follows from the observation that, for nonzero genus, there are diffeomorphisms generated by harmonic vector fields, such that the remaining area-preserving diffeomorphisms constitute an invariant subgroup.

The group of area-preserving diffeomorphisms is also important for another reason. It was shown in [7] that supermembranes can be formulated as one-dimensional supersymmetric gauge theories of area-preserving diffeomorphisms. The supermembrane Hamiltonian can thus be regarded as a one-dimensional reduction of $N = 4$ super-Yang–Mills theory with an infinite-dimensional gauge-group. A supersymmetric regularization of this theory is obtained by approximating the group of area-preserving diffeomorphisms by a finite-dimensional gauge group. In this way, the supermembrane can be analyzed as a system in supersymmetric quantum mechanics. Clearly, a regularization is not only useful but, in fact, necessary if one is to gain any understanding of the nonperturbative dynamics of supermembranes, just as it is necessary for any other interacting (and possibly divergent) field theory. For instance, by employing such a regularization it was demonstrated recently that the spectrum of the associated supersymmetric Hamiltonian is continuous and has no gap [14]. We repeat, however, that the validity of this result is contingent upon the approximability of the group of area-preserving transformations by a finite Lie group.

In Sect. 2 of this paper we will give a systematic treatment of area-preserving transformations (for related discussions, see e.g. [8]). The presentation is set up in a way suited to the treatment of the (classical) Lorentz invariance in the light-cone gauge, which will follow in Sect. 3. Performing a harmonic expansion of all dynamically relevant variables in a basis of scalar harmonics on the membrane, one is naturally led to define the structure constants f_{ABC} of the area-preserving transformations in terms of a certain overlap integral involving these harmonics [2, 3]. Our analysis of the Lorentz generators reveals the existence of two further tensors, which to the best of our knowledge have not appeared in the literature so far. In the $SU(N)$ approximation, one of these, denoted by d_{ABC} , corresponds to the $SU(N)$ -invariant symmetric three-index tensor; the other, denoted by c_{ABC} , has no $SU(N)$ analog. The existence of such a tensor c_{ABC} different from f_{ABC} and d_{ABC} is a genuine property of the infinite-dimensional group that distinguishes it from its finite-dimensional “regulator” subgroups. Namely, we will show explicitly that a tensor with analogous properties can always be redefined to become a linear combination of the known group invariant tensors if the Lie algebra possesses a Killing–Cartan metric (for the infinite-dimensional group of area-preserving diffeomorphisms this metric is singular). Relying on the completeness relations for the scalar and vector harmonics on the membrane, we deduce a number of identities

for these tensors. Using these identities it is thus possible to establish the light-cone Lorentz invariance in an “algebraic” fashion. One of the advantages of the present formulation is that we can treat arbitrary membrane topologies. Furthermore, it is now easy to study the violations of Lorentz invariance that appear in the $SU(N)$ approximation. Not unexpectedly, we find that the boost generators are only conserved modulo terms of order $O(1/N^2)$, but these terms can be controlled and shown to vanish in the $N \rightarrow \infty$ limit. In two appendices we describe how the torus and the sphere algebra can be approximated by $SU(N)$ and give explicit formulae for all the tensors that were introduced before.

Having established the classical Lorentz invariance in the large- N limit, it follows that the Lorentz algebra must close in the usual fashion in this limit. One might wonder whether the formalism is also useful in order to study the closure of the quantum Lorentz algebra and thus could lead to the determination of the critical dimension of the supermembrane. Indeed, treating the variables as quantum operators and paying attention to the order in which they appear one would pick up extra contributions which stay finite or diverge in the limit $N \rightarrow \infty$. However, their determination would not be sufficient for the following reason. Before embarking on such a calculation one would have to ensure first that the Lorentz generators are well-defined as quantum operators. In string theory, this is achieved by normal ordering, which is sufficient to render all operators well-defined, because string theory is essentially free. In contrast, the (super)membrane is described as a three-dimensional interacting quantum field theory. Therefore one cannot a priori expect the operators to become well-defined through normal ordering alone. Rather, one must treat them as composite operators in interacting field theory with all the concomitant complications. This, in turn, requires the order-by-order calculation of the relevant renormalizations and counterterms. As to date no such results are available, it does not make much sense to further pursue the question of the quantum Lorentz algebra at this point. To be sure, one can attempt an analysis of the short distance singularities based on certain plausible “kinematical” assumptions about the operator product expansions [15]. It is indeed possible with rather mild assumptions to predict the space-time dimension $D = 27$ as a necessary condition for the consistency of the bosonic membrane [15] (the same procedure yields $D = 11$ for the supermembrane [16]). In conclusion, there is some evidence of an underlying mathematical structure leading beyond string theory, but much more work is obviously needed to explain the “enigma” of $D = 11$ supergravity.

2. Area-Preserving Diffeomorphisms and Harmonic Decompositions

In this paper we will be considering closed membranes, so that the membrane coordinates and momenta can be expressed as functions of two parameters, σ^1 and σ^2 , which are local coordinates on a closed compact two-dimensional space of the same topology as the membrane. In this parameter space it is convenient to define a metric, which we shall denote by $w_{rs}(\sigma)$ to avoid confusion with the (dynamically relevant) Nambu–Goto metric of the membrane which we denote by g_{rs} . Without loss of generality we may fix the total area of the parameter space to

unity,¹ i.e.,

$$\int d^2\sigma \sqrt{w(\sigma)} = 1. \quad (2.1)$$

Area-preserving diffeomorphisms are diffeomorphisms

$$\sigma^r \rightarrow \sigma^r + \xi^r(\sigma), \quad (2.2)$$

with

$$\partial_r(\sqrt{w(\sigma)}\xi^r(\sigma)) = 0, \quad \text{or, equivalently,} \quad D_r\xi^r(\sigma) = 0, \quad (2.3)$$

where $w(\sigma) = \det w_{rs}(\sigma)$ and D_r is the covariant derivative with respect to the metric w_{rs} . The general solution of (2.3) can be decomposed into co-exact and harmonic vector fields. Furthermore, there can be homotopically nontrivial transformations; these will not be considered in what follows. The co-exact components can be parametrized in terms of globally defined functions $\xi(\sigma)$,

$$\xi^r(\sigma) = \frac{\varepsilon^{rs}}{\sqrt{w(\sigma)}} \partial_s \xi(\sigma). \quad (2.4)$$

Under these diffeomorphisms, the (transverse) membrane coordinates \bar{X} transform as

$$\delta X^a = \{\xi, X^a\}, \quad (2.5)$$

where the bracket $\{A, B\}$ for two functions $A(\sigma)$ and $B(\sigma)$ is defined by

$$\{A, B\}(\sigma) \equiv \frac{1}{\sqrt{w(\sigma)}} \varepsilon^{rs} \partial_r A(\sigma) \partial_s B(\sigma). \quad (2.6)$$

This bracket can be shown to have the requisite properties of a Lie bracket. In particular it satisfies the Jacobi identity,

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0. \quad (2.7)$$

The harmonic component of area-preserving diffeomorphisms can be decomposed into a set of harmonic vectors, denoted by $\Phi_r^{(\lambda)}$, which satisfy

$$D_r \Phi^{(\lambda)r} = D_{[r} \Phi_{s]}^{(\lambda)} = 0. \quad (2.8)$$

For a membrane of genus g there are precisely $2g$ independent harmonic vectors.

It is clear that the diffeomorphisms generated by the co-exact vector fields form a group; this group will be denoted by G . The commutator of two infinitesimal G -transformations characterized by functions ξ_1 and ξ_2 yields a similar transformation characterized by

$$\xi_3 = \{\xi_2, \xi_1\}. \quad (2.9)$$

One can also include the transformations generated by harmonic vector fields. The commutator of such an infinitesimal transformation generated by $\Phi^{(\lambda)r}$ with a G -transformation characterized by ξ_1 is again a G -transformation characterized by

$$\xi_2 = \Phi^{(\lambda)r} \partial_r \xi_1. \quad (2.10)$$

¹ Note that in previous publications (e.g. [7]) we used w rather than \sqrt{w}

Furthermore, the commutator of two infinitesimal transformations generated by $\Phi^{(\lambda)r}$ and $\Phi^{(\lambda')r}$ yields also a G -transformation characterized by

$$\xi = \sqrt{w(\sigma)} \varepsilon_{rs} \Phi^{(\lambda)r} \Phi^{(\lambda')s}. \quad (2.11)$$

The above commutation relations show that G is an invariant subgroup of the full group of (homotopically trivial) area-preserving diffeomorphisms. Hence the latter group is in general not simple. Observe that this remark is not relevant for spherical membranes, where no harmonic vectors exist, while for toroidal membranes (2.11) is constant so that the commutator vanishes. However, in most of this paper we will be concerned with the subgroup G . As we shall discuss below, it is possible to view this group as the $N \rightarrow \infty$ limit of $SU(N)$, at least for spherical and toroidal membranes.

To exhibit the group of area-preserving diffeomorphisms in a convenient fashion one expands the membrane coordinates into a complete orthonormal basis of functions $Y^I(\sigma)$ (where $I = 0, 1, \dots$) according to

$$\bar{X}(\sigma) = \sum_I \bar{X}^I Y_I(\sigma), \quad (2.12)$$

and likewise for all other quantities of interest such as momenta or fermionic coordinates. The basis functions Y_I , which can be chosen real or complex, are normalized according to

$$\int d^2\sigma \sqrt{w(\sigma)} Y^I(\sigma) Y_J(\sigma) = \delta^I_J, \quad (2.13)$$

where

$$Y^I \equiv Y_I^* = \eta^{IJ} Y_J. \quad (2.14)$$

The last equation follows from the completeness of the functions Y_I . The tensor η_{IJ} is thus given by

$$\eta_{IJ} \equiv \int d^2\sigma \sqrt{w(\sigma)} Y_I(\sigma) Y_J(\sigma). \quad (2.15)$$

It is now convenient to split the basis functions into the constant mode $Y_0 = 1$, which characterizes the center-of-mass coordinate of the membrane, i.e.,

$$\bar{X}_0 = \int d^2\sigma \sqrt{w(\sigma)} \bar{X}(\sigma), \quad (2.16)$$

and the remaining functions Y_A , where here and throughout the text the indices A will run over the positive integers. The completeness of the basis functions now implies

$$\sum_A Y^A(\sigma) Y_A(\rho) = \frac{1}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \rho) - 1, \quad (2.17)$$

where the last term is due to the fact that the constant function Y_0 is not included on the left-hand side. Observe that we made use of the normalization (2.1).

For the basis functions Y_I one may choose the eigenfunctions of the covariant Laplacian $\Delta = D' D_r$, so that

$$\Delta Y_0 = 0, \quad \Delta Y_A = -\omega_A Y_A, \quad (2.18)$$

where $\omega_A > 0$. The Green's function of the Laplacian can then be expressed in

terms of this basis, viz.

$$G(\sigma, \rho) = - \sum_A \frac{1}{\omega^A} Y^A(\sigma) Y_A(\rho), \quad (2.19)$$

so that

$$\Delta G(\sigma, \rho) = \frac{1}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \rho) - 1. \quad (2.20)$$

Below we will also need another Green's function in order to solve the equation

$$\partial_r X^- = V_r, \quad (2.21)$$

where X^- is a light-cone coordinate of the membrane, while V_r is defined in terms of the transverse coordinates and momenta as well as fermionic coordinates. As X^- must be a single-valued function of σ the consistency of (2.21) requires that the vector field V^r satisfies the following equations:

$$D_{[r} V_{s]} = 0, \quad \int d^2\sigma \sqrt{w(\sigma)} \Phi^{(\lambda)r}(\sigma) V_r(\sigma) = 0. \quad (2.23)$$

Both these equations are equivalent to requiring that V_r vanishes when integrated along any closed loop, i.e.,

$$\oint \sqrt{w(\sigma)} V_r(\sigma) d\sigma^r = 0. \quad (2.23)$$

To express the solution of (2.21) in terms of V_r , let us first discuss the decomposition of vector fields in terms of ‘‘vector harmonics.’’ From the Hodge decomposition, it follows that a basis for them is given by

$$Y_r^{(1)A}(\sigma) = \frac{1}{\sqrt{\omega^A}} \partial_r Y^A(\sigma), \quad Y_r^{(2)A}(\sigma) = \frac{1}{\sqrt{\omega^A}} \sqrt{w(\sigma)} \varepsilon_{rs} \partial^s Y^A(\sigma), \quad (2.24)$$

and the harmonic vectors $\Phi_r^{(\lambda)}$. These vector fields are eigenfunctions of the Lichnerowicz operator, which, in two dimensions, reads

$$\Delta_L = D^s D_s + \frac{1}{2} R, \quad (2.25)$$

where R is the scalar curvature. The eigenvalues of $Y_r^{(1)A}$, $Y_r^{(2)A}$ and $\Phi_r^{(\lambda)}$ are given by

$$\Delta_L Y_r^{(\alpha)A}(\sigma) = -\omega_A Y_r^{(\alpha)A}(\sigma), \quad \Delta_L \Phi_r^{(\lambda)} = 0 \quad (\alpha = 1, 2). \quad (2.26)$$

With a suitable choice for the harmonic vectors the functions $Y_r^{(1)A}$, $Y_r^{(2)A}$ and $\Phi_r^{(\lambda)}$ are orthonormal,

$$\begin{aligned} \int d^2\sigma \sqrt{w(\sigma)} w^{rs}(\sigma) Y_r^{(\alpha)A}(\sigma) Y_s^{(\beta)B}(\sigma) &= \delta^{\alpha\beta} \eta^{AB}, \\ \int d^2\sigma \sqrt{w(\sigma)} w^{rs}(\sigma) Y_r^{(\alpha)A}(\sigma) \Phi_s^{(\lambda)}(\sigma) &= 0, \\ \int d^2\sigma \sqrt{w(\sigma)} w^{rs}(\sigma) \Phi_r^{(\lambda)}(\sigma) \Phi_s^{(\lambda')}(\sigma) &= \delta^{\lambda\lambda'}. \end{aligned} \quad (2.27)$$

The corresponding completeness relation reads

$$\sum_{A,\alpha} Y_r^{(\alpha)A}(\sigma) Y_{sA}^{(\alpha)}(\rho) = \frac{w_{rs}(\sigma)}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \rho) - \sum_{\lambda} \Phi_r^{(\lambda)}(\sigma) \Phi_s^{(\lambda)}(\rho). \quad (2.28)$$

With the above results it is not difficult to establish that the solution of (2.21) is

$$X^-(\sigma) = \int d^2\rho \sqrt{w(\rho)} G^r(\sigma, \rho) V_r(\rho) + \text{constant}, \quad (2.29)$$

where the constant is arbitrary. The Green's function G^r is given by

$$G^r(\sigma, \rho) = \sum_A \frac{1}{\omega^A} Y^A(\sigma) \partial^r Y_A(\rho). \quad (2.30)$$

It is easy to check that this Green's function obeys

$$\int d^2\sigma \sqrt{w(\sigma)} G^r(\sigma, \rho) = 0, \quad D'_r G^r(\sigma, \rho) = -\frac{1}{\sqrt{w(\sigma)}} \delta^{(2)}(\sigma, \rho) + 1, \quad (2.31)$$

(where D'_r denotes the covariant differentiation with respect to ρ) as one directly verifies from (2.30) and (2.19).

From (2.9) it is obvious that the Lie bracket corresponds to the structure constants of the (subgroup G of the) area-preserving diffeomorphisms. We can now substitute the expansion (2.12) and thereby extract the structure constants of the group of area-preserving diffeomorphisms in this basis. From the completeness of the Y_A (the function Y_0 does not play a role in what follows), we obtain

$$\{Y_A, Y_B\} = f_{ABC} Y^C, \quad (2.32)$$

so that the structure constants are defined by

$$f_{ABC} \equiv \int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) \{Y_B(\sigma), Y_C(\sigma)\}. \quad (2.33)$$

Note that f_{ABC} is totally antisymmetric in A, B and C .

As we have already indicated, the treatment of the Lorentz algebra in this formalism requires further tensors. One of them is the (fully symmetric) overlap integral between three Y 's (readers curious to see where these definitions come from are invited to glance at the following section),

$$d_{ABC} \equiv \int d^2\sigma \sqrt{w(\sigma)} Y_A(\sigma) Y_B(\sigma) Y_C(\sigma). \quad (2.34)$$

A third tensor arises in the harmonic expansion of (2.29) for $V_r \propto Y \partial_r Y'$. It is given by

$$c_{ABC} \equiv -2 \int d^2\sigma \sqrt{w(\sigma)} \frac{w^{rs}(\sigma)}{\omega_A} \partial_r Y_A(\sigma) Y_B(\sigma) \partial_s Y_C(\sigma). \quad (2.35)$$

There are several important identities for these tensors that follow from the above definitions. For instance, the Jacobi identity is simply a consequence of the Jacobi identity (2.7) for the Lie brackets,

$$f_{[AB}{}^E f_{C]DE} = 0, \quad (2.36)$$

where (2.32) has been used. Another relation follows immediately by partial integration,

$$c_{ABC} + c_{ACB} = -2d_{ABC}. \quad (2.37)$$

Two identities similar to (2.36) are

$$c_{DE}{}^{[A} f^{BC]E} = 0, \quad (2.38)$$

$$d_{ABC} f^A{}_{[DE} f^B{}_{FG]} = 0. \quad (2.39)$$

The proof of these identities is simple and follows from writing the left-hand sides as an integral of $\omega_D^{-1} \partial_r Y_D \{Y^{[B}, Y^C\} \partial^r Y^{A]}$ and $\{Y_{[D}, Y_E\} \{Y_{F]}, Y_G\} Y_C$, respectively. Subsequently one makes use of the Schouten identity, which in the present context expresses the vanishing of any tensor antisymmetric in three membrane indices.

The proof of the following identity requires a little more work,

$$f_{AB}{}^E c_{ECD} = c_{EAB} f^E{}_{CD} - 2f_{BD}{}^E d_{ACE} + \sum_{\lambda} c_{\lambda AB} f^{\lambda}{}_{CD}, \quad (2.40)$$

where $f_{\lambda AB}$ and $c_{\lambda AB}$ are antisymmetric in $[AB]$ and defined by

$$f_{\lambda AB} = \int d^2\sigma \sqrt{w(\sigma)} \Phi^{(\lambda)t}(\sigma) \partial_t Y_B(\sigma) Y_A(\sigma), \quad (2.41)$$

$$c_{\lambda AB} = -2 \int d^2\sigma \varepsilon^{rs} \Phi_r^{(\lambda)}(\sigma) \partial_s Y_B(\sigma) Y_A(\sigma). \quad (2.42)$$

Observe that $f_{\lambda AB}$ is precisely the structure constant corresponding to the commutator (2.10). To derive (2.40) one substitutes the definitions (2.33) and (2.35) on the left-hand side, such that the contraction over the functions Y_E takes the form $\partial_r Y^E(\sigma) \partial_s Y_E(\rho) \omega_E^{-1}$. This is just equal to $Y_r^{(1)E}(\sigma) Y_{sE}^{(1)}(\rho)$. Subsequently one uses the completeness relation (2.28), which leads directly to (2.40).

Before continuing let us examine the index structure of (2.40). Using the fact that the $c_{\lambda AB} f_{DC}{}^{\lambda}$ is antisymmetric in $[AB]$ and $[CD]$, symmetrization of (2.40) in either one of these index pairs leads to

$$f_{A(B}{}^E d_{CD)E} = 0, \quad (2.43)$$

where we have made use of (2.37). From this result we can rewrite (2.40) in the form

$$f_{AB}{}^E c_{E[CD]} - c_{E[AB]} f^E{}_{CD} - 2f_{[D[B}{}^E d_{A]C]E} = \sum_{\lambda} c_{\lambda AB} f^{\lambda}{}_{CD}. \quad (2.44)$$

As the left-hand side is now antisymmetric under the interchange of the index pairs $[AB]$ and $[CD]$, we conclude that also the right-hand side of (2.44) must exhibit this antisymmetry. This result can also be justified on the basis of independent geometric arguments.

Furthermore, one may contract (2.40) by $f_{FG}{}^C$ and antisymmetrize in the three indices $[FGD]$. Using the Jacobi identity (2.36), this yields

$$f_{AB}{}^E (c_{EC}{}^{[D} f^{FG]C}) = -2d_{ACE} f^{EB[D} f^{FG]C}. \quad (2.45)$$

This result is obviously consistent with (2.38) and (2.39). Observe that in the derivation of (2.45) we have also used the Jacobi identity for structure constants that involve one index λ .

$$f_{[AB}{}^E f_{C]E\lambda} = 0. \quad (2.46)$$

Clearly there are more identities for products of the various tensors, such as

$$d_{E[A[B} d_{C]D}{}^E = -\eta_{A[B} \eta_{C]D}. \quad (2.47)$$

However, these identities are not needed in what follows, and we will not further elaborate on them.

Let us now return to (2.36) and (2.43), which suggest that f and d are invariant

tensors under the subgroup G of the area-preserving diffeomorphisms. Actually, it is easy to see that f and d , as well as η_{AB} , defined in (2.15), are invariant under all area-preserving transformations, not just under those contained in G . This follows from the observation that f, d and η are defined in terms of integrals that are invariant under general reparametrizations, provided that one also transforms the metric w_{rs} in the usual manner. On the other hand, w_{rs} is supposed to be a given metric. However, since f does not depend on w_{rs} while d and η depend only on the determinant of w_{rs} which is invariant under area-preserving diffeomorphisms, f, d and η must be invariant under the change of the functions Y_A induced by these diffeomorphisms.

However, the above arguments are not applicable to the tensor c_{ABC} , because it depends explicitly on the metric; consequently we have

$$\delta c_{ABC} = 2 \int d^2\sigma \sqrt{w(\sigma)} \frac{1}{\omega_A} \partial_r Y_A(\sigma) Y_B(\sigma) \partial_s Y_C(\sigma) (D^r \xi^s + D^s \xi^r), \quad (2.48)$$

which does not vanish. On the other hand, the covariant variation suggested by the index structure of c_{ABC} , gives

$$\begin{aligned} \delta c_{ABC} &\equiv \xi^E (f_{AE}{}^D c_{DBC} + f_{BE}{}^D c_{ADC} + f_{CE}{}^D c_{ABD}) \\ &= \xi^E (f_{BC}{}^D (c_{DAE} + c_{ADE}) + f_{BC}{}^\lambda c_{\lambda AE}), \end{aligned} \quad (2.49)$$

where we have made use of (2.38) and (2.44). For the transformations generated by the harmonic vector fields, we can derive a similar result, which takes the same form as (2.49) upon changing the index E into λ . This leads to the definition of new tensors $c_{AB\lambda}$ and $c_{\lambda A\lambda'}$. Furthermore we note that there are identities similar to (2.38)–(2.40) in which one index is replaced by λ (for example, see (2.46)).

In Sect. 3 we will argue that c_{ABC} is only relevant up to terms proportional to $f_{BC}{}^D$. On the other hand, (2.49) shows that c_{ABC} is invariant up to precisely such a term. This would seem to suggest that c_{ABC} can be made invariant by making a suitable modification proportional to $f_{BC}{}^D$. Under such a modification the identities (2.37) and (2.38) would remain valid by virtue of the Jacobi identity (2.36), whereas (2.40) and (2.44) would acquire an extra term proportional to $f_{AB}{}^E d_{CD}{}^F$. Assuming the existence of the Killing–Cartan metric

$$g_{AB} = -f_{AC}{}^D f_{DB}{}^C, \quad (2.50)$$

one can indeed write down a modified tensor C'_{ABC} , which is formally invariant. It reads

$$c'_{ABC} = c_{ABC} - f_{BC}{}^D c_{DEF} f^{EFG} (g^{-1})_{GA}. \quad (2.51)$$

From (2.49) and (2.50) one can easily verify the invariance of c'_{ABC} . However, in the case at hand, these conditions cannot be applied because the Killing–Cartan metric is singular for the group G of area-preserving diffeomorphisms. Therefore no modification will render the tensor c_{ABC} invariant.² On the other hand, the

² This conclusion is in agreement with a more general result in Lie algebra cohomology, according to which there are only nontrivial cohomology classes in representations where the second-order Casimir operator is invertible. See, e.g., [17]

modification is possible if the group G is truncated to a finite group such as $SU(N)$. Then any tensor with the same properties as c_{ABC} is actually expressible as a linear combination of the invariant three-index tensors of the finite group, up to the modification given in (2.51). Therefore it appears that the existence of a tensor c_{ABC} different from either f_{ABC} or d_{ABC} is a genuine property of the infinite-dimensional group G .

This brings us to the question of how the definitions and results described so far are affected when the group G is replaced by a finite group. As already mentioned, such an approximation is possible for spherical and toroidal membranes, where the structure constants f_{ABC} can be obtained from the ordinary $SU(N)$ structure constants in the limit $N \rightarrow \infty$. We have demonstrated the necessity of further numerical tensors for the Lorentz generators, and this raises the question whether these, too, can be approximated by $SU(N)$ tensors. The tensor d_{ABC} defined in (2.34) evidently corresponds to the symmetric three-index tensor of $SU(N)$, and in the appendices we will confirm this expected correspondence (which, incidentally shows that the $SU(N)$ groups are the only finite-dimensional simple Lie groups that can possibly serve as a regulator group, since no other simple Lie groups admit an invariant symmetric three-index tensor). As for the tensor c_{ABC} , the arguments presented above show that it has no counterpart for finite-dimensional groups. In accordance with these arguments we can try to represent it as a linear combination of the known $SU(N)$ invariants d_{ABC} and f_{ABC} . Quite independently of the explicit representation, however, it is impossible for finite N to simultaneously satisfy all of the identities that we have derived in this section; it is even impossible to satisfy the ones involving only d_{ABC} and f_{ABC} . This is most conveniently checked by means of the explicit formulae of Appendix A. For instance, (2.39) and (2.47) are invalid for finite N ; substitution of the relevant expressions in Appendix A shows the violations to be of order $O(1/N^2)$ for finite N . As the identities involve only invariant tensors, this conclusion is independent of the explicit basis chosen for the $SU(N)$ matrices. To be sure, there is a certain ambiguity in the choice of c_{ABC} for finite N . Since it has no proper $SU(N)$ analog and since the identities we relied on in deriving (2.51) are anyhow violated for finite N , we can just as well regulate it by any tensor whose components differ from their exact values by $O(1/N^2)$ (further details are given in the appendices). In this way, it is possible to satisfy all identities up to terms of order $O(1/N^2)$.

3. Lorentz Invariance

In this section we will apply the formalism developed in the foregoing section to establish the Lorentz invariance of the classical supermembrane formulated in the light-cone gauge and to show that it holds independently of the topology of the membrane. We will not repeat the basic results about supermembranes here but rather assume that the reader is familiar with them (see e.g. [1]); in particular, we refer the reader to [7] for our conventions and notation as well as a derivation of the results needed here.

The canonical Hamiltonian H in the light-cone gauge is given by

$$H = - \int d^2\sigma P^-(\sigma) = \frac{1}{P_0^+} \int d^2\sigma \sqrt{w(\sigma)} \left(\frac{1}{2} w^{-1} \vec{P}^2 + \frac{1}{4} (\{X^a, X^b\})^2 - \frac{i}{2} \theta \gamma_a \{X^a, \theta\} \right). \quad (3.1)$$

Here, $\vec{P}(\sigma)$ denotes the canonical momenta conjugate to the (transverse) coordinates $\vec{X}(\sigma)$, which transform as a density under reparametrizations of σ^r ; $\theta(\sigma)$ are the fermionic coordinates, which transform as a spinor under the transverse rotations and as a scalar under reparametrizations.³ Our choice for these coordinates is such that the only nonvanishing Dirac brackets take the form

$$\begin{aligned} (X^a(\sigma), P^b(\rho))_{DB} &= \delta^{ab} \delta^{(2)}(\sigma, \rho), \\ (\theta_\alpha(\sigma), \theta_\beta(\rho))_{DB} &= - \frac{i}{\sqrt{w(\sigma)}} \delta_{\alpha\beta} \delta^{(2)}(\sigma, \rho). \end{aligned} \quad (3.2)$$

The quantity P_0^+ represents the total momentum of the supermembrane in one of the directions along the light-cone; as indicated in (3.1) the other momentum component along the light-cone is related to the Hamiltonian. Regarding the light-cone components of the coordinate vector, X^+ is related to the time τ , the dependence on which we have suppressed above, while X^- does not explicitly appear in the Hamiltonian, but is determined through the condition

$$\partial_r X^-(\sigma) = - \frac{1}{P_0^+} \left(\frac{1}{\sqrt{w(\sigma)}} \vec{P}(\sigma) \cdot \partial_r \vec{X}(\sigma) + \frac{i}{2} \theta(\sigma) \partial_r \theta(\sigma) \right). \quad (3.3)$$

As was first observed by Goldstone [2] this condition can be solved by means of the Green's function G^r introduced in Sect. 2, provided the following conditions hold (cf. (2.22)),

$$\left\{ \frac{\vec{P}(\sigma)}{\sqrt{w(\sigma)}}, \cdot \vec{X}(\sigma) \right\} + \frac{i}{2} \{ \theta(\sigma), \theta(\sigma) \} = 0, \quad (3.4)$$

$$\oint \left(\vec{P}(\sigma) \cdot \partial_r \vec{X}(\sigma) + \frac{i}{2} \sqrt{w(\sigma)} \theta(\sigma) \partial_r \theta(\sigma) \right) d\sigma^r = 0. \quad (3.5)$$

We then obtain (cf. (2.29))

$$X^-(\sigma) = q^- - \frac{1}{P_0^+} \int d^2\rho G^r(\sigma, \rho) \left(\vec{P}(\rho) \cdot \partial_r \vec{X}(\rho) + \frac{i}{2} \sqrt{w(\rho)} \theta(\rho) \partial_r \theta(\rho) \right), \quad (3.6)$$

where q^- denotes the center-off-mass coordinate of the membrane, which is canonically conjugate to P_0^+ . Therefore we have an additional nonvanishing Dirac bracket,

$$(q^-, P_0^+)_{DB} = 1. \quad (3.7)$$

³ If the membrane moves in a 11-dimensional space-time, the fermionic coordinates θ are thus 16-component spinors of $SO(9)$. In contrast with the coordinates used in [7], we have rescaled θ by $\sqrt{P_0^+}$ times a numerical factor. The gamma matrices γ_a are real and symmetric

The part of phase space that satisfies the constraints (3.4) and (3.5) will be referred to as the “physical subspace.”

The supermembrane is invariant under Lorentz transformations and space–time supersymmetry, although not all these invariances are manifest in the light-cone gauge. Furthermore, the invariance under general diffeomorphisms of the three-dimensional world tube is broken in this gauge and one is left with a residual invariance under area-preserving diffeomorphisms, which we have extensively discussed in Sect. 2. According to [18] the first-class constraints (3.4) and (3.5) are also the generators of the area-preserving diffeomorphisms. These generators therefore take the form

$$\varphi(\sigma) = - \left\{ \frac{\bar{P}(\sigma)}{\sqrt{w(\sigma)}}, \bar{X}(\sigma) \right\} - \frac{i}{2} \{ \theta(\sigma), \theta(\sigma) \}, \quad (3.8)$$

$$\varphi^\lambda = - \int d^2\sigma \Phi^{(\lambda)r}(\sigma) \left(\bar{P}(\sigma) \cdot \partial_r \bar{X}(\sigma) + \frac{i}{2} \sqrt{w(\sigma)} \theta(\sigma) \partial_r \theta(\sigma) \right). \quad (3.9)$$

To calculate the Lorentz and supersymmetry generators of the supermembrane in the light-cone gauge, we apply the Noether procedure to the original supermembrane action, pass to light-cone coordinates and impose the light-cone gauge conditions. This straightforward but somewhat tedious calculation leads to the following result for the Lorentz generators:

$$\begin{aligned} M^{ab} &= \int d^2\sigma \left(-P^a X^b + P^b X^a - \frac{i}{4} \theta \gamma^{ab} \theta \right) \\ M^{+-} &= \int d^2\sigma (-P^+ X^- + P^- X^+), \\ M^{+a} &= \int d^2\sigma (-P^+ X^a + P^a X^+), \\ M^{-a} &= \int d^2\sigma \left(P^a X^- - P^- X^a - \frac{i}{4P_0^+} \theta \gamma^{ab} \theta P_b - \frac{i\sqrt{w}}{8P_0^+} \{X_b, X_c\} \theta \gamma^{abc} \theta \right). \end{aligned} \quad (3.10)$$

The supersymmetry generators read

$$\begin{aligned} Q^+ &= \frac{1}{\sqrt{P_0^+}} \int d^2\sigma \left(P^a \gamma_a + \frac{\sqrt{w}}{2} \{X^a, X^b\} \gamma_{ab} \right) \theta, \\ Q^- &= \sqrt{P_0^+} \int d^2\sigma \sqrt{w(\sigma)} \theta, \end{aligned} \quad (3.11)$$

where Q^+ and Q^- transform according to inequivalent spinor representations of the transverse rotation group.

Following the analysis of the previous section, we expand the membrane coordinates into a complete set of orthonormal functions, so that all relevant quantities can be expressed with the help of the tensors $f_{ABC}, d_{ABC}, c_{ABC}$ and the metric η_{AB} . For instance, the generators of the area-preserving reparametrizations now become

$$\varphi_A = f_{ABC} \left(\bar{X}^B \cdot \bar{P}^C - \frac{i}{2} \theta^B \theta^C \right), \quad (3.12)$$

$$\varphi_\lambda = f_{\lambda BC} \left(\bar{X}^B \cdot \bar{P}^C - \frac{i}{2} \theta^B \theta^C \right), \quad (3.13)$$

and the constraints

$$\varphi_A = 0, \quad \varphi^\lambda = 0 \quad (3.14)$$

hold on the physical subspace.

In this formulation the canonical brackets are given by

$$\begin{aligned} (X_A^a, P_B^b)_{DB} &= \delta^{ab} \eta_{AB}, & (\theta_{\alpha A}, \theta_{\beta B})_{DB} &= -i \delta_{\alpha\beta} \eta_{AB}, \\ (q^-, P_0^+)_{DB} &= 1, & (\theta_{\alpha 0}, \theta_{\beta 0})_{DB} &= -i \delta_{\alpha\beta}. \end{aligned} \quad (3.15)$$

The solution (3.3) for the light-cone coordinate X^- can now be written explicitly with the help of the coefficient c_{ABC} defined in Sect. 2,

$$X_A^- = -\frac{1}{P_0^+} \left(\bar{P}_0 \cdot \bar{X}_A + \frac{i}{2} \theta_0 \theta_A \right) + \frac{1}{2P_0^+} c_{ABC} \left(\bar{P}^B \cdot \bar{X}^C + \frac{i}{2} \theta^B \theta^C \right). \quad (3.16)$$

In view of the remarks made towards the end of Sect. 2, it is clear that X_A^- does not transform covariantly under the group of area-preserving diffeomorphisms. Using (2.49) we find

$$\delta X_A^- = \xi^B f_{AB}{}^C X_C^- + \frac{\xi^B}{2P_0^+} ((c_{CAB} + c_{ACB}) \varphi^C + c_{\lambda AB} \varphi^\lambda), \quad (3.17)$$

and a similar result for the diffeomorphisms generated by harmonic vectors. However, the anomalous terms in the transformation rules of X_A^- are proportional to the constraints φ^A and φ^λ , so that X_A^- transforms covariantly in the physical subspace. Of course, the same remark applies to the Lorentz generators M^{-a} , which depend on X_A^- .

The Hamiltonian (3.1) reads in this basis,

$$H = \frac{\bar{P}_0^2}{2P_0^+} + \frac{\mathcal{M}^2}{2P_0^+}, \quad (3.18)$$

where \mathcal{M} is the invariant mass of the supermembrane, which is given by

$$\mathcal{M}^2 = \bar{P}_A^2 + \frac{1}{2} (f_{ABC} X_a^B X_b^C)^2 - i f_{ABC} \theta^A \gamma^a X_a^B \theta^C. \quad (3.19)$$

Observe that \mathcal{M} does *not* depend on the center-of-mass coordinates and momenta, and neither does it depend on P_0^+ .

It is straightforward to substitute the mode expansions and (3.16) into the Lorentz generators listed in (3.10). The first three generators take a simple form,

$$M^{ab} = -P_0^a X_0^b + P_0^b X_0^a - \frac{i}{4} \theta_0 \gamma^{ab} \theta_0 - P_A^a X^{bA} + P_A^b X^{aA} - \frac{i}{4} \theta_A \gamma^{ab} \theta^A, \quad (3.20)$$

$$M^{+-} = -P_0^+ q^- - H\tau, \quad (3.21)$$

$$M^{+a} = -P_0^+ X_0^a + \tau P_0^a, \quad (3.22)$$

where we have substituted $X^+ = \tau$. It is convenient to write M^{-a} , which contains many more terms, in the following form:

$$M^{-a} = (M^{-a})^{(0)} + \frac{1}{P_0^+} \left(P_{0b} \tilde{M}^{ab} - \frac{i}{2} \theta_0 \gamma^a \tilde{Q}^+ \right) + \frac{1}{P_0^+} \tilde{M}^{-a}, \quad (3.23)$$

where $(M^{-a})^{(0)}$ represents the contributions of the center-of-mass modes,

$$(M^{-a})^{(0)} = q^- P_0^a + X_0^a H - \frac{i}{8P_0^+} \theta_0 \gamma^{ab} \theta_0 P_0^b, \quad (3.24)$$

and \tilde{Q}^+ , \tilde{M}^{ab} and \tilde{M}^{-a} refer to contributions of the nonzero modes; those are defined by

$$\tilde{Q}^+ = (P_A^a \gamma_a + \frac{1}{2} f_{ABC} X_a^B X_b^C \gamma^{ab}) \theta^A, \quad (3.25)$$

$$\tilde{M}^{ab} = -P_A^a X^{bA} + P_A^b X^{aA} - \frac{i}{4} \theta_A \gamma^{ab} \theta^A, \quad (3.26)$$

$$\begin{aligned} \tilde{M}^{-a} = & \frac{1}{2} d^{ABC} X_A^a (\vec{P}_B \cdot \vec{P}_C + \frac{1}{2} (f_B^{DE} X_D^b X_E^c)) (f_C^{FG} X_F^b X_G^c) - i f_C^{DE} X_D^b \theta_B \gamma_b \theta_E \\ & - \frac{1}{4} i d^{ABC} P_{Ab} \theta_B \gamma^{ab} \theta_C + \frac{1}{2} c^{ABC} P_A^a (\vec{P}_B \cdot \vec{X}_C + \frac{1}{2} i \theta_B \theta_C) \\ & - \frac{1}{8} i f^{ABC} d_A^{DE} X_{Bb} X_{Cc} \theta_D \gamma^{abc} \theta_E. \end{aligned} \quad (3.27)$$

In principle, the construction of the Lorentz generators ensures that they are conserved charges i.e.,

$$\frac{d}{d\tau} M = \frac{\partial}{\partial \tau} M + (M, H)_{DB} = 0, \quad (3.28)$$

where M generically denotes all the Lorentz generators.⁴ Explicit verification of (3.28) can thus be viewed as a check on the consistency of the light-cone gauge. However, in the formulation that we employ the classical check of the Lorentz invariance is reduced to a purely algebraic problem for the tensors defined in the previous section: what we intend to show is that Lorentz invariance is a consequence of some of the tensor identities derived in Sect. 2. This has the advantage that we can explicitly see where the Lorentz invariance breaks down in the $SU(N)$ approximation.

The verification of (3.28) for the generators M^{ab} , M^{+-} and M^{+a} is straightforward, and does not depend on the properties of the various tensors. The proof of the τ -independence of M^{-a} is more involved. In order to simplify the discussion somewhat, we first note that the Dirac bracket of $(M^{-a})^{(0)}$ and H vanishes. Using the decomposition (3.23) one can then easily show that M^{-a} is conserved, provided that

$$(\tilde{M}^{ab}, \mathcal{M}^2)_{DB} = 0, \quad (3.29)$$

⁴ In principle one should use the total Hamiltonian, but as the Lorentz generators are invariant under area-preserving diffeomorphisms in the physical subspace (cf. (3.17)) it is sufficient to consider the canonical Hamiltonian

$$(\tilde{Q}^+, \mathcal{M}^2)_{DB} = 0, \quad (3.30)$$

$$(\tilde{M}^{-a}, \mathcal{M}^2)_{DB} = 0. \quad (3.31)$$

The first equation is obviously satisfied as a consequence of the manifest invariance of \mathcal{M}^2 under transverse rotations. While the second equation represents the well-known fact that the invariant supermembrane mass is supersymmetric (see, e.g. [7]), its explicit verification is not entirely straightforward, and depends on the Jacobi identity (2.36) and an identity for anticommuting spinors,

$$\begin{aligned} & \theta_1 \gamma^b \theta_2 \theta_3 \gamma_{ab} \theta_4 + \theta_3 \gamma^b \theta_4 \theta_1 \gamma_{ab} \theta_2 - \theta_1 \gamma^b \theta_4 \theta_3 \gamma_{ab} \theta_2 \\ & - \theta_3 \gamma^b \theta_2 \theta_1 \gamma_{ab} \theta_4 + 2\theta_4 \theta_2 \theta_3 \gamma_a \theta_1 - 2\theta_1 \theta_3 \theta_2 \gamma_a \theta_4 = 0. \end{aligned} \quad (3.32)$$

This identity is derived from a similar identity for Lorentz spinors in the covariant formulation of the supermembrane [5]. As is well-known, the necessity for these identities leads to the restriction that supermembranes can only live in a $d = 4, 5, 7$ or 11 dimensional space-time. Using (3.32) one derives

$$(\tilde{Q}^+, \mathcal{M}^2)_{DB} = 2\theta_A \varphi^A, \quad (3.33)$$

so that (3.30) is satisfied in the physical subspace.

Finally, we are left with the proof of (3.31). We find it convenient to write the result in the following way,

$$(\tilde{M}^{-a}, \mathcal{M}^2)_{DB} = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7, \quad (3.34)$$

with

$$A_1 = P_A^a (c^{ABC} + d^{ABC}) \bar{P}_B \cdot \bar{P}_C, \quad (3.35a)$$

$$\begin{aligned} A_2 = & P_A^a (\frac{1}{2} d^{ABC} f_B^{DE} f_C^{FG} \bar{X}_D \cdot \bar{X}_F \bar{X}_E \cdot \bar{X}_G \\ & - c^{ABC} f^{DEF} f_{BD}^G \bar{X}_C \cdot \bar{X}_F \bar{X}_E \cdot \bar{X}_G), \end{aligned} \quad (3.35b)$$

$$\begin{aligned} A_3 = & iP_A^a (-d^{ABC} f_C^{DE} X_D^b \theta_B \gamma_b \theta_E - \frac{1}{2} c^{ABC} f_B^{DE} X_C^b \theta_D \gamma_b \theta_E \\ & - \frac{1}{2} c^{ABC} f_C^{DE} X_D^b \theta_B \gamma_b \theta_E + \frac{1}{2} c^{ABC} f_B^{DE} X_D^b \theta_C \gamma_b \theta_E), \end{aligned} \quad (3.35c)$$

$$\begin{aligned} A_4 = & \frac{1}{2} i (d^{BDE} f^{AGF} f_{AB}^C X_{Gb} \bar{X}_C \cdot \bar{X}_F \theta_D \gamma^{ab} \theta_E \\ & - c^{BDE} f^{AFG} f_{AB}^C X_F^a \bar{X}_C \cdot \bar{X}_G \theta_D \theta_E \\ & + d^{ABC} f_A^{FG} f_C^{DE} X_{Fb} X_{Gc} X_{Dd} \theta_B \gamma^{abc} \gamma^d \theta_E \\ & + 2d^{ABC} f_C^{DE} f_E^{FG} X_A^a X_D^b X_F^c \theta_B \gamma_b \gamma_c \theta_G), \end{aligned} \quad (3.35d)$$

$$\begin{aligned} A_5 = & X_A^a (2d^{ABC} f_B^{DE} f_C^{FG} \bar{X}_D \cdot \bar{X}_F \bar{X}_E \cdot \bar{P}_G \\ & - 2d^{ABC} f_E^{GF} f_C^{ED} \bar{X}_F \cdot \bar{P}_B \bar{X}_D \cdot \bar{X}_G \\ & - c^{BCD} f^{AEF} f_{BE}^G \bar{X}_D \cdot \bar{P}_C \bar{X}_F \cdot \bar{X}_G), \end{aligned} \quad (3.35e)$$

$$\begin{aligned} A_6 = & -iX_A^a d^{ABC} (f_C^{DE} P_D^b \theta_B \gamma_b \theta_E + f_C^{DE} P_B^b \theta_D \gamma^b \theta_E) \\ & + id^{ABC} (f_B^{DE} P_{Cb} X_{Dc} \theta_A \gamma^{ab} \gamma^c \theta_E - \frac{1}{2} f_A^{DE} P_{Db} X_{Ec} \theta_B \gamma^{abc} \theta_C) \\ & - \frac{1}{2} ic^{ABC} f_A^{DE} \bar{P}_B \cdot \bar{X}_C \theta_D \gamma^a \theta_E, \end{aligned} \quad (3.35f)$$

$$A_7 = \frac{1}{4} (c^{ABE} f_A^{CD} + 2d^{ABC} f_A^{DE}) \theta_B \theta_E \theta_C \gamma^a \theta_D. \quad (3.35g)$$

Observe that the evaluation of the θ^4 -terms requires the use of the identity (3.32).

We will now show that (3.34) vanishes in the physical subspace by virtue of the

tensor identities derived in Sect. 2. Using (2.37) and (2.38) one first shows that A_1, A_2 and A_3 vanish. Subsequently, one rewrites A_4 and A_6 in the following form:

$$A_4 = i f_A^{FC} (d^{ABD} f_C^{GE} - \frac{1}{2} c^{ABE} f_C^{DG}) X_D^a \bar{X}_F \cdot \bar{X}_G \theta_B \theta_E, \quad (3.36)$$

$$A_6 = -i (\frac{1}{2} c_A^{BE} f^{ACD} + d_A^{BC} f^{ADE}) \theta_C \gamma^a \theta_D \bar{X}_E \cdot \bar{P}_B, \quad (3.37)$$

where, for A_4 , we made use of (2.36–39) and (2.43), while, for A_6 , we only used (2.43). Combining the previous results, and using again (2.36) and (2.43), one finds

$$\begin{aligned} (\tilde{M}^{-a}, \mathcal{M}^2)_{DB} = & \left\{ -d_C^{DG} f^{ACF} f_A^{EB} X_F^a \bar{X}_D \cdot \bar{X}_G \right. \\ & + \left(f_C^{FG} X_F^a \bar{X}_D \cdot \bar{X}_G - \frac{i}{2} \theta_C \gamma^a \theta_D \right) (c^{ABE} f_A^{CD} - 2d^{ABD} f_A^{CE}) \left. \right\} \\ & \times \left(\bar{X}_E \cdot \bar{P}_B - \frac{i}{2} \theta_E \theta_B \right). \end{aligned} \quad (3.38)$$

It is now easy to see that (3.38) becomes proportional to the constraints φ^A and φ^λ , as result of (2.40),

$$(\tilde{M}^{-a}, \mathcal{M}^2)_{DB} = - \left(f_B^{DE} X_D^a \bar{X}_C \cdot \bar{X}_E - \frac{i}{2} \theta_B \gamma^a \theta_C \right) (c^{ABC} \varphi_A + c^{\lambda BC} \varphi_\lambda), \quad (3.39)$$

so that also the Lorentz boost operators M^{-a} are conserved in the physical subspace. We should stress that this result is valid irrespective of the topology of the membrane. In the $SU(N)$ approximation we get of course additional terms and Lorentz invariance is inevitably broken for finite N . As the relations (2.37–40) hold for finite N modulo terms of order $O(1/N^2)$, these additional terms are of order $O(1/N^2)$, too. This shows that in the limit $N \rightarrow \infty$ the Lorentz symmetry is restored.

From the fact that the Lorentz generators are conserved, it follows that these generators must close with respect to the Dirac brackets, at least in the physical subspace. On the other hand, an explicit verification of this closure seems to require various new identities, such as identities for products of the c -tensors. As the closure is already guaranteed by the calculation of this section, we must conclude that these new identities are already contained in the set of identities used above. Another issue concerns the evaluation of the quantum Lorentz algebra, but as we have pointed out in Sect. 1, this requires more insight in the structure of the quantum divergences of supermembranes.

Appendix A: Area-Preserving Diffeomorphisms on T^2

Choosing torus coordinates $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1, \sigma_2 \in [0, 2\pi)$, we label the basis functions Y_A by two-dimensional *nonzero* vectors with integer components $\mathbf{A} = (A_1, A_2)$. They are given by

$$Y_{\mathbf{A}}(\sigma) = \exp(i(A_1 \sigma_1 + A_2 \sigma_2)) = (Y^{\mathbf{A}}(\sigma))^*. \quad (\text{A.1})$$

With $w_{rs}(\sigma) = (4\pi^2)^{-1} \delta_{rs}$ it is straightforward to check that the basis functions are normalized and to compute the various tensors defined in Sect. 2. We find

$$\eta_{\mathbf{AB}} = \delta_{\mathbf{A+B}}, \quad (\text{A.2})$$

$$f_{\mathbf{ABC}} = -4\pi^2 (\mathbf{A} \times \mathbf{B}) \delta_{\mathbf{A+B+C}}, \quad (\text{A.3})$$

$$d_{\mathbf{ABC}} = \delta_{\mathbf{A+B+C}}, \quad (\text{A.4})$$

$$c_{\mathbf{ABC}} = 2 \frac{\mathbf{A} \cdot \mathbf{C}}{\mathbf{A} \cdot \mathbf{A}} \delta_{\mathbf{A+B+C}}, \quad (\text{A.5})$$

where $\mathbf{A} \times \mathbf{B} \equiv A_1 B_2 - A_2 B_1$. The normalized harmonic vectors are

$$\Phi_r^{(\lambda)}(\sigma) = \frac{1}{2\pi} \delta_r^\lambda. \quad (\lambda = 1, 2). \quad (\text{A.6})$$

From these we compute the tensors (2.41) and (2.42),

$$f_{\lambda\mathbf{BC}} = 2\pi i C_\lambda \delta_{\mathbf{B+C}}, \quad (\text{A.7})$$

$$c_{\lambda\mathbf{BC}} = -4\pi i e^{i\lambda r} C_r \delta_{\mathbf{B+C}}. \quad (\text{A.8})$$

The identities of Sect. 2 and, in particular, the noninvariance of c can now be explicitly verified.

It has been recognized only recently that the structure constants of the torus algebra can also be approximated by $SU(N)$ structure constants [19, 10, 12]. The essential trick is to exploit 't Hooft's twist matrices [20]. These are $SU(N)$ matrices Ω_1, Ω_2 , obeying [20, 21]

$$\Omega_1 \Omega_2 = z \Omega_2 \Omega_1, \quad (\text{A.9})$$

where the phase factor z must satisfy $z^N = 1$ (explicit representations for these matrices can be found in [20, 21]). For later convenience we choose N odd and put

$$z \equiv \exp\left(-\frac{4\pi i}{N}\right). \quad (\text{A.10})$$

To each vector \mathbf{A} we assign a $N \times N$ matrix

$$T_{\mathbf{A}} \equiv N z^{(1/2)A_1 A_2} \Omega_2^{A_1} \Omega_1^{A_2}. \quad (\text{A.11})$$

Making use of the explicit representation of the twist matrices given in [20, 21] we can now show that

$$T_{\mathbf{A}}^\dagger = T_{-\mathbf{A}}, \quad \text{Tr } T_{\mathbf{A}} = 0, \quad (\text{A.12})$$

$$T_{\mathbf{A}} = T_{\mathbf{B}} \quad \text{iff} \quad A_r = B_r \text{ mod } N. \quad (\text{A.13})$$

According to (A.13) we can restrict the vectors \mathbf{A} to some fundamental lattice, for instance the sub-lattice defined by $A_1, A_2 = 0, \dots, N-1$, with the exception of the origin $A_1 = A_2 = 0$. Hence we conclude that there are precisely $N^2 - 1$ traceless independent matrices $T_{\mathbf{A}}$, which we will use as the generators of $SU(N)$. Furthermore, if we also include the identity matrix, $T_0 = N\mathbf{1}$, we have a complete

set of $N \times N$ matrices, which close under multiplication. One easily derives

$$T_{\mathbf{A}} T_{\mathbf{B}} = N z^{-(1/2)(\mathbf{A} \times \mathbf{B})} T_{\mathbf{A} + \mathbf{B}}, \quad (\text{A.14})$$

$$\text{Tr}(T_{\mathbf{A}} T_{\mathbf{B}}) = N^3 \delta_{\mathbf{A} + \mathbf{B}}, \quad (\text{A.15})$$

where $\delta_{\mathbf{A} + \mathbf{B}}$ equals 1 iff $A_r = B_r \pmod N$, and zero otherwise. From (A.15) we define an $SU(N)$ -invariant tensor $\tilde{\eta}_{\mathbf{AB}}$,

$$\tilde{\eta}_{\mathbf{AB}} \equiv \frac{1}{N^3} \text{Tr}(T_{\mathbf{A}} T_{\mathbf{B}}) = \delta_{\mathbf{A} + \mathbf{B}}, \quad (\text{A.16})$$

which coincides with (A.2). The coefficients appearing in the product rule (A.14) decompose into two $SU(N)$ -invariant tensors. One corresponds to the $SU(N)$ structure constants

$$\tilde{f}_{\mathbf{ABC}} \equiv -\frac{i}{N^3} \text{Tr}([T_{\mathbf{A}}, T_{\mathbf{B}}] T_{\mathbf{C}}) = 2N \sin \frac{2\pi(\mathbf{A} \times \mathbf{B})}{N} \delta_{\mathbf{A} + \mathbf{B} + \mathbf{C}}, \quad (\text{A.17})$$

the other to the symmetric invariant tensor

$$\tilde{d}_{\mathbf{ABC}} \equiv \frac{1}{N^4} \text{Tr}(\{T_{\mathbf{A}}, T_{\mathbf{B}}\} T_{\mathbf{C}}) = 2 \cos \frac{2\pi(\mathbf{A} \times \mathbf{B})}{N} \delta_{\mathbf{A} + \mathbf{B} + \mathbf{C}}. \quad (\text{A.18})$$

In (A.18) we have extracted an extra factor N in order to ensure that $\tilde{d}_{\mathbf{ABC}}$ remains finite in the limit $N \rightarrow \infty$. At this point, the advantage of choosing the phase factor as in (A.10) becomes clear: the tensors $\tilde{f}_{\mathbf{ABC}}$ and $\tilde{d}_{\mathbf{ABC}}$ now satisfy the proper periodicity requirements for shifts by N unit lattice vectors. In the large- N limit it is obvious that

$$\tilde{f}_{\mathbf{ABC}} = -\frac{1}{\pi} f_{\mathbf{ABC}} + O\left(\frac{1}{N^2}\right), \quad (\text{A.19})$$

$$\tilde{d}_{\mathbf{ABC}} = 2d_{\mathbf{ABC}} + O\left(\frac{1}{N^2}\right). \quad (\text{A.20})$$

Thus, the torus tensors $f_{\mathbf{ABC}}$ and $d_{\mathbf{ABC}}$ can be approximated by the $SU(N)$ tensors $\tilde{f}_{\mathbf{ABC}}$ and $\tilde{d}_{\mathbf{ABC}}$, and moreover the large- N corrections can be determined explicitly. As an example, let us look at (2.39) in the finite- N approximation. Substituting (A.17) and (A.18), we find

$$\tilde{d}_{\mathbf{ABC}} \tilde{f}_{[\mathbf{DE}}^{\mathbf{A}} \tilde{f}_{\mathbf{F}]\mathbf{G}}^{\mathbf{B}} = O\left(\frac{1}{N^2}\right) \delta_{\mathbf{C} + \mathbf{D} + \mathbf{E} + \mathbf{F} + \mathbf{G}}, \quad (\text{A.21})$$

so that (2.39) is only valid in the limit $N \rightarrow \infty$.

As already pointed out in Sect. 2, the definition of $c_{\mathbf{ABC}}$ for finite N is ambiguous. We may, for instance, adopt the following definition,

$$\tilde{c}_{\mathbf{ABC}} = \frac{2\pi}{N} \sin \frac{2\pi \mathbf{A} \cdot \mathbf{C}}{N} \left[2 - \cos \frac{2\pi A_1}{N} - \cos \frac{2\pi A_2}{N} \right]^{-1} \delta_{\mathbf{A} + \mathbf{B} + \mathbf{C}}, \quad (\text{A.22})$$

which is periodic and tends to (A.5) in the large- N limit. Another question is

whether, for finite N , one can also incorporate the structure constants $f_{\lambda\mathbf{A}\mathbf{B}}$ associated with the vectors (A.6) by an extension of the $SU(N)$ algebra. However, imposing the Jacobi identity (2.46) for the case of $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$, we find that

$$\tilde{f}_{\lambda\mathbf{A}(-\mathbf{A})} + \tilde{f}_{\lambda\mathbf{B}(-\mathbf{B})} + \tilde{f}_{\lambda(-\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B})} = 0. \quad (\text{A.23})$$

Furthermore $\tilde{f}_{\lambda\mathbf{A}\mathbf{B}}$ must be antisymmetric under the interchange of \mathbf{A} and \mathbf{B} . Under these conditions we can show that there is no periodic solution to (A.23) so that no consistent extension of the $SU(N)$ structure constants exists that includes $\tilde{f}_{\lambda\mathbf{A}\mathbf{B}}$. Nonetheless, at the level of the group, it is possible to include the corresponding finite transformations. For the infinite lattice, the transformations generated by (A.6) are simply the shifts $\sigma^r \rightarrow \sigma^r + c^r$, where c^r is a constant vector. On the basis functions (A.1) this induces the transformation

$$Y_{\mathbf{A}}(\sigma) \rightarrow \exp(i\mathbf{c} \cdot \mathbf{A}) Y_{\mathbf{A}}(\sigma). \quad (\text{A.24})$$

For the finite lattice, the phase factors for finite shifts by $(4i\pi/N)\mathbf{e}$, where \mathbf{e} is a unit lattice vector, are obtained by conjugation with Ω_1 and Ω_2 ,

$$\begin{aligned} \Omega_1 T_{\mathbf{A}} \Omega_1^{-1} &= \exp\left(\frac{4\pi i A_1}{N}\right) T_{\mathbf{A}}, \\ \Omega_2^{-1} T_{\mathbf{A}} \Omega_2 &= \exp\left(\frac{4\pi i A_2}{N}\right) T_{\mathbf{A}}. \end{aligned} \quad (\text{A.25})$$

Appendix B: Area-Preserving Diffeomorphisms on S^2

Several discussions of area-preserving diffeomorphisms on S^2 have appeared in the literature [3, 7, 9, 11], which mainly deal with the structure constants and their $SU(N)$ approximation. In this appendix we present a uniform treatment of all the tensors that were introduced in Sect. 2. To facilitate their derivation we make use of a two-component ($SU(2)$) notation for the spherical harmonics.

In terms of the standard angular coordinates on S^2 , $\sigma_1 = \phi \in [0, 2\pi)$ and $\sigma_2 = \theta \in [0, \pi)$, the induced metric for a two-dimensional sphere of unit area reads

$$w_{rs}(\phi, \theta) = \frac{1}{4\pi} \begin{pmatrix} \sin^2 \theta & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{B.1})$$

The basis functions $Y_{\mathbf{A}}$ are now the spherical harmonics, which can be represented as symmetric traceless polynomials of a three-dimensional vector $\vec{X}(\phi, \theta)$ of unit length. It is convenient to adopt a two-component notation, where three-vectors are written as

$$X^{\alpha\beta} = X^{\beta\alpha} \equiv X^i (\sigma_2 \sigma_i)^{\alpha\beta}, \quad (\text{B.2})$$

with $\alpha, \beta, \dots = 1, 2$ and σ_i the Pauli spin matrices. For a real vector \vec{X} we have a corresponding reality condition

$$X_{\alpha\beta} \equiv (X^{\alpha\beta})^* = \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} X^{\gamma\delta}. \quad (\text{B.3})$$

Consider now the set of functions $Y^{\alpha(2n)}(\phi, \theta)$, where $\alpha(2n)$ denotes an array of $2n$

symmetrized indices $\alpha_1, \alpha_2, \dots, \alpha_{2n}$, which are defined as the symmetrized product of n coordinates $X^{\alpha\alpha'}(\phi, \theta)$,

$$Y^{\alpha(2n)} \equiv X^{(\alpha_1\alpha_2} X^{\alpha_3\alpha_4} \dots X^{\alpha_{2n-1}\alpha_{2n})}. \quad (\text{B.4})$$

For given n we have thus $2n + 1$ independent polynomials, which transform according to an irreducible representation of $SO(3)$. Observe that the $Y^{\alpha(2n)}$ are *not* normalized,

$$\int d\phi d\theta \sqrt{w(\phi, \theta)} Y^{\alpha(2m)} Y^{\beta(2n)} = \frac{(m!2^m)^2}{(2m+1)!} \underbrace{\delta^{mn} \varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} \dots \varepsilon^{\alpha\beta}}_{2m}, \quad (\text{B.5})$$

where symmetrization (with unit strength) over the indices in each array $\alpha(2n)$, $\beta(2n)$, etc., is understood henceforth. Furthermore, we have

$$D^r D_r Y^{\alpha(2n)} = -\vec{L}^2 Y^{\alpha(2n)} = -n(n+1) Y^{\alpha(2n)}, \quad (\text{B.6})$$

where \vec{L} are the angular momentum operators, $L_i = -i\varepsilon_{ijk} X^j \partial / \partial X^k$, or, in two-component notation,

$$L^{\alpha\beta} = 2i\varepsilon^{\gamma(\alpha} X^{\beta)\delta} \frac{\partial}{\partial X^{\gamma\delta}}. \quad (\text{B.7})$$

Here $\partial / \partial X^{\alpha\beta} \equiv \frac{1}{2}(\sigma^i \sigma_2)^{\alpha\beta} \partial / \partial X^i$, so that

$$\frac{\partial}{\partial X^{\alpha\beta}} X^{\gamma\delta} = \delta_{(\alpha}^{\gamma} \delta_{\beta)}^{\delta}, \quad (\text{B.8})$$

$$[L^{\alpha\beta}, L^{\gamma\delta}] = -2i\varepsilon^{(\alpha(\gamma} L^{\delta)\beta)}. \quad (\text{B.9})$$

Any product of two functions $Y^{\alpha(2m)}$ and $Y^{\beta(2n)}$ can be decomposed into the same basis functions,

$$Y^{\alpha(2m)} Y^{\beta(2n)} = \sum_k D_k^{mn} \underbrace{\varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} \dots \varepsilon^{\alpha\beta}}_{m+n-k} Y^{\alpha(m-n+k)\beta(n-m+k)}, \quad (\text{B.10})$$

where $k = m + n, m + n - 2, \dots, |m - n|$. The real tensor D_k^{mn} is directly related to the tensor d_{ABC} , after the appropriate normalization factors for the basis functions Y_A are taken into account, and satisfies the following relations,

$$D_k^{mn} = D_k^{nm}, \quad D_{m+n}^{mn} = 1, \quad D_{m-1}^{m1} = \frac{2m}{2m+1}, \quad (\text{B.11})$$

$$\frac{m+n-k+2}{m+n+k+1} D_{k-2}^{mn} = \frac{k^2 - (m-n)^2}{4k^2 - 1} D_k^{mn}, \quad (\text{B.12})$$

$$\frac{n-m+k}{2n} D_k^{mn} = \frac{2k+1}{m+n+k+1} D_{k-1}^{mn-1}. \quad (\text{B.13})$$

The above equations are sufficient to determine D_k^{mn} . While (B.11) is still rather obvious, the proof of (B.12–13) is more involved and will be discussed later. After multiplication of (B.10) with another spherical harmonic and integration over S^2 ,

one readily deduces that $((k!2^k)^2/(2k+1!)D_k^{mn})$ must be symmetric in m , n and k .

For the evaluation of the tensors f_{ABC} and c_{ABC} we first derive the following expressions for arbitrary functions A and B of $X^{\alpha\beta}(\phi, \theta)$,

$$\{A, B\} = 4\pi i \frac{\partial A}{\partial X^{\alpha\beta}} (L^{\alpha\beta} B) = -8\pi X^{\alpha\beta} \varepsilon^{\gamma\delta} \frac{\partial A}{\partial X^{\alpha\gamma}} \frac{\partial B}{\partial X^{\beta\delta}}, \quad (\text{B.14})$$

$$w^{rs} D_r A D_s B = -\frac{1}{2} (L^{\alpha\beta} A) (L_{\alpha\beta} B) = (X^{\alpha\beta} X^{\gamma\delta} - 2X^{\alpha\gamma} X^{\beta\delta}) \frac{\partial A}{\partial X^{\alpha\beta}} \frac{\partial B}{\partial X^{\gamma\delta}}. \quad (\text{B.15})$$

Choosing $A = Y^{\alpha(2m)}$ and $B = Y^{\beta(2n)}$ and decomposing into the same set of basis functions leads to two real tensors, F_k^{mn} and C_k^{mn} , which are related to the tensors f_{ABC} and c_{ABC} after proper normalization,

$$\{Y^{\alpha(2m)}, Y^{\beta(2n)}\} = \sum_k F_k^{mn} \underbrace{\varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} \dots \varepsilon^{\alpha\beta}}_{m+n-k} Y^{\alpha(m-n+k)\beta(n-m+k)}, \quad (\text{B.16})$$

$$\frac{-2}{m(m+1)} D^r Y^{\alpha(2m)} D_r Y^{\beta(2n)} = \sum_k C_k^{mn} \underbrace{\varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} \dots \varepsilon^{\alpha\beta}}_{m+n-k} Y^{\alpha(m-n+k)\beta(n-m+k)}, \quad (\text{B.17})$$

where

$$F_k^{mn} = -8\pi \frac{mn(2k+1)}{m+n+k} D_{k-1}^{m-1} n^{-1}, \quad (\text{B.18})$$

$$C_k^{mn} = \frac{k(k+1) - m(m+1) - n(n+1)}{m(m+1)} D_k^{mn}. \quad (\text{B.19})$$

In deriving (B.18) and (B.19) we made use of (B.10) and (B.13). Note that F_k^{mn} and C_k^{mn} are zero whenever $m+n-k$ is even or odd, respectively. Obviously the Jacobi identity (2.7) for the Lie brackets gives rise to a corresponding identity for the tensor F_k^{mn} .

The structure constants can be approximated by those of $SU(N)$. To show this one introduces the matrices [3]

$$T^{\alpha(2n)} \equiv \left(\frac{4}{N^2 - 1} \right)^{(1/2)(n-1)} L^{(\alpha_1 \alpha_2} L^{\alpha_3 \alpha_4} \dots L^{\alpha_{2n-1} \alpha_{2n}}), \quad (\text{B.20})$$

where the $L^{\alpha\beta}$ are the generators of $SO(3)$ in the N -dimensional representation, so that $L^{\alpha\beta} L_{\alpha\beta} = \frac{1}{2}(N^2 - 1)$. These matrices satisfy the following conditions:

$$T^{\alpha(2n)} = 0 \quad \text{for } n \geq N, \quad (\text{B.21})$$

$$(T^{\alpha(2n)})^\dagger = \varepsilon_{\alpha\beta} \dots \varepsilon_{\alpha\beta} T^{\beta(2n)}, \quad (\text{B.22})$$

$$\text{Tr } T^{\alpha(2n)} = 0 \quad \text{for } n \neq 0. \quad (\text{B.23})$$

As all the $T^{\alpha(2n)}$ are independent, there are $\sum_{n=1}^{N-1} (2n+1) = N^2 - 1$ traceless independent matrices $T^{\alpha(2n)}$, $0 < n < N$, which we may choose as the generators of $SU(N)$. If we also include the identity matrix, $T^0 = \sqrt{((N^2 - 1)/4)} \mathbf{1}$, then these

matrices close under multiplication,

$$T^{\alpha(2m)} T^{\beta(2n)} = \sum_k \left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\} \underbrace{\varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} \dots \varepsilon^{\alpha\beta}}_{m+n-k} T^{\alpha(m-n+k)\beta(n-m+k)}, \quad (\text{B.24})$$

where $k = m + n, m + n - 1, \dots, |m - n|$. We note the following properties of the coefficients $\left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\}$,

$$\left(\left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\} \right)^* = (-)^{m+n-k} \left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\}, \quad (\text{B.25})$$

$$\left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\} = \left\{ \begin{matrix} n & m \\ k & k \end{matrix} \right\}, \quad (\text{B.26})$$

$$\left\{ \begin{matrix} m & n \\ m+n & \end{matrix} \right\} = \sqrt{\frac{N^2 - 1}{4}}, \quad (\text{B.27})$$

$$\left\{ \begin{matrix} m & n \\ m+n-1 & \end{matrix} \right\} = -imn, \quad (\text{B.28})$$

$$\left\{ \begin{matrix} m & 1 \\ m-1 & \end{matrix} \right\} = \frac{2m}{2m+1} \frac{N^2 - m^2}{N^2 - 1} \sqrt{\frac{N^2 - 1}{4}}, \quad (\text{B.29})$$

$$\text{Tr}(T^{\alpha(2m)} T^{\beta(2n)}) = \frac{(m!2^m)^2}{(2m+1)!} \delta^{mn} \underbrace{\varepsilon^{\alpha\beta} \varepsilon^{\alpha\beta} \dots \varepsilon^{\alpha\beta}}_{2m} \left(\frac{N(N^2 - 1)}{4} \prod_{k=1}^m \frac{N^2 - k^2}{N^2 - 1} \right). \quad (\text{B.30})$$

From (B.30) we may define an $SU(N)$ -invariant metric, which, up to normalization factors, yields the tensor η_{AB} for the spherical harmonics in the $N \rightarrow \infty$ limit (cf. (B.5)).

If the matrices $L^{\alpha\beta}$ were commuting then $\left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\}$ would coincide with D_k^{mn} , up to an overall factor $\sqrt{(N^2 - 1)/4}$. The fact that the $L^{\alpha\beta}$ are not commuting gives rise to additional contributions, but those will be suppressed by powers of $\sqrt{4/(N^2 - 1)}$. Using this observation and the intermediate result shown in (B.14), one can establish the following result,

$$\left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\} = \begin{cases} \sqrt{\frac{N^2 - 1}{4}} \left[D_k^{mn} + O\left(\frac{1}{N^2}\right) \right], & \text{for } m + n - k \text{ even} \\ -\frac{i}{8\pi} \left[F_k^{mn} + O\left(\frac{1}{N^2}\right) \right], & \text{for } m + n - k \text{ odd} \end{cases} \quad (\text{B.31})$$

Observe that (B.31) is indeed consistent with various results that have been listed above. As far as the large- N behaviour is concerned, there is a qualitative agreement between (B.31) and the coefficients that arise in the products of the $SU(N)$ generators for the torus (cf. (A.19–20)). The $SU(N)$ approximation to the tensors f_{ABC} and d_{ABC} is therefore given by the coefficients $\left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\}$ and $\sqrt{4/(N^2 - 1)} \left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\}$, multiplied by the appropriate product of ε -symbols and the normalization factors for the spherical harmonics. The $SU(N)$ approximation of the tensor c_{ABC} then follows from (B.19).

The coefficients $\left\{ \begin{matrix} m & n \\ k & k \end{matrix} \right\}$ are subject to the following recurrency relations, which,

when combined with (B.26–28), allow their complete determination,

$$\begin{aligned} & \frac{m+n-k+2}{m+n+k+1} \begin{Bmatrix} m & n \\ k-2 & \end{Bmatrix} - \frac{k^2 - (m-n)^2}{4k^2-1} \frac{N^2 - k^2}{N^2-1} \begin{Bmatrix} m & n \\ k & \end{Bmatrix} \\ &= -\frac{i}{\sqrt{N^2-1}} \frac{k(k-1) - m(m+1) - n(n+1)}{m+n+k+1} \begin{Bmatrix} m & n \\ k-1 & \end{Bmatrix}, \end{aligned} \quad (\text{B.32})$$

$$\begin{aligned} & \frac{n-m+k}{2n} \begin{Bmatrix} m & n \\ k & \end{Bmatrix} - \frac{2k+1}{m+n+k+1} \begin{Bmatrix} m & n-1 \\ k-1 & \end{Bmatrix} \\ &= -\frac{i}{\sqrt{N^2-1}} \frac{k(k+1) - n(n-1) + m(n+k)}{m+n+k+1} \begin{Bmatrix} m & n-1 \\ k & \end{Bmatrix}, \end{aligned} \quad (\text{B.33})$$

$$\begin{aligned} & \frac{m+n-k}{2n} \begin{Bmatrix} m & n \\ k & \end{Bmatrix} - \frac{m-n+k+2}{2k+3} \frac{N^2 - n^2}{N^2-1} \begin{Bmatrix} m & n-1 \\ k+1 & \end{Bmatrix} \\ &= -\frac{i}{\sqrt{N^2-1}} \frac{(m+n+k+2)(m-n-k-1)(n-k-1)}{2n(2k+3)} \begin{Bmatrix} m & n \\ k+1 & \end{Bmatrix}. \end{aligned} \quad (\text{B.34})$$

These relations have been derived by using the associativity of the matrix product, $(T^{\alpha(2m)}T^{\beta(2n)})T^{\gamma(2)} = T^{\alpha(2m)}(T^{\beta(2n)}T^{\gamma(2)})$, and substituting (B.24) and the explicit expressions for $\{m_k^1\}$. The latter have been calculated separately (cf. (B.27–29)). For $m+n-k$ even, (B.32–34) become degenerate in the large- N limit and yield (B.12–13). Obviously, the associativity of the matrix product can be used to derive more identities for products of the $\{m_k^n\}$.

Finally, we come to the relation between the area-preserving diffeomorphisms generated in the $N \rightarrow \infty$ limit of the torus and the sphere. Clearly, for finite N , the generators T_A of the torus algebra can be decomposed in terms of the generators $T^{\alpha(2n)}$ of the sphere algebra. For instance, using the results given above, we can explicitly compare the diagonal generators of both algebras. In a basis where L^{12} (which is equal to iL^3 in Cartesian coordinates) is diagonal, we may choose $\Omega_1 = \exp(4\pi L^{12}/N)$, so that

$$T_{(A,0)} = \sqrt{\frac{4N^2}{N^2-1}} \sum_{n=0}^{N-1} M_{n0} T^{1(n)2(n)}, \quad (\text{B.35})$$

where M_{n0} are the elements of an $N \times N$ matrix M , defined by

$$\begin{aligned} (\ln M)_{n-1n} &= -2\pi A \frac{n^2}{4n^2-1} \frac{N^2 - n^2}{N\sqrt{N^2-1}}, \\ (\ln M)_{nm-1} &= 2\pi A \frac{\sqrt{N^2-1}}{N}, \quad n=0, \dots, N-1, \end{aligned} \quad (\text{B.36})$$

and all remaining elements of $\ln M$ equal to zero. From the left-hand side of (B.35) it is clear that $M_{00} = 0$, unless $A = 0 \pmod N$, although this property is not manifest for the right-hand side of this equation. For $N \rightarrow \infty$, (B.35) defines a map between the torus functions $Y_{(A,0)}$ and functions on the sphere, expressed as a series

expansion in terms of the spherical harmonics $Y^{1(n)2(n)}$. Although the functions corresponding to (B.35) are relatively simple, as they are only functions of one coordinate of the torus and the sphere, respectively, this example demonstrates the complexity of the relation between the two sets of functions. However, in view of the different topological structure of the torus and the sphere, it is unlikely that the torus functions Y_A will always lead to functions that are differentiable on the sphere.

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Note added. Recently, the relation between the algebra of area-preserving diffeomorphisms on S^2 and the algebras used for the description of higher-spin gauge fields [22] received some attention [11, 23].

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