Commun. Math. Phys. 116, 1–22 (1988)

Communications in
Mathematical
Physics

Renormalization Group Approach to Lattice Gauge Field Theories
II. Cluster Expansions*

Tadeusz Balaban
Department of Mathematics, Boston University, Boston, MA 02215, USA

Abstract. The fluctuation field integral, constructed in Part I, is represented by the exponentiated cluster expansion. It is proved that the terms of the expansion satisfy the inductive assumptions. This completes the construction of the sequence of effective actions in the small field approximation.

Introduction

In the first paper of this series we have considered the fluctuation field integral defined by the k^{th} renormalization transformation. We have shown there that the fluctuation field effective action is a small perturbation of the basic quadratic form, in the small field approximation. This is the main part of the analysis of this integral, and it includes the analysis of renormalization. Now it remains to construct a localized representation for the new term in the effective action defined by the fluctuation field integral. This is done by an application of the exponentiated cluster expansion. This expansion is constructed in two steps.

At first we localize the fluctuation field effective action by a procedure similar to a cluster expansion, using the generalized random walk expansions for propagators and minimizers occurring in the action. This procedure is described in Sect. 1. It yields the integral in the form to which we can apply in a straightforward way the exponentiated cluster expansion. This expansion is described in Sect. 2. It yields the desired localized expansion of the new effective action. We prove also that terms of this expansion satisfy the inductive assumptions formulated in the first paper. Thus we complete the proof of Theorem 3 of that paper. The two expansions constructed here are quite general, as will become clear from their descriptions. They can be, and will be, applied in many other situations, such as for the expressions constructed with the help of more general propagators described in [13], or for integrals conditioned to subdomains of the lattice.

* Work supported in part by the Air Force under Grant AFOSR-86-0229 and by the National Science Foundation under Grant DMS-86-02207

The first paper is referred to as *I*, so we refer to its sections and formulas by writing I before the corresponding numbers, e.g. Sect. I.3, or the formula (I.1.18). We also refer the reader to the list of references at the end of that paper.

1. Localizations and Bounds of Terms in the Fluctuation Field Action

In Sects. I.3-5 we have constructed the expansions of terms in the original fluctuation field effective action in (I.2.12). The newly created terms, after the cancellations described at the end of Sect. I.4, satisfy much better bounds of the type (I.0.29), bounds assuring convergence of the sum of all these terms. They are analytic and nonlocal functions of the configurations $\mathbf{U}, \mathbf{J}, B$. The nonlocality is, however, of a simple origin, it is introduced by the functions $\mathbf{H}_j, \mathbf{H}_k$, propagators and other expressions appearing after the expansions. Analyzing the terms (I.3.7), (I.3.21) or (I.3.34), we see that they have the following structure: there exists a function $\mathbf{E}(X, \mathbf{U}, \mathbf{J}, \mathbf{A})$ analytic and localized to X in $\mathbf{U}, \mathbf{J}, \mathbf{A}$, such that a given term is obtained substituting a proper nonlocal expression in the place of $\mathbf{A}$, e.g., the function $\mathbf{H}_j(B(t)) + t_\square \delta \mathbf{H}_j$ in (I.3.7), $(t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(B')$ in (I.3.34), and so on. Instead of trying to cover all possible cases, let us consider one important and typical term, and then discuss necessary changes for other terms. For example let us consider the last term on the right-hand side of (I.3.34). We have proved that it is an analytic function of $\mathbf{U}, \mathbf{J}, \mathbf{A}$, for $(\mathbf{U}, \mathbf{J}) \in U_{k+1}^c(\square_0, (1+2\beta)\alpha_0, (1+2\beta)\alpha_1, \alpha_0)$ and $\mathbf{A}$ satisfying (I.3.31). Substituting

$$\mathbf{A} = (t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(B') \tag{1.1}$$

we introduce, through the function $\mathbf{H}_k(B')$, a dependence on $\mathbf{U}, \mathbf{J}, B$ on the whole lattice. Our problem is to localize the obtained function, more precisely to represent it as a sum of terms, which are localized in domains from $\mathbf{D}_k$, and which satisfy bounds of the type (I.1.18).

We construct such an expansion by a procedure similar to decoupling procedures in cluster expansions. To this purpose we use extensively the generalized random walk expansions for propagators constructed in [13]. The function $\mathbf{H}_k(B')$ is determined by the propagators through the equations it satisfies. By the results of Sect. G [15] this function is given by

$$\mathbf{H}_k(B') = H_0 B' + \mathbf{A}_0 - HD(H_0 B' + \mathbf{A}_0), \tag{1.2}$$

where the linearizing transformation $D(A')$ is a solution of the equation

$$D(A') = C(A' - HD(A')), \tag{1.3}$$

and $\mathbf{A}_0$ is a solution of the equation

$$\mathbf{A}_0 + \tilde{G}\left(\frac{\delta}{\delta A'} V\right)(H_0 B' + \mathbf{A}_0) = 0. \tag{1.4}$$

We have suppressed the subscript k in the equations, and also the dependence on the variables $\mathbf{U}, \mathbf{J}$. The function $V(A')$ is a nonlinear and nonlocal function depending on H and $D(A')$. It is given by the formula (80) [15]. Thus Eqs. (1.3), (1.4) are determined by local, analytic functions, and nonlocal propagators $H, H_0, \tilde{G}$.

In these equations we can replace the propagators by arbitrary operators having the same regularity properties and satisfying the same bounds. Only these bounds were important in the analysis of Sect. C, E [15], hence the solutions $D, \mathbf{A}_0$ can be considered as functionals of these operators, having all the properties proved in [15]

$$D(A')=D(H,A'), \quad \mathbf{A}_0=\mathbf{A}_0(H,\tilde{G},H_0B'). \tag{1.5}$$

The nonlocality of these solutions is a result of a nonlocality of the operators, hence we localize the solutions by localizations of the operators. We start with the original propagators, and we decompose them into generalized random walk expansions. A propagator is represented by the sum (3.107) [13]

$$\sum_{\omega} R_0(X_0)R_{\alpha_1}(X_1)\cdots R_{\alpha_n}(X_n), \tag{1.6}$$

where $\omega=((0,X_0)(\alpha_1,X_1),\ldots,(\alpha_n,X_n)), X_0, X_1,\ldots,X_n$ are simple localization domains, unions of connected families containing several cubes from $\pi_k(1)$, i.e. of the size M_1 instead of M. For more detailed explanations of (1.6) see Sect. C [13]. Localization properties and bounds are important for us. Each factor in (6.6) depends on external gauge field configurations restricted to the corresponding domain $\tilde{X}_j^5$, and its kernel vanishes in a sufficiently thick neighborhood of ∂X_j, e.g. in a neighborhood of the width $\frac{1}{3}M_1$. The term corresponding to a walk ω in (1.6) can be bounded, as (3.108) [13], by

$$O(1)O(M_1^{-1/2})^{|\omega|}M_1^{-1/2|\omega|}\exp(-\delta_0 d(\omega,y,y')), \tag{1.7}$$

where $\Delta(y), \Delta(y')$ are localization cubes of the term. This bound holds for all operator norms in formulations of theorems in [13], e.g. in Theorem 3.1. The first two factors in (1.7) are used to control the sum over ω, and they determine the constant B_0. We will use the remaining two factors to produce exponential bounds for localized terms.

To construct the decoupling we introduce a regular partition σ_k of the space T, in the scale corresponding to the lattice T_η, into cubes Δ of the size R_1M_1. The number R_1 is a power of L satisfying the condition $R_1M_1M^{-1}\leqq 1$. For the particular example under consideration we take cubes of the size M, which are disjoint with the interior of $\tilde{\square}^4$. We denote this family by σ_0. To each cube Δ in this family we assign a variable $s(\Delta)$, and we denote by s the system of all these variables. We introduce a dependence on s in propagators in the following way: for a random walk ω localized in $\tilde{X}_0^5\cup\tilde{X}_1^5\cup\cdots\cup\tilde{X}_n^5$ we take the $\{\Delta_1,\ldots,\Delta_m\}$ of all cubes from σ_0 which intersect this localization domain, and we multiply the term in (1.6) corresponding to ω by $s(\Delta_1)\cdots s(\Delta_m)$. This way the s-dependent propagators $H(s), \tilde{G}(s), H_0(s)$ are defined. They coincide with the original ones for $s=1$. The symbol B' in (1.1) denotes the local function of B defined by (I.3.2).

Now let us consider the last term in (I.3.34), or an arbitrary expression obtained by the transformations of one of the terms on the right-hand side of (I.3.34). We denote the corresponding function of $\mathbf{U},\mathbf{A}$ by $\mathbf{E}(\square_0,\mathbf{U},\mathbf{A})$, hence making the substitution (1.1) we have

$$\begin{aligned}&\mathbf{E}(\square_0,\mathbf{U},(t\tilde{\zeta}_\square+t_\square\zeta_\square)\mathbf{H}_k(\mathbf{U},\mathbf{J},B'))\\&\quad=\mathbf{E}(\square_0,\mathbf{U},(t\tilde{\zeta}_\square+t_\square\zeta_\square)\mathbf{H}_k(\mathbf{U},\mathbf{J},H(s),\tilde{G}(s),H_0(s)B'))|_{s=1}.\end{aligned} \tag{1.8}$$

To simplify the formulas let us omit the symbols $\mathbf{U}, \mathbf{J}$. Now, as in cluster expansions, we apply the fundamental theorem of calculus

$$\mathbf{E}(\square_0, (t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(B'))$$
$$= \sum_{\sigma \subset \sigma_0} \prod_{\Delta \in \sigma} \int_0^1 ds(\Delta) \frac{\partial}{\partial s(\Delta)} \mathbf{E}(\square_0, (t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(H(s), \tilde{G}(s), H_0(s)B'))|_{s(\sigma^c)=0}. \quad (1.9)$$

Consider a term in the last sum. The set $Y(\sigma) = \tilde{\square}^4 \cup (\cup_{\Delta \in \sigma} \Delta)$ is a sum of connected components, with the following notion of connectedness. We consider domains Y which are unions of subfamilies of σ_k, and which satisfy one of the following two conditions: either Y is disjoint with the cube $\tilde{\square}^4$, or it contains this cube. In the first case Y is connected if and only if for any two cubes $\Delta, \Delta' \subset Y$ there exists a sequence $\Delta, \Delta_1, \ldots, \Delta_n, \Delta'$ of cubes contained in Y and such, that boundaries of two successive terms in the sequence have a common $d-1$-dimensional wall, and there is no cube in Y having this property with respect to $\tilde{\square}^4$. In the second case Y is connected if and only if for any cube $\Delta \subset Y$ there exists a sequence having the above property and connecting Δ with $\tilde{\square}^4$, i.e. the last term in the sequence intersects $\tilde{\square}^4$ along a $d-1$-dimensional wall. We denote by Y_0 the connected component of the domain $Y(\sigma)$ containing the cube $\tilde{\square}^4$.

Consider the function $(t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(H(s), \tilde{G}(s), H_0(s)B')$ for s satisfying $s(\sigma^c) = 0$. We prove that it depends on B, s is restricted to the component Y_0. At first we remark that kernels of the operators $H(s), \tilde{G}(s), H_0(s)$ vanish, unless both arguments are in the interior of one component of $Y(\sigma)$, in fact with distances to the boundary of this component bigger than M_1. Consider Eq. (1.3) with H replaced by $H(s)$. By the above remark this equation reduces to the equality $D(A') = C(A')$ on the neighborhood of $Y(\sigma)^c$, and to separate equations in components of $Y(\sigma)$. The solution $D(H(s), A')$ restricted to the interior of a component depends on $A', H(s)$, hence s, restricted to this component. This implies that the function $(\delta/\delta A')V(H(s), A')$ has similar properties, i.e. it coincides with the local function $(\delta/\delta A')V_0(A')$ on the neighborhood of $Y(\sigma)^c$, and considered on a component of $Y(\sigma)$ it depends on $A', H(s), s$ restricted to this component. Thus Eq. (1.4) represents a system of separated equations in components of $Y(\sigma)$. The solution $\mathbf{A}_0(H(s), \tilde{G}(s), H_0(s)B')$ is equal to 0 on the neighborhood of $Y(\sigma)^c$, and again, on a component of $Y(\sigma)$ it depends on the propagators and B' restricted to the component. Finally, the function B' defined by (I.3.2) is a local function of B. As a consequence of these properties we obtain that the term in the sum on the right-hand side of (1.9), corresponding to the set σ, depends on propagators and function B, s restricted to the component Y_0, the component of $Y(\sigma)$ containing $\tilde{\square}^4$. If there are other components, then the derivatives with respect to s restricted to these components render the term equal to 0. This simplifies the sum in (1.9). We can write it as a sum over connected domains Y_0 containing $\tilde{\square}^4$, with parameters $s = 0$ on Y_0^c. We denote s by $s(Y_0)$, hence we have

$$\mathbf{E}(\square_0, (t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(B'))$$
$$= \sum_{Y_0} \prod_{\Delta \subset Y_0 \setminus \tilde{\square}^4} \int_0^1 ds(\Delta) \frac{\partial}{\partial s(\Delta)} \mathbf{E}(\square_0, (t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(H(s(Y_0)), \tilde{G}(s(Y_0)), (H_0 B')(s(Y_0)))). \quad (1.10)$$

In the term corresponding to Y_0 in the above sum all propagators and the function B are restricted to Y_0. This implies that this term depends on $\mathbf{U}, \mathbf{J}$ restricted to $Y_0 \cup \square_0$.

To estimate a term in the sum (1.10) we use the Cauchy formula, hence we have to investigate analyticity properties of the functions $\mathbf{E}(\cdots)$ with respect to the variables $s(Y_0)$. We review successively the functions and the operators in (1.10). Let us start with a bound for the function $H(s(Y_0))X, X$ is an arbitrary $\mathbf{g}^c$-valued function on $Y_0^{(k)}$, the parameters $s(Y_0)$ are complex valued and satisfy the bound $|s(\Delta)| \leqq e^{\kappa_1}$ for $\Delta \subset Y_0 \setminus \tilde{\square}^4$, κ_1 is a sufficiently big positive number. Using the bound (1.7) we estimate the function by

$$B_0 |X| \sup_{\omega} M_1^{-1/2|\omega|} \exp(-\delta_0 d(\omega)) e^{m\kappa_1}, \tag{1.11}$$

where $d(\omega)$ is a length of a shortest tree graph intersecting all localization domains of ω, m is the number of the parameters s connected with the walk ω. If m is big enough, for example $m > 2^4$, then $\delta_0 d(\omega) \geqq \delta_1 m M$, for a positive constant δ_1 depending on δ_0 only, and we can bound the expression under the supremum in (1.11) by 1, for $\delta_1 M \geqq \kappa_1$. If $m \leqq 2^4$, then we can have short walks, in fact with $|\omega| = 0$, and the expression can be bounded by $e^{16\kappa_1}$ only. Thus the function $H(s(Y_0))X$ is an analytic function of the variables $s(Y_0)$ on the domain $|s(Y_0)| \leqq e^{\kappa_1}$, bounded by $B_0 e^{16\kappa_1} |X|$ in all norms of Theorems 3.1.-3.10 [13].

Consider the change of variables

$$A = A' - H(s(Y_0)) D(H(s(Y_0)), A'). \tag{1.12}$$

The function D is the solution of Eq. (1.3) with $H(s(Y_0))$ instead of H. Let us introduce an auxiliary constant ε_2, and consider this equation on the space of functions satisfying $|A'| < \varepsilon_2$. Let us recall the most important points from the discussion of Sect. C [15]. If we replace the function D in (1.3) by a configuration X, then for the complex $s(Y_0)$,

$$|C(A' - H(s(Y_0))X)| \leqq 2C_2 \varepsilon_2^2 + 2C_2 B_0^2 e^{32\kappa_1} |X|^2, \tag{1.13}$$

and for X satisfying $|X| < 4C_2\varepsilon_2^2 \leqq (4C_2 B_0^2 e^{32\kappa_1})^{-1}$, hence for ε_2 satisfying $4C_2 B_0 e^{16\kappa_1} \varepsilon_2 \leqq 1$, the right-hand side above is bounded by $4C_2\varepsilon_2^2$. Thus the transformation defined by the function on the left-hand side of (1.13) maps the domain $\{X : |X| < 4C_2\varepsilon_2^2\}$ into itself. We prove similarly that it is contractive on this domain, hence the fixed point is an analytic function of $A', s(Y_0)$, bounded by $4C_2\varepsilon_2^2$. Because ε_2 can be chosen arbitrarily close to $|A'|$, so we have the inequality

$$|D(H(s(Y_0)), A')| \leqq 4C_2 |A'|^2. \tag{1.14}$$

It is the same as the inequality (55) [15]. Similarly the other inequalities and statements of Sect. C [15] hold in this case, for example the transformation (1.12) can be bounded as in (57) [15]: $|A| < \varepsilon_2 + B_0 e^{16\kappa_1} 4C_2 \varepsilon_2^2 \leqq 2\varepsilon_2$. This implies that the function $V(H(s(Y_0)), A')$, defined by (80) [15] with the help of the transformation (1.12), is an analytic function of $A', s(Y_0)$, and Proposition 4 [15] holds for it (with ε_3 replaced by ε_2). Consider now Eq. (1.4) with the above function V, and with $\tilde{G}, H_0$ replaced by $\tilde{G}(s(Y_0)), H_0(s(Y_0))$. The last propagators are analytic functions of $s(Y_0)$, and they have the same bound (1.11) as $H(s(Y_0))$, hence their norms are bounded by $B_0 e^{16\kappa_1}$. Using this, and the inequality (98) in Proposition 4 [15], we obtain

$$\begin{aligned}
&|\tilde{G}(s(Y_0))\left(\frac{\delta}{\delta A'} V\right)(H(s(Y_0)), H_0(s(Y_0))B' + \mathbf{A})| \\
&\leqq 2C_4(B_0 e^{16\kappa_1})^3 |B'|^2 + 2C_4 B_0 e^{16\kappa_1}(\max\{|\mathbf{A}|, |\nabla \mathbf{A}|\})^2 \\
&< 2C_4(B_0 e^{16\kappa_1})^3 \varepsilon_3^2 + \tfrac{1}{2}\max\{|\mathbf{A}|, |\nabla \mathbf{A}|\} < 4C_4(B_0 e^{16\kappa_1})^3 \varepsilon_3^2, \qquad (1.15)
\end{aligned}$$

if

$$\max\{|\mathbf{A}|, \nabla \mathbf{A}|\} < 4C_4(B_0 e^{16\kappa_1})^3 \varepsilon_3^2 \leqq (4C_4 B_0 e^{16\kappa_1})^{-1},$$

and if $|B'| < \varepsilon_3$, where ε_3 is another auxiliary constant. The last inequality holds if $4C_4 B_0^2 e^{32\kappa_1} \varepsilon_3 \leqq 1$. The norm of the function $H_0(s(Y_0))B' + \mathbf{A}$ is bounded by

$$B_0 e^{16\kappa_1} \varepsilon_3 + 4C_4(B_0 e^{16\kappa_1})^3 \varepsilon_3^2 \leqq 2B_0 e^{16\kappa_1} \varepsilon_3,$$

and we assume that $2B_0 e^{16\kappa_1} \varepsilon_3 \leqq \varepsilon_2$, so the function satisfies the assumption of Proposition 4 in this case. The inequality (1.15) holds for all admissible norms. It implies that the fixed point, i.e. the solution of Eq. (1.4), is an analytic function of $B', s(Y_0)$ on the considered domains, it satisfies the bound

$$\begin{aligned}
|\mathbf{A}_0(s(Y_0), H_0(s(Y_0))B')| &\leqq 4C_4(B_0 e^{16\kappa_1})^3 |B'|^2 \\
&< 4C_4(B_0 e^{16\kappa_1})^3 \varepsilon_3^2 \leqq B_0 e^{16\kappa_1} \varepsilon_3 \leqq \tfrac{1}{2}\varepsilon_2, \qquad (1.16)
\end{aligned}$$

and the same bound for the other norms. This finally implies the desired result for the function

$$\begin{aligned}
\mathbf{H}_k(s(Y_0), B') = {} & H_0(s(Y_0))B' + \mathbf{A}_0(s(Y_0), H_0(s(Y_0))B') \\
& - H(s(Y_0))D(H(s(Y_0)), H_0(s(Y_0))' + \mathbf{A}_0(s(Y_0), H_0(s(Y_0))B')). \qquad (1.17)
\end{aligned}$$

It is an analytic function of $s(Y_0), B'$, satisfying the bound

$$|\mathbf{H}_k(s(Y_0), B')| \leqq 4B_0 e^{16\kappa_1} |B'| < 4B_0 e^{16\kappa_1} \varepsilon_3 \leqq 2\varepsilon_2, \qquad (1.18)$$

and the same bound for all admissible norms. Let us recall that all the considered functions and operators are analytic functions of the configurations $\mathbf{U}, \mathbf{J}$ in a domain given by the conditions I.(i)-(iii) with some α_0', α_1', and all the bounds above are uniform on the domain.

We have to consider the function B' yet. Notice that in the preceding considerations B' was a variable field. The function is given by the formula

$$B' = g_k CB - h\tilde{D}(g_k CB). \qquad (1.19)$$

It satisfies the bound

$$|B'| \leqq O(1) g_k |B| + 4C_2(O(1) g_k |B|)^2 \leqq C_1 g_k |B| < C_1 \varepsilon_1, \qquad (1.20)$$

where C_1 is an absolute constant, and $g_k |B| < \varepsilon_1$.

Gathering together the above statements and estimates we obtain that $\mathbf{H}_k(s(Y_0), B'))$ is an analytic function of $s(Y_0), B$, for $|s(Y_0)| \leqq e^{\kappa_1}$ and $g_k |B| < \varepsilon_1$. It has the estimate

$$|\mathbf{H}_k(s(Y_0), B')| \leqq 4B_0 C_1 e^{16\kappa_1} g_k |B| < 4B_0 C_1 e^{16\kappa_1} \varepsilon_1, \tag{1.21}$$

and the same for all admissible norms, if $e^{32\kappa_1}\varepsilon_1$ is smaller than an absolute constant. This constant can be easily obtained by inspection of all the above conditions. We have yet another condition for the function $\mathbf{H}_k$ in (1.21), namely this function multiplied by $t\tilde{\zeta}_\square + t_\square \zeta_\square$ has to satisfy the bounds (I.3.31). We assume that $4B_0 C_1 e^{16\kappa_1}\varepsilon_1 \leqq \frac{1}{4}\alpha_2$, and this implies that the product with $t\tilde{\zeta}_\square$ satisfies (I.3.31) with $\frac{1}{2}\alpha_2$. The product with $\zeta_\square$ has the norms in (I.3.31) bounded by $4B_0 C_1 e^{16\kappa_1} g_k |B| < 4B_0 C_1 e^{16\kappa_1}\varepsilon_1$, and we extend the expression in (1.10) analytically with respect to $t_\square$ satisfying $|t_\square| 4B_0 C_1 e^{16\kappa_1} g_k |B| \leqq \frac{1}{2}\alpha_2$. Taking $t_\square$ for which the equality holds, we get

$$\frac{1}{|t_\square|} = 8B_0 C_1 e^{16\kappa_1}\alpha_2^{-1} g_k |B| < 8B_0 C_1 e^{16\kappa_1}\alpha_2^{-1}\varepsilon_1. \tag{1.22}$$

Let us come back to the expansion (1.10). We differentiate it with respect to $t_\square$, at $t_\square = 0$, and we represent all derivatives by the Cauchy formula. The term in (1.10) corresponding to a domain Y_0 is represented as

$$\frac{1}{2\pi i}\int \frac{dt_\square}{t_\square^2} \prod_{\Delta \subset Y_0 \setminus \tilde{\square}^4} \int ds(\Delta) \frac{1}{2\pi i}\int \frac{d\sigma(\Delta)}{(\sigma(\Delta) - s(\Delta))^2} \cdot \mathbf{E}(\square_0, (t\tilde{\zeta}_\square + t_\square \zeta_\square)\mathbf{H}_k(\sigma(Y_0), B')), \tag{1.23}$$

where the $t_\square$-integration is over the circle (1.22), and the $\sigma(\Delta)$-integrations are over the circles $|\sigma(\Delta)| = e^{\kappa_1}$. We were doing all the considerations for the last term in (I.3.34), hence we estimate the above expression using (I.3.54). We get

$$|(1.23)| \leqq 8B_0 C_1 e^{16\kappa_1}\alpha_2^{-1} g_k |B| E_0 \left(\frac{\alpha_1}{\alpha_3}\right)^5 (L^j \eta)^5$$
$$\cdot \exp(-(\kappa_1 - 1)M^{-4}|Y_0 \setminus \tilde{\square}^4|)\exp(-\kappa d_j(X)). \tag{1.24}$$

Adding and subtracting $\frac{1}{8}\kappa_1 d_k(\square_0)$ under the first exponential above, we can bound it by

$$\exp(-\tfrac{1}{8}(\kappa_1 - 1)d_k(Y) + \tfrac{1}{8}\kappa_1 d_k(\square_0) - \tfrac{1}{2}(\kappa_1 - 1)M^{-4}|Y_0 \setminus \tilde{\square}^4|), \tag{1.25}$$

where $Y = Y_0 \cup \square_0$. Of course Y is a localization domain form $\mathbf{D}_k$. The inequality (1.24) supplemented by the above bound is enough to control the sum over all terms (1.23) having a common localization domain Y. Before considering such a sum we have to take a common domain of analyticity for all terms in it. From the results of Sect. I.3 it follows that the term (1.23) is an analytic function of configurations $\mathbf{U}, \mathbf{J}$, defined on the space $\mathbf{U}_{k+1}^{\prime c}(\square_0, (1+2\beta)\alpha_0, (1+2\beta)\alpha_1, \alpha_0)$. By the construction it depends on $\mathbf{U}, \mathbf{J}$ restricted to $Y_0 \cup \square_0 = Y$, and the conditions outside Y are unessential. Thus we consider this space defined by the conditions I.(i)-(iii) on the domain Y, and we take its subspace $\mathbf{U}_{k+1}^{c}(Y, (1+\beta)\alpha_0, (1+\beta)\alpha_1, \alpha_0)$. All terms (1.23) with the localization domain Y are defined and analytic on this space.

Now we consider the sum of all terms (1.23) having the same localization domain Y. It is a sum over all admissible $\square_0, Y_0, j$ and X. The expression defined by this sum can be bounded using (1.24), (1.25). At first we consider the sum over X. We take $X \in \mathbf{D}_j, X \subset \tilde{\square}^2$. This sum can be bounded by two sums, the first is over $\square' \in \pi_j, \square' \subset \tilde{\square}^2$. This sum can be bounded by two sums, the first is over $\square' \in \pi_j$, $\square' \subset \tilde{\square}^2$, the second over $X \in \mathbf{D}_j$ such that $\square' \subset X$. For the second sum we have

$$\sum_{X \in \mathbf{D}_j, X \supset \square'} \exp(-\kappa d_j(X)) \leqq O(1), \tag{1.26}$$

for κ sufficiently large. The number $O(1)$ is in fact small, because we sum over X with $d_j(X) \neq 0$, as it follows from our inductive construction. This inequality was used many times in convergence proofs for cluster expansions, for example see [48, 50, 40, 26, 3]. To bound the first sum, over $\square' \subset \tilde{\square}^2$, we use the factor $(L^j\eta)^5$ in (1.24). This yields $(6L)^4 L^j \eta$, and the sum over j is bounded by $2(6L)^4$. The sum over Y_0 is simply a sum over subsets of the family of cubes Δ contained in $\square_0 \setminus \tilde{\square}^4$. The number of terms in this sum is an absolute number, but we get a better bound using the last term under the exponential in (1.25). The sum is bounded by

$$\exp(8 \cdot 12^3 \exp(-\tfrac{1}{2}(\kappa_1 - 1))) \leqq e \tag{1.27}$$

for κ_1 sufficiently large. Finally, the sum over $\square_0$ can be bounded by

$$M^{-4}|Y| \leqq 3 \cdot 2^3 d_k(Y) \leqq \exp \tfrac{1}{16}(\kappa_1 - 2) d_k(Y). \tag{1.28}$$

Gathering together the above bounds we obtain

$$\left|\sum(1.23)\right| \leqq E_0 \varepsilon_1 O(M^q) \exp O(1)\kappa_1 \exp(-\tfrac{1}{16}\kappa_1 d_k(Y)), \tag{1.29}$$

with absolute constants $O(1), q$. The constant q is a small, nonnegative integer, which can be easily calculated from the conditions on $\alpha_1, \alpha_2, \alpha_3$. Another possibility is to use the expression $g_k|B|$ instead of ε_1. It gives a better bound, but the above is simpler.

These considerations and bounds were done on the example of the last term on the right-hand side of (I.3.34), but almost all of them are quite general and can be applied to all terms in the fluctuation field action. Let us discuss the other terms briefly. The simplest situation is for the remaining terms on the right-hand side of (I.3.34), or rather for expressions obtained by the transformations described in Sect. I.4. We apply the expansion (1.10) to them, and the terms in this expansion can be bounded similarly as in (1.24), using the inequalities (I.4.5), (I.4.22), (I.4.36), (I.5.44), and other inequalities mentioned in the previous sections. The summation over all possible choices of $\square_0, Y_0, j, X$ yields an expression satisfying (1.29).

Consider now the expression (I.3.7), or rather its analytic extension (I.3.15). It is determined by the functions $H_j, (\delta/\delta B)\mathbf{H}_j$, besides the function $\mathbf{H}_k$, and it depends on the first two functions restricted to X. We introduce the parameters s in the same way as before, only the family σ_0 is different. It is defined now as the family of all cubes Δ disjoint with the interior of X_0, where X_0 is the smallest localization domain from $\mathbf{D}_k$ containing X. We apply the expansion (1.10). Terms in this expansion can be bounded using the inequalities (I.3.17), (1.21), and we obtain the bound

$$E_0 24 B_0^2 C_1 B_3 e^{16\kappa_1} \alpha_2^{-1} \varepsilon_1 \exp(-\tfrac{1}{2}\delta_0 M(L^j\eta)^{-1} - \tfrac{1}{2}\delta_0 \operatorname{dist}^{(\xi)}(X, \square))$$

$$\cdot \exp(-(\kappa_1 - 1)M^{-4}|Y \backslash X_0|)\exp(-\kappa d_j(X)). \tag{1.30}$$

By the definition of the linear size functions they satisfy the following fundamental scaling inequality

$$d_j(X) \geqq (L^j\eta)^{-1} d_k(X_0). \tag{1.31}$$

It implies that the last two exponentials in (1.30) can be bounded by

$$\exp(-(1-\delta)\kappa d_k(Y) - \delta\kappa d_j(X)), \tag{1.32}$$

if $\frac{1}{4}(\kappa_1 - 1) \geqq (1-\delta)\kappa, 0 < \delta < 1$. Now we sum all the terms having the same localization domain Y. At first we notice that each term is defined and analytic on the corresponding space (I.3.16) restricted to the domain Y. All these spaces contain the subspace $\mathbf{U}^c_{k+1}(Y, (1+\beta)\alpha_0, (1+\beta)\alpha_1, \alpha_0)$, and all the terms are defined and analytic on it. The summation over X is controlled in several steps. We fix a cube $\square' \in \pi_j$ such, that $\square' \subset X$, $\mathrm{dist}^{(\xi)}(X, \square) = \mathrm{dist}^{(\xi)}(\square', \square)$, and we sum over X containing $\square'$, using the second exponential in (1.32), and the inequality (1.26) for $\delta\kappa$ sufficiently large. Next we sum over the cubes $\square'$, and this sum is controlled by the first exponential in (1.30). The first term under the exponential gives also the factor $L^j\eta$, which controls the sum over j. Finally, the sum over all possible cubes $\square$ can be bounded by

$$M^{-4}|Y| \leqq 3 \cdot 2^3 d_k(Y) \leqq \exp \delta\kappa d_k(Y).$$

These estimates yield again a bound of the form (1.29) for the considered sum, with the last exponential replaced by $\exp(-(1-2\delta)\kappa d_k(Y))$. We assume that $(\frac{1}{16})\kappa_1 \geqq (1-2\delta)\kappa$, hence we can bound both sums by the above exponential.

The expression in (I.3.21), given by the last integral, is expanded and analyzed in almost the same way as the last term in (I.3.34), so we do not repeat these considerations.

The results obtained for the expression in the curly bracket $\{\cdots\}$ can be summarized as follows.

Lemma 1. *The second expression in the fluctuation field action in (I.2.13) is represented as the sum*

$$\mathbf{E}_k(U_k(\exp iB' V^{(k)})) - \mathbf{E}_k(U_k(V^{(k)})) = \sum_{Y \in \mathbf{D}_k} \mathbf{V}'_k(Y, U_{k+1}, B). \tag{1.33}$$

For each term in the sum there exists a function $\mathbf{V}'_k(Y, \mathbf{U}, \mathbf{J}, B)$, *defined and analytic on the space*

$$\mathbf{U}^c_{k+1}(Y, (1+\beta)\alpha_0, (1+\beta)\alpha_1, \alpha_0) \times \{B : |B| < \varepsilon_1 g_k^{-1} \quad \text{on} \quad Y\}, \tag{1.34}$$

i.e. *it depends on configurations* $\mathbf{U}, \mathbf{J}, B$ *restricted to the interior of* Y, *and such that*

$$\mathbf{V}'_k(Y, U_{k+1}, B) = \mathbf{V}'_k(Y, U_{k+1}, J_{k+1}, B). \tag{1.35}$$

There exist absolute constants C_1, C_2, q, *for which*

$$|\mathbf{V}'_k(Y, \mathbf{U}, \mathbf{J}, B)| \leqq E_0 \varepsilon_1 C_1 M^q \exp C_2 \kappa_1 \exp(-(1-2\delta)\kappa d_k(Y)). \tag{1.36}$$

We would like to prove a similar result for the first term in the fluctuation field action in (I.2.13). It is the function $\mathbf{P}^{(k)}(g_k, U_{k+1}, B)$ and, as we have remarked already at the beginning of Sect. I.3, this problem is much simpler for it, because of good regularity and boundedness properties on the unit lattice $T_1^{(k)}$. Nevertheless, there is one aspect of the problem which we have to discuss here. It is connected with the factor $1/g_k^2$ at many terms in this function. We have to cancel this factor, and this implies that we can get a bounded polynomial in B only, not the absolute bound of the type (1.36). Let us discuss it on the most important example of the expression $(1/g_k^2)V(H_1 B')$, where B' is given by (I.3.2). The function V is given by the formula (80) [15], which comes from the expansion (41) [15] of the Wilson action by the change of variables (47). In this function, as in all other terms, we replace the configurations U_{k+1}, J_{k+1} by the variable $\mathbf{U}, \mathbf{J}$. We obtain an analytic function on the space $\mathbf{U}_{k+1}^{\prime c}(T_\eta, \alpha_0', \alpha_1')$, with α_0', α_1' much bigger than α_0, α_1. Let us recall that it is the space of $\mathbf{U}, \mathbf{J}$ satisfying the conditions I. (i)–(iii) on the whole lattice T_η. The function V can have a natural localization connected with the sum over plaquettes in the Wilson action. We take the decomposition of unity $1 = \sum_{\square} \zeta_{\square}$ constructed in Sect. I.3. This decomposition is introduced into the sum over plaquettes, and we obtain

$$V(H_1 B') = \sum_{\square} V_{\square}(H_1 B'), \tag{1.37}$$

where in each term the sum over plaquettes is localized by the function ζ. Now we apply the decomposition (1.10) to each term in the sum (1.37). For a fixed cube $\square$ we take σ_0 as the family of all cubes Δ disjoint with the interior of $\square$. From the definition (80) [15] it follows that the sum in the decomposition is over all localization domains $Y \subset \mathbf{D}_k$ containing the cube $\square$. The term corresponding to a domain Y is represented as

$$\prod_{\Delta \subset Y \setminus \square} \int ds(\Delta) \frac{1}{2\pi i} \int \frac{d\sigma(\Delta)}{(\sigma(\Delta) - s(\Delta))^2} V_{\square}(\sigma(Y), H_1(\sigma(Y))B'). \tag{1.38}$$

The above expression is localized in the interior of Y, with respect to $\mathbf{U}, \mathbf{J}, B'$ or B. It is an analytic function of $(\mathbf{U}, \mathbf{J})$ in the space $\mathbf{U}_{k+1}^{\prime c}(T_\eta, \alpha_0', \alpha_1')$, and of B' in the domain $\{B' : e^{16\kappa_1}|B'| \leqq a_1 \text{ on } Y\}$, as it follows from the considerations of the beginning of the section. The underintegral expression is analytic in $\sigma(Y)$ on the polydisc $|\sigma(Y)| \leqq e^{\kappa_1}$. The estimates (33), (37), (55), (57), (58) [15] imply the bound

$$|(1.38)| \leqq C_3 (e^{16\kappa_1}|B'|)^3 M^4 \exp(-(\kappa_1 - 1)M^{-4}|Y \setminus \square|), \tag{1.39}$$

where C_3 is an absolute constant. To cancel the factor $1/g_k^2$ we expand (1.38) with respect to B' up to the second order. The terms of zeroth and first order vanish, and the second order term is written as a quadratic form with coefficients given by second order derivatives of the function (1.38). The coefficients satisfy the bound

$$\left| \int_0^1 dt(1-t) \frac{\partial^2}{\partial B'_\mu(x) \partial B'_\nu(y)} ((1.38) \text{ with } B' \text{ replaced by } tB') \right|$$
$$\leqq \tfrac{1}{2} C_3 3^3 e^{7 \cdot 8\kappa_1} |B'| \exp(-\tfrac{1}{8}(\kappa_1 - 1) d_k(Y) - \tfrac{1}{2}(\kappa_1 - 1) M^{-4}|Y|), \tag{1.40}$$

if $e^{16\kappa_1}|B'| \leqq \frac{1}{3} a_1$. Taking B' as in (1.19) we obtain the above bound with $|B'|$

replaced by $C_1\varepsilon_1$. We fix a localization domain Y and we sum up all the expressions (1.38) with the domain Y, i.e. we sum over all admissible $\square \subset Y$. This yields an expression satisfying the bounds (1.39), (1.40) with the additional factor $M^{-4}|Y| \leqq \exp M^{-4}|Y|$.

The above analysis was done on the example of the expression $(1/g_k^2)V(H_1 B')$, but it can be done in the same way for all terms in $\mathbf{P}^{(k)}(g_k, \mathbf{U}, \mathbf{J}, B)$, and we obtain the same decompositions and bounds, possibly with other absolute constants. In fact many of these terms are much simpler, and the above results can be obtained applying the generalized random walk expansions directly, or even more elementary means. Our last step is to sum up all the expressions with the same localization domain. The constructions and the results of this section are gathered together in the folloiwng lemma.

Lemma 2. *The fluctuation field action is represented as the sum*

$$\mathbf{P}^{(k)}(g_k, U_{k+1}, B) + \{\cdots\} = \sum_{Y \in \mathbf{D}_k} \mathbf{V}_k(Y, U_{k+1}, B). \tag{1.41}$$

For each term in the sum there exists a function $\mathbf{V}_k(Y, \mathbf{U}, \mathbf{J}, B)$, defined and analytic on the space (1.34), and satisfying the corresponding equality (1.35). This function is a sum of two terms

$$\mathbf{V}_k(Y, B) = \tfrac{1}{2}\langle Q(Y, B), B, B\rangle + \mathbf{V}_k''(Y, B). \tag{1.42}$$

The matrix elements of the operator of the quadratic form satisfy the bound

$$|Q(Y, B, b, b')| \leqq C_3\varepsilon_1 M^4 \exp C_2\kappa_1 \exp(-\tfrac{1}{8}(\kappa_1 - 1)d_k(Y) - \tfrac{1}{2}(\kappa_1 - 1)M^{-4}|Y|), \tag{1.43}$$

and the function $\mathbf{V}_k''(Y, B)$ satisfies the bound (1.36). The functions $\mathbf{V}_k(Y, \mathbf{U}, \mathbf{J}, B)$, and both terms in (1.42), are gauge invariant with respect to the simultaneous gauge transformations (I.3.29), for G^c-valued transformations u in a sufficiently small neighborhood of all G-valued transformations.

Let us remark that the last statement is a simple consequence of the statement in Sect. I.3, in the paragraph containing (I.3.29), and of the fact that the operations in this section preserve the gauge invariance.

2. A Final Localization by a Cluster Expansion. Analyticity Properties and Bounds for Terms in the Effective Action

In the last section we have represented the fluctuation field action in the form (I.1.7) for $j = k + 1$, with the analyticity properties and bounds slightly better than demanded by the inductive assumption. Thus we have prepared the integral in (I.2.12), or (I.2.13), to the last step in our construction, to a cluster expansion yielding the decomposition (I.1.7). Let us suppress the dependence on the external gauge fields in the formulas below. We consider the integral in (I.2.13), with the fluctuation field action represented by (1.41). The first step is the Mayer expansion of the action density

$$\int d\mu_{C^{(k)}}(B)\chi_k \exp\left[\sum_{Y\in \mathbf{D}_k} \mathbf{V}_k(Y,B)\right]$$

$$= \sum_{\mathbf{D}\subset \mathbf{D}_k} \prod_{Y\in \mathbf{D}} \int_0^1 dt(Y) \int d\mu_{C^{(k)}}(B)\chi_k \exp\left[\sum_{Y\in \mathbf{D}} t(Y)\mathbf{V}_k(Y,B)\right] \prod_{Y\in \mathbf{D}} \mathbf{V}_k(Y,B). \tag{2.1}$$

Each term in the sum over subfamilies **D** has the underintegral expression localized in the domain

$$Y_0 = \bigcup_{Y\in \mathbf{D}} Y. \tag{2.2}$$

Next, for a fixed domain Y_0 we define the set of bonds $Y_0^{c*} = \{b\in T^{(k)}: b \subset Y_0^c\} \setminus \{b_0(c): c\in T^{(k+1)}\}$, and we take the following decomposition of the characteristic functions χ_k:

$$\chi_k = \chi_{k,Y_0}\chi_{k,Y_0^c} = \sum_{P\subset Y_0^{c*}} (-1)^{|P|}\chi_{k,Y_0}\chi_{k,P}^c, \tag{2.3}$$

$$\chi_{k,P}^c = \prod_{b\in P} \chi\left(\left\{|B(b)| \geqq \frac{\varepsilon_1}{g_k}\right\}\right).$$

Here the symbol $|P|$ means the number of bonds in the set P. For a given set P we take the smallest localization domain $Z_0 \in \mathbf{D}_k$ containing Y_0 and P. Let us stress that bonds of P have to be contained in the interior of Z_0, they cannot intersect the boundary ∂Z_0. We insert the decompositions (2.3) into the integrals in (2.1), and we write the sums over **D** and P as a sum over localization domains Z_0, resumming all terms with **D** and P determining a given domain Z_0. Let us denote by $F(Z_0,B)$ the resummed underintegral expression corresponding to the domain Z_0,

$$(2.1) = \sum_{Z_0\in \mathbf{D}_k} \int d\mu_{C^{(k)}}(B)F(Z_0,B). \tag{2.4}$$

The integrals above are represented in a similar way to (2.28) [6], namely as conditioning on Z_0^c. Thus, we write the term corresponding to a domain Z_0 as

$$(Z^{(k)})^{-1}\int dB \exp(-\tfrac{1}{2}\langle B, C^*\Delta_k CB\rangle)F(Z_0,B) = \int d\mu_{C^{(k)}}(B')$$

$$\cdot \frac{\int dB|_{Z_0} \exp(-\langle Z_0^c B', C^*\Delta_k CZ_0 B\rangle - \tfrac{1}{2}\langle Z_0 B, C^*\Delta_k CZ_0 B\rangle)F(Z_0,B)}{\int dB|_{Z_0} \exp(-\langle Z_0^c B', C^*\Delta_k CZ_0 B\rangle - \tfrac{1}{2}\langle Z_0 B, C^*\Delta_k CZ_0 B\rangle)}$$

$$= \int d\mu_{C^{(k)}}(B')\exp(-\tfrac{1}{2}\langle Z_0^c B', C^*\Delta_k CC^{(k)}(Z_0)C^*\Delta_k CZ_0^c B'\rangle)$$

$$\cdot \int d\mu_{C^{(k)}(Z_0)}(B)\exp(-\langle Z_0^c B', C^*\Delta_k CZ_0 B\rangle)F(Z_0,B). \tag{2.5}$$

In the integral with respect to B' we make the linear change of variables $B' = (C^{(k)})^{1/2}X$. This yields

$$\int d\mu_{C^{(k)}}(B)F(Z_0,B) = \prod_{b\in T_1^{(k)*}} \int \frac{dX(b)e^{-1/2|X(b)|^2}}{(2\pi)^{1/2d(\mathrm{g})}}$$

$$\cdot \exp(-\tfrac{1}{2}\langle C^*\Delta_k CZ_0^c(C^{(k)})^{1/2}X, C^{(k)}(Z_0)C^*\Delta_k CZ_0^c(C^{(k)})^{1/2}X\rangle)$$

$$\cdot \int d\mu_{C^{(k)}(Z_0)}(B) \exp(-\langle B, C^* \Delta_k C Z_0^c (C^{(k)})^{1/2} X \rangle) F(Z_0, B)$$
$$= \int d\mu_0(X) G(Z_0, X, C^{(k)}(Z_0), (C^{(k)})^{1/2}, \Delta_k), \tag{2.6}$$

where the last equality is a definition of the function G, and the measure $d\mu_0$.

The above expression depends yet on the external gauge field configurations on the whole lattice T_η, through the propagators and the operator in the definition of the function G. We localize it again by the method used in Sect. 1, introducing the parameters s and differentiating. There is a difference in comparison with the construction of Sect. 1, we take cubes from π_{k+1}, i.e. cubes of the size LM in the scale corresponding to the lattice T_η. We define σ_0 as the family of such cubes Δ disjoint with the interior of $\tilde{Z}_0$, or with the interior of Z_0', where Z_0' is a union of the smallest family of such cubes containing $\tilde{Z}_0$. The parameters $s(\Delta), \Delta \in \sigma_0$, are introduced into the operators as before, but the generalized random expansions are constructed now in a slightly different way. We use the possibility mentioned in Sect. C [13], that the localization domains in expansions can be chosen to a large extent arbitrarily, they have to be only unions of connected families of M_1-cubes. We choose $\tilde{Z}_0$ as one of the domains, more exactly we take a domain X_0 such, that $\tilde{X}_0^5 = \tilde{Z}_0$ (let us notice the inconsistency in the notation, the tilde over X_0 means that M_1-cubes are adjoined, and the tilde over Z_0 concerns M-cubes). Other localization domains X are chosen as in Sect. C [13], i.e. they are unions of small, connected families of M_1-cubes, and we assume that $\mathrm{dist}(X, Z_0) > \frac{2}{3}M$. For this class of localization domains we construct the generalized random walk expansions. This construction was discussed in [13] for all operators determining Δ_k, and for $C^{(k)}(Z_0)$, but not for $(C^{(k)})^{1/2}$. To expand the last operator we use a method similar to the method of Sect. C [16]. There it was applied to expand the determinant of this operator, see (63) [16]. Now we can simplify it a bit using a better decay property of an underintegral function, and we have

$$(C^{(k)})^{1/2} = (C^* \Delta_k C)^{-1/2} = \frac{1}{\pi} \int_0^\infty dx\, x^{-1/2} (xI + C^* \Delta_k C)^{-1}$$
$$= \frac{1}{\pi} \int_0^{\gamma_1} dx\, x^{-1/2} (xI + C^* \Delta_k C)^{-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1/2} \gamma_1^{-n-1/2} (C^* \Delta_k C)^n. \tag{2.7}$$

Expanding the operator Δ_k into the generalized random walks, we obtain an expansion of the series above, if γ_1 is sufficiently large. The resolvent $(xI + C^* \Delta_k C)^{-1}$ has a representation similar to $(C^* \Delta_k C)^{-1}$. More exactly, the operator $C(xI + C^* \Delta_k C)^{-1} C^*$ is representated by the integral (3.185) [13] with the additional term $-\frac{1}{2} x \| \chi^*(QA + \bar{D}\mu(QA)) \|^2$ under the exponential function, where χ^* is the characteristic function of the set of bonds $T_1^{(k)} \setminus \{b_0(c) : c \in T^{(k+1)}\}$. This term determines a nonnegative, bounded and almost local operator. The integral yields the representation (3.185) [13], with the operator $\tilde{G}_2$ replaced by $\tilde{G}_3(x)$, which is defined as $\tilde{G}_2$, but with this additional operator. The operator $\tilde{G}_3(x)$ has the same properties as $\tilde{G}_2$, especially it can be expanded into a generalized random walk expansion. This yields an expansion of the integral above, hence an expansion of $(C^{(k)})^{1/2}$ also.

Using these expansions we introduce the parameters s, and we apply the decomposition (1.10),

$$\int d\mu_0(X) G(Z_0, X, C^{(k)}(Z_0), (C^{(k)})^{1/2}, \Delta_k) = \sum_Z \prod_{\Delta \subset Z \setminus \tilde{Z}'_0} \int_0^1 ds(\Delta) \frac{\partial}{\partial s(\Delta)}$$
$$\cdot \int d\mu_0(X) G(Z_0, X, C^{(k)}(Z_0, s(Z)), (C^{(k)})^{1/2}(s(Z)), \Delta_k(s(Z)))$$
$$= \sum_Z H(Z, Z_0). \tag{2.8}$$

The sums are over Z such, that each connected component of Z contains a component of Z'_0. The equalities (2.4), (2.6), (2.8) imply

$$(2.1) = \sum_Z H(Z), \quad \text{where} \quad H(Z) = \sum_{Z_0 \cdot \tilde{Z}_0 \subset Z} H(Z, Z_0). \tag{2.9}$$

This is the desired expansion into localized quantities. The function $H(Z)$ is localized in the interior of Z with respect to the external gauge fields. From the definition of $H(Z)$ it is also clear, that if $Z = Z_1 \cup \dots \cup Z_n$, where Z_i is a connected component of Z, then

$$H(Z) = H(Z_1) \cdots H(Z_n). \tag{2.10}$$

Thus finally we obtain the polymer expansion

$$(2.1) = \sum_{\{Z_1, \dots, Z_n\}} H(Z_1) \cdots H(Z_n) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Z_1, \dots, Z_n)} \prod_{\{i,j\}, i<j} \zeta(Z_i, Z_j) H(Z_1) \cdots H(Z_n), \tag{2.11}$$

where the function $\zeta(Z, Z')$ is defined by the condition: $\zeta(Z, Z') = 0$ if $Z \cap Z'$ contains a cube, or a wall of a cube, and $\zeta(Z, Z') = 1$ otherwise.

If the activities $H(Z)$ of the above polymer expansion are sufficiently small, then the polymer expansion can be exponentiated according to the well-known formula, see [36, 60, 26, 25, 67, 50]. We obtain by (I.2.13), (1.41), (2.11)

$$\mathbf{E}^{(k+1)} = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Z_1, \dots, Z_n)} \rho^T(Z_1, \dots, Z_n) H(Z_1) \cdots H(Z_n), \tag{2.12}$$

where $\rho^T(Z) = 1$, and

$$\rho^T(Z_1, \dots, Z_n) = \sum_{g \in C_n} \prod_{\{i,j\} \in g} (\zeta(Z_i, Z_j) - 1),$$

C_n is the set of connected graphs on the set $\{1, \dots, n\}$. The representation (I.1.7) for $\mathbf{E}^{(k+1)}$ is constructed by taking

$$\mathbf{E}^{(k+1)}(X) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(Z_1, \dots Z_n) \cdot \cup Z_i = X} \rho^T(Z_1, \dots, Z_n) H(Z_1) \cdots H(Z_n), \tag{2.13}$$

where $X \in \mathbf{D}_{k+1}$. To end the proof of the first part of Theorem 3 we have to prove the bounds (I.1.18). We also need bounds to prove a convergence of (2.12), (2.13). In the rest of this section we will prove bounds and discuss analyticity properties of the functions (2.13).

We start with a discussion of analyticity properties of (2.13). The activities $H(Z)$ are sums of many terms, more exactly the sums in (2.9), (2.1), (2.3) over $Z_0, \mathbf{D}, P$.

The subfamily $\mathbf{D}$ is important for the analyticity properties, because it determines the potentials $\mathbf{V}_k(Y,B)$. They can be extended to functions of $(\mathbf{U},\mathbf{J})$, analytic on the space $\mathbf{U}^c_{k+1}(Y,(1+\beta)\alpha_0,(1+\beta)\alpha_1,\alpha_0)$. Of course $Y\subset Z_0$, and $\tilde{Z}_0\subset Z\subset X$ for the activities in (2.13). The potentials are also analytic functions on the subspace $\mathbf{U}^c_{k+1}(X,\alpha_0,\alpha_1)$. The quadratic forms and covariances in $H(Z)$ are analytic functions on the space of configurations $(\mathbf{U},\mathbf{J})$ satisfying the conditions I.(i)–(iii) on the domain Z, with constants α'_0,α'_1 much bigger than α_0,α_1, therefore we can restrict them, as analytic functions, to the above subspace. Thus the activities in (2.13), and the whole sum $\mathbf{E}^{(k+1)}(X)$, are analytic functions of $(\mathbf{U},\mathbf{J})$, on the space $\mathbf{U}^c_{k+1}(X,\alpha_0,\alpha_1)$. This is the analyticity statement in the inductive assumptions.

To get a bound for $H(Z)$ we consider a term in the sum over $\mathbf{D},P$. This term can be written in the following form:

$$\prod_{\Delta\subset Z\setminus Z'_0}\int_0^1 ds(\Delta)\frac{1}{2\pi i}\int\frac{d\sigma(\Delta)}{(\sigma(\Delta)-s(\Delta))^2}\prod_{Y\in\mathbf{D}}\int_0^1 dt(Y)\frac{1}{2\pi i}\int\frac{d\tau(Y)}{(\tau(Y)-t(Y))^2}$$
$$\cdot\int d\mu_0(X)|_Z\exp(-\tfrac{1}{2}\langle\Gamma_k(Z_0,\sigma(Z))X,C^{(k)}(Z_0,\sigma(Z))\Gamma_k(Z_0,\sigma(Z))X\rangle)$$
$$\cdot\int d\mu_{C^{(k)}(Z_0,\sigma(Z))}(B)\exp(-\langle B,\Gamma_k(Z_0,\sigma(Z))X\rangle)$$
$$\cdot(-1)^{|P|}\chi_{k,Y_0}(B)\chi^c_{k,P}(B)\exp\left[\sum_{Y\in\mathbf{D}}\tau(Y)\mathbf{V}_k(Y,B)\right],\tag{2.14}$$

where

$$\Gamma_k(Z_0,\sigma(Z))=C^*\Delta_k(\sigma(Z))CZ^c_0(C^{(k)})^{1/2}(\sigma(Z)).$$

We consider it as an analytic function of $(\mathbf{U},\mathbf{J})$ in the space $\mathbf{U}^c_{k+1}(X,\alpha_0,\alpha_1)$, and of the complex parameters $\sigma(Z),\tau$. This complicates estimates of this expression, because the operators in it are not symmetric, and the second measure is complex. We use the fact that, by the definition of the space, the configuration $\mathbf{U}$ can be written as $\mathbf{U}=U'U$, $U'=\exp iL^{-1}\eta A'$, and $A',\mathbf{J}$ have values in $\mathbf{g}^c$, but they are small. More precisely we have $|A''|,|\nabla_U^{L^{-1}\eta}A'|<\alpha_1,|\mathbf{J}|<\alpha_0$. For the pair $(U,0)$ the operators are symmetric, and the measure is positive, and then the estimates are simpler. The general case is handled by a perturbative argument. The first estimate is

$$|(7.14)|\leqq\exp(-(\kappa_1-1)(LM)^{-4}|Z\setminus Z'_0|)\prod_{Y\in\mathbf{D}}\frac{2}{|\tau(Y)|}$$
$$\cdot\int d\mu_0(X)|_Z\exp(-\tfrac{1}{2}\operatorname{Re}\langle\Gamma_k(Z_0,\sigma(Z))X,C^{(k)}(Z_0,\sigma(Z))\Gamma_k(Z_0,\sigma(Z))X\rangle)$$
$$\cdot\left|\frac{\det(C^{(k)}(Z_0,\sigma(Z))^{-1})}{\det(\operatorname{Re}C^{(k)}(Z_0,\sigma(Z))^{-1})}\right|^{1/2}\int d\mu_{(\operatorname{Re}C^{(k)}(Z_0\sigma(Z))^{-1})^{-1}}(B)$$
$$\cdot\exp(-\langle B,\operatorname{Re}\Gamma_k(Z_0,\sigma(Z))X\rangle)\chi_{k,Y_0}\chi^c_{k,P}\exp\left[\sum_{Y\in\mathbf{D}}|\tau(Y)||\mathbf{V}_k(Y,B)|\right].\tag{2.15}$$

In the expression on the right-hand side we replace the operators by the corresponding operators with $\sigma(Z)=0$, $\mathbf{U}=U,\mathbf{J}=0$, and we estimate the error. For the quadratic form in the first exponential the difference is a quadratic form

$\frac{1}{2}\langle X, R_1 X\rangle$, with matrix elements satisfying the bound

$$|R_1(b,b')| \leqq (O(1)e^{-1/3\delta_0 M} + O(\alpha_0+\alpha_1))\exp(-\tfrac{1}{2}\delta_0|b_- - b'_-|). \tag{2.16}$$

We have assumed, as in Sect. 1, that M is much bigger than κ_1, especially that $e^{-1/3\delta_0 M}e^{16\kappa_1} < 1$. The determinants in the next factor are equal for the new operators, hence this factor can be estimated by

$$\exp((O(1)e^{-1/3\delta_0 M} + O(\alpha_0+\alpha_1))|Z_0|). \tag{2.17}$$

Similarly, the next Gaussian measure is replaced by the measure with the new covariance, multiplied by a quotient of determinants, which can be estimated by (2.17), and by the function $\exp\frac{1}{2}\langle B, R_2 B\rangle$ with R_2 satisfying (2.16). In the second exponential the difference between the new bilinear form and the form in (2.15) is a bilinear from $-\langle B, R_3 X\rangle$ with R_3 satisfying (2.16). The expression in the last exponential can be estimated using (1.42), and the inequalities (1.43), (1.36). We take a small, positive number α_4, to be chosen later, and

$$\frac{1}{|\tau(Y)|} = E_0\varepsilon_1 C_1 \alpha_4^{-1} M^q \exp C_2\kappa_1 \exp(-(1-3\delta)\kappa d_k(Y)). \tag{2.18}$$

We assume that $\frac{1}{8}(\kappa_1 - 1) \geqq (1-3\delta)\kappa$, $C_3 \leqq E_0 C_1$, and $q \geqq 8$. The quadratic form in (1.42), after multiplication by $|\tau(Y)|$, can be bounded by

$$\frac{1}{2}\sum_{b,b'\subset Y} \alpha_4 M^{-4}\exp(-\tfrac{1}{4}(\kappa_1-1)M^{-4}|Y| - \tfrac{1}{16}(\kappa_1-1)M^{-1}|b_- - b'_-|)|B(b)||B(b')|. \tag{2.19}$$

The sum of these quadratic forms over $Y\in\mathbf{D}$ is bounded by a quadratic form with the above matrix elements resummed over all $Y\in\mathbf{D}_k$ containing, for example, the point b_-. We use the first exponential factor in (2.19) to bound the sum, and this yields a constant $O(1)$. In fact the constant is small for κ_1 large, hence we can bound it by 1. Using (1.28) we obtain

$$\begin{aligned}\sum_{Y\in\mathbf{D}} |\tau(Y)||\mathbf{V}_k(Y,B)| &\leqq \frac{1}{2}\sum_{b,b'\subset Y_0} \alpha_4 M^{-4}\exp(-\tfrac{1}{16}(\kappa_1-1)M^{-1}|b_- - b'_-|)|B(b)||B(b')| \\ &\quad + \sum_{Y\in\mathbf{D}}\alpha_4\exp(-\delta\kappa d_k(Y)) \leqq \tfrac{1}{2}O(1)\alpha_4\sum_{b\subset Y_0}|B(b)|^2 \\ &\quad + O(1)\alpha_4 M^{-4}|Y_0|.\end{aligned} \tag{2.20}$$

The other quadratic forms, i.e. the forms R_1, R_2, R_3, can be bounded in a similar way using (2.16), by the forms

$$\tfrac{1}{2}(O(1)e^{-(1/3)\delta_0 M} + O(\alpha_0+\alpha_1))(\|ZX\|^2 + \|Z_0 B\|^2). \tag{2.21}$$

Finally we estimate

$$\chi_{k,Y_0}(B)\chi^c_{k,P}(B) \leqq \exp\left(-\frac{1}{2}\gamma_2\frac{\varepsilon_1^2}{g_k^2}|P| + \tfrac{1}{2}\gamma_2\|PB\|^2\right), \tag{2.22}$$

where γ_2 is a small, positive constant. Applying the above estimates to the expressions under the integral in (2.15), we obtain the following integral:

$$\int d\mu_0(X)|_Z \exp(-\tfrac{1}{2}\langle \Gamma_k(Z_0,0)X, C^{(k)}(Z_0,0)\Gamma_k(Z_0,0)X\rangle + \tfrac{1}{2}\alpha_5 \|ZX\|^2) \\ \cdot \int d\mu_{C^{(k)}(Z_0 0)}(B) \exp(-\langle B, \Gamma_k(Z_0,0)X\rangle + \tfrac{1}{2}\alpha_5 \|Z_0 B\|^2), \tag{2.23}$$

where $\alpha_5 = O(1)e^{-1/3\delta_0 M} + O(\alpha_0 + \alpha_1) + O(1)\alpha_4 + \gamma_2$. It is a Gaussian integral, and can be easily calculated. At first we calculate the integral with respect to B, including the last two quadratic forms under the exponential into the Gaussian measure. It is equal to

$$\left|\frac{\det(C^{(k)}(Z_0,0)^{-1})}{\det(C^{(k)}(Z_0,0)^{-1} - \alpha_5 I)}\right|^{1/2} \\ \cdot \exp(\tfrac{1}{2}\langle \Gamma_k(Z_0,0)X, (C^{(k)}(Z_0,0)^{-1} - \alpha_5 I)^{-1}\Gamma_k(Z_0,0)X\rangle). \tag{2.24}$$

Of course we have assumed that α_5 is sufficiently small, e.g. $\alpha_5 \|C^{(k)}(Z_0,0)\| < \frac{1}{2}$. The factor with the determinants can be estimated by $\exp O(1)\alpha_5 |Z_0|$. The quadratic form under the exponential is expanded with respect to α_5, and the zeroth order term cancels the first quadratic form under the first exponential in (2.23). The remainder can be estimated by $\frac{1}{2}O(\alpha_5)\|ZX\|^2$. In fact a better bound can be proved using localizations and the exponential decay of operators in the above expressions, but we do not need such a bound. These calculations and estimates yield the following integral

$$\int d\mu_0(X)|_Z \exp \tfrac{1}{2}O(\alpha_5)\|ZX\|^2 = \prod_{b\in Z}(1 - O(\alpha_5))^{1/2d(g)} \leqq \exp(O(\alpha_5)|Z|). \tag{2.25}$$

This ends the estimate of the expression (2.14). Gathering together all the bounds we get

$$|(2.14)| \leqq \exp(-(\kappa_1 - 1)(LM)^{-4}|Z\setminus Z_0'|) \\ \cdot\left[\prod_{Y\in \mathbf{D}} 2E_0\varepsilon_1 C_1 \alpha_4^{-1} M^q \exp C_2\kappa_1 \exp(-(1-3\delta)\kappa d_k(Y))\right] \exp\left(-\frac{1}{2}\gamma_2 \frac{\varepsilon_1^2}{g_k^2}|P|\right) \\ \cdot \exp O(1)\alpha_5 |Z|. \tag{2.26}$$

As we have remarked already it is possible to get a better bound, e.g. with $|Z_0|$ instead of $|Z|$ in the last exponential, by more careful estimates of the quadratic forms.

To get a bound for $H(Z)$ we have to perform the resummation of the terms (2.14) over $\mathbf{D}, P$ and Z_0. We do it in the following order. For a fixed Y_0 we sum over all $\mathbf{D}$ satisfying (2.2). Next, we sum over Y_0, P determining a fixed Z_0. Further, for a fixed Z_0', we sum over all possible Z_0 determining this fixed Z_0'. Finally we sum over all $Z_0' \subset Z$. To bound $H(Z)$ we use the estimate (2.26) for terms of these sums, and we bound the sums using the factors on the right-hand side of (2.26).

Let us start with the sum over $\mathbf{D}$ satisfying (2.2) with a fixed Y_0. The set Y_0 is a union of its connected components, $Y_0 = \bigcup Y_i$, and this decomposition induces the decomposition of the families $\mathbf{D}, \mathbf{D} = \bigcup \mathbf{D}_i$, $\mathbf{D}_i$ satisfy $\bigcup_{Y\in\mathbf{D}_i} Y = Y_i$. The sum over $\mathbf{D}$ factorizes into independent sums over $\mathbf{D}_i$, similarly the product over $Y\in\mathbf{D}$ in

(2.26) factorizes into products over $Y \in \mathbf{D}_i$. For simplicity let us denote by Y_0 one of the components. Consider the product over $Y \in \mathbf{D}$ in (2.26). We extract the expression $\alpha_6 \exp(-\delta\kappa d_k(Y))$ from each factor in it, where α_6 is a sufficiently small, positive constant. Because $\bigcup_{Y \in \mathbf{D}} Y = Y_0$ and Y_0 is a connected domain, hence the definition of $d_k(Y)$ implies the inequality

$$\sum_{Y \in \mathbf{D}} (d_k(Y) + 5) \geqq d_k(Y_0) + 5. \tag{2.27}$$

Assuming $2E_0 \varepsilon_1 C_1 \alpha_4^{-1} \alpha_6^{-1} M^q \exp C_2 \kappa_1 \exp 5\kappa \leqq 1$, we have

$$\begin{aligned} \prod_{Y \in \mathbf{D}} \cdots (\text{in } (2.26)) &\leqq \prod_{Y \in \mathbf{D}} \alpha_6 \exp(-\delta\kappa d_k(Y)) \\ &\cdot 2E_0 \varepsilon_1 C_1 \alpha_4^{-1} \alpha_6^{-1} M^q \exp C_2 \kappa_1 \exp(-(1-4\delta)\kappa d_k(Y_0)). \end{aligned} \tag{2.28}$$

Now we estimate the sum over $\mathbf{D}$ satisfying (2.2). We use the product on the right-hand side above. For κ sufficiently large and α_6 sufficiently small we have

$$\sum_{\mathbf{D}} \prod_{Y \in \mathbf{D}} \alpha_6 \exp(-\delta\kappa d_k(Y)) \leqq 1. \tag{2.29}$$

Inequalities of this type were proved many times, the above can be proved, for example, by a simple modification of the argument in [26]. In connection with this notice the following useful inequality

$$(3 \cdot 2^3)^{-1} M^{-4} |Y| \leqq d_k(Y) \leqq M^{-4} |Y| - 1 \tag{2.30}$$

holding for localization domains $Y \in \mathbf{D}_k$. The inequalities (2.28), (2.29) yield a bound of the sum over $\mathbf{D}$.

Now we consider the sum over Y_0, P, with fixed Z_0. It is controlled by the exponential factor with $|P|$ in (2.26). The definition of Z_0 yields $|P| \geqq \frac{1}{2} M^{-4} |Z_0 \setminus Y_0|$, because one bond in P may connect two cubes in $Z_0 \setminus Y_0$. We decompose the exponential factor into a product of five equal factors. Four of them are bounded using the above inequality, the fifth is used to bound the sum over P, with a fixed Y_0. We have

$$\begin{aligned} &\sum_P \exp\left(-\frac{1}{20}\gamma_2 \frac{\varepsilon_1^2}{g_k^2} M^{-4} |Z_0 \setminus Y_0|\right) \exp\left(-\frac{1}{10}\gamma_2 \frac{\varepsilon_1^2}{g_k^2} |P|\right) \\ &\leqq \exp\left\{-M^{-4} |Z_0 \setminus Y_0| \left[\frac{1}{20}\gamma_2 \frac{\varepsilon_1^2}{g_k^2} - 4M^4 \exp\left(-\frac{1}{10}\gamma_2 \frac{\varepsilon_1^2}{g_k^2}\right)\right]\right\} \leqq 1 \end{aligned} \tag{2.31}$$

for g_k^2, i.e. γ^2 sufficiently small, depending on M and κ. In general the set Z_0 is a union of connected components. Let us denote one of the components by Z_0. It contains $Y_0 = \bigcup Y_i$. By a simple geometric argument we have

$$\sum_i d_k(Y_i) + 4M^{-4} |Z_0 \setminus Y_0| \geqq d_k(Z_0). \tag{2.32}$$

Assuming

$$\frac{1}{20}\gamma_2 \frac{\varepsilon_1^2}{g_k^2} \geqq \frac{1}{20}\gamma_2 \frac{\varepsilon_1^2}{\gamma^2} \geqq 4\kappa,$$

and denoting $\varepsilon_2 = 2E_0\varepsilon_1 C_1 \alpha_4^{-1}\alpha_6^{-1} M^q \exp C_2\kappa_1$, we obtain

$$\left[\prod_i \varepsilon_2 \exp(-(1-4\delta)\kappa d_k(Y_i))\right]\exp\left(-\frac{1}{10}\gamma_2\frac{\varepsilon_1^2}{g_k^2}M^{-4}|Z_0\setminus Y_0|\right)$$
$$\leqq \max\left\{\varepsilon_2, \exp\left(-\frac{1}{20}\gamma_2\frac{\varepsilon_1^2}{\gamma^2}\right)\right\}\exp(-(1-4\delta)\kappa d_k(Z_0)). \tag{2.33}$$

We have used the assumption $\varepsilon_2 \leqq 1$, and the fact that if Y_0 is empty, then we have the exponential factor. For simplicity let us assume that

$$\exp\left(-\frac{1}{20}\gamma_2\frac{\varepsilon_1^2}{\gamma^2}\right) \leqq \varepsilon_2.$$

Finally, we have the sum over $Y_0 \subset Z_0$, or over $Z_0 \setminus Y_0$. The remaining factor is used to bound this sum

$$\sum_{Z_0\setminus Y_0} \exp\left(-\frac{1}{20}\gamma_2\frac{\varepsilon_1^2}{g_k^2}M^{-4}|Z_0\setminus Y_0|\right) \leqq \exp\left(\exp\left(-\frac{1}{20}\gamma_2\frac{\varepsilon_1^2}{\gamma^2}\right)M^{-4}|Z_0|\right). \tag{2.34}$$

The exponential on the right-hand side multiplied by $\exp(-\delta\kappa d_k(Z_0))$ can be estimated by 1.

Let us write the inequality we have obtained for the partially resummed terms (2.14). The set Z_0 is a union of connected components, $Z_0 = \bigcup Z_i$, and we have

$$\left|\sum_{\mathbf{D},P}(2.14)\right| \leqq \exp(-(\kappa_1 - 1)(LM)^{-4}|Z\setminus Z_0'|)$$
$$\cdot\left[\prod_i \varepsilon_2 \exp(-(1-5\delta)\kappa d_k(Z_i))\right]\exp O(1)\alpha_5|Z|. \tag{2.35}$$

The next step is to bound the sum over Z_0, with Z_0' fixed. Again the set Z_0' is a union of connected components, which are localization domains from $\mathbf{D}_{k+1}$, and we denote by Z_0' one of the components. The set Z_0 determining it is a union of connected components, $Z_0 = \bigcup Z_i$, and we denote by Z_i' the smallest localization domain from $\mathbf{D}_{k+1}$ containing $\tilde{Z}_i$. The sum over Z_0 is decomposed into several sums. For each Z_i' we sum over all possible components of Z_0 determining this Z_i'. The we sum over all families of domains Z_i' such, that $\bigcup Z_i' = Z_0'$. Consider the sum over the components with a fixed Z_i'. We extract $\exp(-\delta\kappa d_k(Z_i))$ from each exponential in (2.35), corresponding to one of these components. The remaining exponential is bounded using the following inequality:

$$2d_k(Z_i) \geqq L d_{k+1}(Z_i'). \tag{2.36}$$

This inequality can be obtained by simple, but awkward, geometric and combinatoric considerations. It follows by considering locally many possible cases. Now, the sum over the components can be decomposed into a sum over one component. plus a sum over two components, and so on. A sum over n components is estimated by a product of n sums, each of them is a sum over independently changing components. The last sum is estimated using (1.28), with κ replaced by $\delta\kappa$, and with an additional sum over $(L+2)^4$ cubes $\square'$ from π_k, the cubes touching a fixed LM-cube in Z_i'. This yields a bound similar to (2.35), with ε_2 replaced by $(L+2)^4O(1)\varepsilon_2$, and $(1-5\delta)\kappa d_k(Z_i)$ in the exponentials replaced by $(1-6\delta)\frac{1}{2}L\kappa d_{k+1}(Z_i')$. For $n>1$ we leave only one exponential, estimating by 1 the remaining ones with the same domain Z_i'. The sum over $n \geqq 1$ is now bounded by $2(L+2)^4O(1)\varepsilon_2$, assuming that $(L+2)^4O(1)\varepsilon_2 \leqq \frac{1}{2}$. Finally, the sum over all families of different localization domains Z_i' from $\mathbf{D}_{k+1}$, satisfying the condition $\bigcup Z_i' = Z_0'$, is estimated using (2.29). We have $\delta\frac{1}{2}L\kappa d_{k+1}(Z_i')$ instead of $\delta\kappa d_k(Y)$, but the inequality (2.29) is valid for all k. The inequality (2.27) is used for the remaining exponential factors. Assuming that $2(L+2)^4O(1)\varepsilon_2 \exp 5\kappa \leqq 1$, we obtain for a fixed Z_0',

$$\left|\sum_{\mathbf{D},P,Z_0}(2.14)\right| \leqq \exp(-(\kappa_1-1)(LM)^{-4}|Z\setminus Z_0'|)$$
$$\cdot\left[\prod_i 2(L+2)^4O(1)\varepsilon_2\exp(-(1-7\delta)\tfrac{1}{2}L\kappa d_{k+1}(Z_i'))\right]\exp O(1)\alpha_5|Z|, \tag{2.37}$$

where now Z_i' denote connected components of Z_0'.

The last sum to estimate is the sum over Z_0', or over $Z\setminus Z_0'$. Using the inequality (2.32), properly adapted to the new situation, we bound the exponential factors in the square bracket above, and half of the first exponential factor, by $\exp(-(1-7\delta)\frac{1}{2}L\kappa d_{k+1}(Z))$. Of course, we assume that $\frac{1}{2}(\kappa_1-1)\geqq 2L\kappa$. The sum over $Z\setminus Z_0'$ is bounded, using the remaining factor and an inequality similar to (2.34), by $\exp(\exp(-\frac{1}{2}(\kappa_1-1))(LM)^{-4}|Z|)$. This exponential is of the same type as the last exponential in (2.37), which can be written as $\exp O(1)(LM)^4\alpha_5(LM)^{-4}|Z|$. Let us recall the definition of the constant α_5:

$$\alpha_5 = O(1)e^{-1/3\delta_0 M} + O(\alpha_0+\alpha_1) + O(1)\alpha_4 + \gamma_2 .$$

We assume that $(LM)^4\alpha_0, (LM)^4\alpha_1, (LM)^4\alpha_4, (LM)^4\gamma_2$ are bounded by a constant independent of M, for example by 1. Then $O(1)(LM)^4\alpha_5 + \exp(-\frac{1}{2}(\kappa_1-1))$ is bounded by an absolute constant. We use the factor $\exp(-\delta\frac{1}{2}L\kappa d_{k+1}(Z))$, and the inequality (2.30), to bound the exponentials by 1. We leave one factor $2(L+2)^4O(1)\varepsilon_2$, and the remaining factors are estimated by 1. We define the constant $C_3 = 2(L+2)^4O(1)2E_0C_1\alpha_4^{-1}\alpha_6^{-1}M^q\exp C_2\kappa_1$.

Thus we have finished the estimate of the resummed terms (2.14). The sum defines the activity $H(Z)$, and we have proved the following lemma.

Lemma 3. *Under all the above restrictions on the constants $M,\kappa,\kappa_1,\alpha_0,\alpha_1,\alpha_4,\alpha_6,\gamma_2,\gamma,\varepsilon_1$, the activity $H(Z)$ for a localization domain $Z\in\mathbf{D}_{k+1}$ satisfies the inequality*

$$|H(Z)| \leqq C_3\varepsilon_1\exp(-(1-8\delta)\tfrac{1}{2}L\kappa d_{k+1}(Z)). \tag{2.38}$$

The above lemma implies that sufficient conditions for convergence of the series (2.12), (2.13) are satisfied, see [26, 67, 25, 50]. We want to prove the inequality (I.1.18) for $\mathbf{E}^{(k+1)}(X)$. The series (2.13), defining $\mathbf{E}^{(k+1)}(X)$, is estimated in the standard way, each factor $|H(Z)|$ is replaced by the right-hand side of (2.38) in the bound. Consider a term in the sum. We have a product of n exponentials from (2.38). We extract the exponential $\exp(-\delta\frac{1}{2}L\kappa d_{k+1}(Z_i))$ from the i-th factor, and the remaining product is estimated using (2.27), and the condition $\cup Z_i = X, X$ is a connected domain. This yields

$$|\mathbf{E}^{(k+1)}(X)| \leqq \exp(-(1-9\delta)\tfrac{1}{2}L\kappa d_{k+1}(X))\exp(-5\kappa) \cdot \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{(Z_1,\dots,Z_n):\cup Z_i = X} |\rho^T(Z_1,\dots,Z_n)|\prod_{i=1}^{n} C_3\varepsilon_1 \exp 5\kappa \exp(-\delta\tfrac{1}{2}L\kappa d_{k+1}(Z_i)). \tag{2.39}$$

To the above sum we can repeat all the considerations and bounds of the paper [26], for κ sufficiently large, and ε_1 sufficiently small. We obtain

$$|\mathbf{E}^{(k+1)}(X)| \leqq \exp(-(1-9\delta)\tfrac{1}{2}L\kappa d_{k+1}(X))\exp(-5\kappa) \cdot \sum_{Z\subset X} C_3\varepsilon_1 \exp 5\kappa \exp(-\tfrac{1}{2}\delta L\kappa d_{k+1}(Z))O(1)\exp 2(LM)^{-4}|Z|. \tag{2.40}$$

The last sum is bounded by $C_3\varepsilon_1 \exp 5\kappa O(1)(LM)^{-4}|X| \leqq C_3\varepsilon_1 \exp 5\kappa O(1)$ $\exp(LM)^{-4}|X|$, and the last exponential multiplied by $\exp(-\frac{1}{2}\delta L\kappa d_{k+1}(X))$ is bounded by 1. This yields

$$|\mathbf{E}^{(k+1)}(X)| \leqq O(1)C_3\varepsilon_1 \exp(-(1-10\delta)\tfrac{1}{2}L\kappa d_{k+1}(X)). \tag{2.41}$$

Now we make our last assumptions. At first we assume that $(1-10\delta)\frac{1}{2}L = 1$, or $\delta = \frac{1}{10}(1-2L^{-1})$. Next, we assume that $O(1)C_3\varepsilon_1 \leqq \frac{1}{2}E_0$. In fact this assumption is unessential, because the constant $C_3\varepsilon_1$ is small anyway, and we can take E_0 such, that the assumption is satisfied. The assumptions allow finally us to fix all the constants, or rather bounds on these constants.

The inequality (2.41) and the assumptions imply the inequality (I.1.18), with $\frac{1}{2}E_0$ instead of E_0, for the terms of the effective action $\mathbf{E}^{(k+1)}$ in (I.1.3). The effective action in (I.1.6) is obtained by adding to the above action the expression

$$[\log Z^{(k)}(U_{k+1}) - \log Z^{(k)}(1)].$$

For this expression we construct the representation (I.1.7) using the generalized random walk expansion for $Z^{(k)}(U_{k+1})$. The expansion was constructed in [16], see the formula (63) there, and the discussion after it. We gather all terms in the expansions, localized in X, and we extend them to analytic functions of $\mathbf{U}, \mathbf{J}$. The expression localized in X satisfies the bound (I.1.18) with κ replaced by $\delta_0 M$, and with an absolute constant instead of E_0. We define $\frac{1}{2}E_0$ as equal to this constant, and we take M sufficiently large, so that $\delta_0 M \geqq \kappa$. This yields the bounds (I.1.18) for terms of the effective action in the representation (I.1.6) also.

Let us make a last remark about the expressions (2.14). By Lemma 2, and the transformation properties of the operators in (2.14) with respect to gauge transformations, e.g. see (3.28)–(3.34) [13], the expressions (2.14) are gauge invariant with respect to all G-valued transformations. The expressions are analytic functions of $(\mathbf{U}, \mathbf{J})$, hence the invariance can be extended, by the analyticity, to G^c-valued gauge transformations in a small neighborhood of the space of G-valued ones. This means that the expressions are constant on intersections of orbits with the corresponding space of configurations $(\mathbf{U}, \mathbf{J})$ satisfying the conditions I.(i)–(iv). We extend them to constant functions on whole orbits having non-empty intersections with the space.

The above remark completes the proof of the inductive assumptions for the action A_{k+1}, hence the proof of Theorem I.3.

References

1. Balaban, T.: Renormalization group approach to lattice gauge field theoreis. I. Generation of effective actions in a small field approximation and a coupling constant renormalization in four dimensions. Commun. Math. Phys. **109**, 249–301 (1987) and references therein.

Communicated by A. Jaffe

Received September 23, 1985; in revised form October 5, 1987