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One-Dimensional Schrödinger Operators with Random Decaying Potentials

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Abstract. We investigate the spectrum of the following random Schrödinger operators:

$$H(\omega)=-\frac{d^2}{dt^2}+a(t)F(X_t(\omega)),$$

where $F(X_t(\omega))$ is a Markovian potential studied by the Russian school [8]. We completely describe the transition of the spectrum from pure point type to absolutely continuous type as the decreasing order of $a(t)$ grows. This is an extension to a continuous case of the result due to Delyon-Simon-Souillard [6], who deal with the lattice case.

1. Introduction

In this paper, we will study the one-dimensional Schrödinger operator:

$$H(\omega)=-\frac{d^2}{dt^2}+a(t)F(X_t(\omega)) \tag{1.1}$$

on $L^2(\mathbf{R}, dt)$, where $\{X_t(\omega); t\in\mathbf{R}\}$ is a Brownian motion on a compact Riemannian manifold M with the normalized Riemannian volume element μ as its marginal distribution. Then $\{X_t(\omega); t\in\mathbf{R}\}$ becomes a stationary ergodic process on M. We assume that $F\in C^\infty(M)$, $a\in C^\infty(\mathbf{R})$, $a(t)$ is non-increasing on $\mathbf{R}_+=(0,\infty)$, $a(t)=a(-t)$, and $a(t)\to 0$ as $|t|\to\infty$. It is known that $H(\omega)$ defines a self-adjoint operator on $L^2(\mathbf{R}, dt)$.

For a self-adjoint operator H on a Hilbert space, we denote by ΣH, $\Sigma_{\mathrm{p}}H$, $\Sigma_{\mathrm{sc}}H$, and $\Sigma_{\mathrm{ac}}H$ spectrum, point spectrum, singular continuous spectrum and absolutely continuous spectrum of H respectively (see Kato [12]). Our interest here is to investigate the existence or non-existence of these components of the spectrum of $H(\omega)$. Since $a(t)\to 0$ as $|t|\to\infty$, $H(\omega)$ has only discrete spectrum on $(-\infty,0)$ if any and $\Sigma H(\omega)\cap[0,\infty)=[0,\infty)$ (Reed and Simon [17]).

Let L be the generator of $\{X_t;\ t\in\mathbf{R}\}$, that is, $L=\frac{1}{2}$ of the Laplace-Beltrami operator on M. For $\lambda>0$, set:

$$\gamma(\lambda)=-\frac{1}{4\lambda}\operatorname{Re}\int_M F(x)(L+2i\lambda^{1/2})^{-1}F(x)\mu(dx),$$

which is equal to

$$=\int_0^\infty \frac{\xi}{4\lambda(\xi^2+4\lambda)}\sigma_F(d\xi), \tag{1.2}$$

where σ_F is the spectral measure of $-L$ associated with F. Therefore $\gamma(\lambda)$ is positive and strictly decreasing unless $\sigma_F(d\xi)$ concentrates on $\{0\}$, namely, unless F is a constant. We set:

$$A(t)=\left|\int_0^t a(s)^2 ds\right|,$$

$$\lambda_0=\sup\left\{\lambda\geqq 0;\ \int_{\mathbf{R}}\exp(-2\gamma(\lambda)A(t))dt<\infty\right\}.$$

We say that a smooth function $f: M\to\mathbf{R}$ is non-flattening if there exists an $n_0\geqq 1$ such that at any point $x\in M$ some differential d^kf $(1\leqq k\leqq n_0)$ is non-zero (cf. [8]). Then our theorem is the following.

Theorem. (1) *If F is non-flattening and $A(t)\to\infty$ as $|t|\to\infty$, then the spectrum of $H(\omega)$ is of pure point type on $[0,\lambda_0]$, and purely singular continuous type on $[\lambda_0,\infty)$ with probability one. Moreover, every generalized eigenfunction $\psi_\lambda(t)$ of $H(\omega)$ decays as follows:*

$$\lim_{|t|\to\infty} A(t)^{-1}\log\{\psi_\lambda(t)^2+\psi_\lambda'(t)^2\}^{1/2}=-\gamma(\lambda)$$

with probability one, which especially shows that the spectral multiplicity is one on $(0,\infty)$.

(2) *If* $\int_M F(x)\mu(dx)=0$, *$A(t)$ is bounded and* $\int_{\mathbf{R}} a(t)^2|t|^\delta dt<\infty$ *for a $\delta\in(0,1)$, then the spectrum of $H(\omega)$ is of purely absolutely continuous type on $[0,\infty)$ with probability one. Moreover, we have two independent solutions φ_λ and $\bar\varphi_\lambda$ ($=$complex conjugate of φ_λ) satisfying compact uniformly in $\lambda\in(0,\infty)$ with probability one:*

$$\varphi_\lambda(t)=\exp(i\lambda^{1/2}t)+o(1),$$

$$\varphi_\lambda'(t)=i\lambda^{1/2}\exp(i\lambda^{1/2}t)+o(1),$$

as $t\to\infty$, which especially shows that the spectral multiplicity is two on $(0,\infty)$.

When $H(\omega)=-\dfrac{d^2}{dt^2}+F(X_t(\omega))$ and F is non-flattening, it is well known that $H(\omega)$ has only point spectrum and every eigenfunction decays exponentially fast (Goldsheid et al. [8], Molchanov [16]). When $a(t)=(1+|t|)^{-\alpha}$ for sufficiently large $|t|$, the case (1) occurs if $0<\alpha\leqq\frac{1}{2}$. (A) If $0<\alpha<\frac{1}{2}$, then $\lambda_0=\infty$. (B) If $\alpha=\frac{1}{2}$, then $0<\lambda_0<\infty$. (C) If $\alpha>\frac{1}{2}$, then the case (2) occurs. Delyon et al. [6] considered a discrete model of the form: $(H(\omega)\psi)(n)=\psi(n+1)+\psi(n-1)+|n|^{-\alpha}V_n(\omega)\psi(n)$, where $V_n(\omega)$ is i.i.d., and proved the following: (A) If $0<\alpha<\frac{1}{2}$, then the spectrum is of pure point type and the eigenfunction decays fractionally exponentially fast. However, the estimate is a little rougher than the above theorem. They did not obtain the Lyapounov exponent $\gamma(\lambda)$. (B) If $\alpha=\frac{1}{2}$, then the transition from point spectrum to continuous spectrum occurs. However, they did not identify λ_0 and the unsettled region of the spectrum has remained. (C) If $\alpha>\frac{1}{2}$, then the spectrum is continuous. Delyon [5] showed in this discrete model that if $\alpha=\frac{1}{2}$, then the spectrum has no absolutely continuous part. Therefore our theorem can be regarded as a completion of their work at least in the continuous system. In the discrete system also, it is possible to show that the assertion (2) in the theorem holds, however, we have not so far succeeded in obtaining the result corresponding to (1).

To prove the theorem, we have to study the growth order of solutions of:

$$H(\omega)\psi=\lambda\psi\,, \tag{1.3}$$

for $\lambda\in\mathbf{R}$. However, since our random potentials are not stationary ergodic, Kotani's argument [13] which uses Osceledec' theorem is not available. Hence we must investigate the Lyapounov behavior of the solutions by explicit calculations, imposing a rather strong condition of randomness on the potentials. This will be done by the martingale analysis in Sect. 2. To show the existence of a decaying solution when $0<\lambda_0<\infty$ in the case (1), it is necessary to study the asymptotics of the ratio of the amplitudes of two linearly indepednent solutions of (1.3). In the case (2), once we show the compact uniform boundedness of the solution of (1.3) in λ uniformly with respect to $t\geqq 0$, we can deduce the assertion applying Carmona's lemma [3, 4].

2. Generalized Lyapounov Behavior

Equation (1.3) is equivalent to:

$$\frac{d}{dt}\begin{pmatrix}x(t,\omega,\lambda)\\ y(t,\omega,\lambda)\end{pmatrix}=\begin{pmatrix}0 & \lambda^{1/2}\\ \lambda^{-1/2}a(t)F(X_t(\omega))-\lambda^{1/2} & 0\end{pmatrix}\begin{pmatrix}x(t,\omega,\lambda)\\ y(t,\omega,\lambda)\end{pmatrix}, \tag{2.1}$$

where $y(t,\omega,\lambda)=\lambda^{-1/2}\dfrac{d}{dt}x(t,\omega,\lambda)$. We sometimes omit the dependence on parameters from now on. For this solution ${}^t(x(t,\omega),y(t,\omega))$, set:

$${}^t(x(t),y(t))=r_t{}^t(\sin\theta_t,\ \cos\theta_t)\,.$$

We want to show that the limit of $A(t)^{-1}\log r_t^2$ as $t\to\infty$ exists and it depends sensitively on a random initial phase $\theta_0(\omega)$ in the case (1) of the theorem, and the asymptotics of ${}^t(r_t,\theta_t)$ can be obtained in the case (2) of the theorem. However, instead of doing so, we treat the following vector:

$$\begin{pmatrix}\tilde{x}(t)\\ \tilde{y}(t)\end{pmatrix}=\begin{pmatrix}\cos(\lambda^{1/2}t) & -\sin(\lambda^{1/2}t)\\ \sin(\lambda^{1/2}t) & \cos(\lambda^{1/2}t)\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}, \tag{2.2}$$

and set:

$${}^t(\tilde{x}(t),\tilde{y}(t))=\tilde{r}_t{}^t(\sin\tilde{\theta}_t,\cos\tilde{\theta}_t)\,. \tag{2.3}$$

Noting that $r_t = \tilde{r}_t$ and $\theta_t = \tilde{\theta}_t + \lambda^{1/2}t$, it is enough to study the asymptotics of $\tilde{r}_t$ and $\tilde{\theta}_t$. After a short calculation, we have

$$\frac{d}{dt}\begin{pmatrix} \log \tilde{r}_t^2 \\ \tilde{\theta}_t \end{pmatrix} = \begin{pmatrix} \lambda^{-1/2}\sin 2(\tilde{\theta}_t + \lambda^{1/2}t)a(t)F(X_t) \\ -\lambda^{-1/2}\sin^2(\tilde{\theta}_t + \lambda^{1/2}t)a(t)F(X_t) \end{pmatrix}. \tag{2.4}$$

The key step to show the theorem is related to the following proposition. This will be proved through two lemmas related to the martingale analysis.

Proposition 2.1. (1) *Fix* $\lambda > 0$. *Under the assumption of* (1) *in the theorem, for any fixed* $\tilde{\theta}_0$ *(non-random), the limit* $\lim_{t\to\infty} A(t)^{-1}\log \tilde{r}_t = \gamma(\lambda)$ *exists with probability one. Moreover, there exists a unique initial phase* $\tilde{\theta}_0(\omega)$ *such that if a solution of* (2.4) *satisfies* $\tilde{\theta}_0 = \tilde{\theta}_0(\omega)$, *then*

$$\lim_{t\to\infty} A(t)^{-1}\log \tilde{r}_t = -\gamma(\lambda),$$

and any other solution grows as:

$$\lim_{t\to\infty} A(t)^{-1}\log \tilde{r}_t = \gamma(\lambda)$$

with probability one.

(2) *Under the assumption of* (2) *in the theorem, for any solution of* (2.4), $\tilde{r}_\infty = \lim_{t\to\infty} \tilde{r}_t$ *and* $\tilde{\theta}_\infty = \lim_{t\to\infty} \tilde{\theta}_t$ *exist and satisfy:*

$$\tilde{r}_t = \tilde{r}_\infty + o(1), \quad \text{and} \quad \tilde{\theta}_t = \tilde{\theta}_\infty + o(1)$$

as $t \to \infty$, *compact uniformly in* $\lambda \in (0, \infty)$ *with probability one.*

Lemma 2.1. *Let* $f \in C^1(M)$ *and* $\beta \in \mathbf{R} - \{0\}$. *Then there exist* $g(t, x, \lambda)$ *and* $g_1(x)$ *satisfying:*

$$g \in C^{\infty,2,\infty}(\mathbf{R}_+ \times M \times \mathbf{R}_+), \qquad g_1 \in C^2(M),$$

and complex martingales $M(t, \omega, \lambda)$ *and* $M_1(t, \omega)$ *satisfying:*

$$\begin{cases} \exp(i\beta\lambda^{1/2}t)f(X_t)dt = dg(t, X_t) - dM(t), \\ (f(X_t) - \bar{f})dt = dg_1(X_t) - dM_1(t), \end{cases}$$

where $\bar{f} = \int_M f(x)\mu(dx)$. *Moreover we have:*

$$\langle \operatorname{Re} M\rangle_t = \int_0^t \{L(\operatorname{Re} g)^2 - 2\cdot \operatorname{Re} g \cdot L(\operatorname{Re} g)\}(s, X_s)ds,$$

and so on for $\operatorname{Im} M$, $\operatorname{Re} M_1$, *and* $\operatorname{Im} M_1$, *which especially shows that*

$$d\langle \operatorname{Re} M\rangle_t \leqq \text{const}\, dt,$$

and so on, where the const *depends on* $\lambda \in (0, \infty)$ *compact uniformly. Here we denote by* $\langle\cdot\rangle$ *the bracket in the martingale theory.*

Proof. Set g and g_1 as follows:

$$g = \exp(i\beta\lambda^{1/2}t)(L + i\beta\lambda^{1/2})^{-1}f, \qquad g_1 = L^{-1}(f - \bar{f}).$$

Here we note that since $\int_M (f(x)-\bar{f})\mu(dx)=0$, g_1 exists uniquely by Fredholm's alternative theorem. Then this lemma is nothing but a consequence of Ito's formula (Ikeda and Watanabe [9]).

Lemma 2.2. *Let $f\in C^1(M)$, $\beta\in\mathbf{R}-\{0\}$ and $k\in\mathbf{Z}_+=\{1,2,\ldots\}$.*

(1) *Fix $\lambda>0$. Under the assumption of* (1) *in the theorem, the following holds with probability one:*

$$\lim_{t\to\infty} A(t)^{-1}\int_0^t \exp(i\beta(\tilde{\theta}_s+\lambda^{1/2}s))f(X_s)a(s)^k ds$$
$$=\begin{cases} 0 & \text{when } k\neq 1 \text{ or } \beta\neq \pm 2,\\ -\frac{1}{4}i\beta\lambda^{-1/2}\int_M F(x)(L+i\beta\lambda^{1/2})^{-1}f(x)\mu(dx) & \text{otherwise}.\end{cases}$$

(2) *Under the assumption of* (2) *in the theorem,*

$$\lim_{t\to\infty}\int_0^t \exp(i\beta(\tilde{\theta}_s+\lambda^{1/2}s))f(X_s)a(s)ds$$

exists and

$$\int_t^\infty \exp(i\beta(\tilde{\theta}_s+\lambda^{1/2}s))f(X_s)a(s)ds=o(1)$$

as $t\to\infty$ holds compact uniformly in $\lambda\in(0,\infty)$ with probability one.

Proof. (1) Taking g in Lemma 2.1, we have:

$$\int_0^t \exp(i\beta(\tilde{\theta}_s+\lambda^{1/2}s))f(X_s)a(s)^k ds=\int_0^t \exp(i\beta\tilde{\theta}_s)a(s)^k(dg-dM)_s:=l(t)-m(t). \quad (2.5)$$

We first show that $A(t)^{-1}m(t)\to 0$ as $t\to\infty$. By Lemma 2.1, it is easy to see that:

$$\langle \mathrm{Re}\, m\rangle_t+\langle \mathrm{Im}\, m\rangle_t \leqq c\int_0^t a(s)^{2k}ds \quad (2.6)$$

with a constant c depending only on λ. On the other hand, for a real continuous martingale $n(t)$, there exists a one-dimensional Brownian motion $B(t)$ such that $n(t)=B(\langle n\rangle_t)$ holds (Ikeda and Watanabe [9]). Since $B(t)=o(t)$ as $t\to\infty$, $m(t)=o\left(\int_0^t a(s)^{2k}ds\right)$. Therefore $A(t)^{-1}m(t)\to 0$ as $t\to\infty$ follows for every $k\in\mathbf{Z}$.

As for $l(t)$ in (2.5), integration by parts shows that the following holds:

$$l(t)=[\exp(i\beta\tilde{\theta}_s)a(s)^k g(s)]_0^t - i\beta\int_0^t \exp(i\beta\tilde{\theta}_s)a(s)^k g(s)\tilde{\theta}'_s ds$$
$$-\int_0^t \exp(i\beta\tilde{\theta}_s)ka(s)'a(s)^{k-1}g(s)ds. \quad (2.7)$$

We treat only the second term and set it as $l_1(t)$. It is easy to see that other terms are of $o(A(t))$ as $t\to\infty$. Noting that $\tilde{\theta}'$ satisfies (2.4) and g is defined in Lemma 2.1, we have:

$$l_1(t)=\tfrac{1}{2}i\beta\lambda^{-1/2}\int_0^t \exp(i\beta(\widetilde{\theta}_s+\lambda^{1/2}s))F(X_s)(L+i\beta\lambda^{1/2})^{-1}f(X_s)a(s)^{k+1}ds$$

$$-\tfrac{1}{4}i\beta\lambda^{-1/2}\int_0^t \exp(i(\beta+2)(\widetilde{\theta}_s+\lambda^{1/2}s))F(X_s)(L+i\beta\lambda^{1/2})^{-1}f(X_s)a(s)^{k+1}ds$$

$$-\tfrac{1}{4}i\beta\lambda^{-1/2}\int_0^t \exp(i(\beta-2)(\widetilde{\theta}_s+\lambda^2 s))F(X_s)(L+i\beta\lambda^{1/2})^{-1}f(X_s)a(s)^{k+1}ds. \tag{2.8}$$

When $\beta \neq \pm 2$, every term in (2.8) is of the form:

$$\int_0^t \exp(i\beta(\widetilde{\theta}_s+\lambda^{1/2}s))f(X_s)a(s)^{k+1}ds \tag{2.9}$$

with another $\beta \neq 0$ and f. However, we can repeat the above argument and replace k by $k+1$, to see that (2.9) is of $o(A(t))$ for every $k\in\mathbf{Z}_+$. When $\beta=2$, the main term in (2.8) is:

$$-\tfrac{1}{4}i\beta\lambda^{-1/2}\int_0^t F(X_s)(L+i\beta\lambda^{1/2})^{-1}f(X_s)a(s)^{k+1}ds. \tag{2.10}$$

If $k\geqq 2$, then this is of $o(A(t))$ as $t\to\infty$. When $k=1$, setting $f_1=F\cdot(L+i\beta\lambda^{1/2})^{-1}f$, we apply Lemma 2.1 to f_1 and obtain:

$$(2.10)=-\tfrac{1}{4}i\beta\lambda^{-1/2}\int_0^t \bar{f}_1 a(s)^2ds-\tfrac{1}{4}i\beta\lambda^{-1/2}\int_0^t a(s)^2(dg_1-dM_1).$$

The second term is of $o(A(t))$ by the same procedure as before. Noting:

$$\bar{f}_1=\int_M F(x)(L+i\beta\lambda^{1/2})^{-1}f(x)\mu(dx),$$

we come to the conclusion in the case (1).

(2) We return to (2.5). We deal with the case $k=1$. It is almost obvious that the assertion holds for $l(t)$ noting that every term in (2.8) is absolutely convergent. As for $m(t)$, we see that for fixed $\lambda>0$, $\lim_{t\to\infty} m(t)$ exists finitely with probability one because we can represent $\mathrm{Re}\,m(t)$ as $B(\langle \mathrm{Re}\,m\rangle_t)$, where B is a 1-dim. Brownian motion and $\langle \mathrm{Re}\,m\rangle_t$ is bounded in this case in view of (2.6), and the same argument is valid for $\mathrm{Im}\,m(t)$. To proceed, we need a more detailed analysis. Noting that:

$$\mathrm{Re}\,m(t)=-\int_0^t \cos(\beta\widetilde{\theta}_s)a(s)d(\mathrm{Re}\,M)_s+\int_0^t \sin(\beta\widetilde{\theta}_s)a(s)d(\mathrm{Im}\,M)_s,$$

we deal only with the second term and set:

$$I(t,\omega,\lambda)=\int_0^t \sin(\beta\widetilde{\theta}_s)a(s)d(\mathrm{Im}\,M)_s. \tag{2.11}$$

We must prove the compact uniform boundedness of I in λ. To this end, we regard $I(t,\omega,\lambda)$ as a continuous bounded function space-valued stochastic process, i.e., $\lambda\to I(\cdot,\omega,\lambda)\in C_b([0,\infty))$, and apply Kolmogorov's modification theorem in this

Banach space (C_b, sup norm $\|\cdot\|_\infty$). We have:

$$I(t,\omega,\lambda)-I(t,\omega,\lambda')=\int_0^t \{\sin(\beta\tilde\theta_s(\lambda))-\sin(\beta\tilde\theta_s(\lambda'))\}a(s)d(\operatorname{Im}M(\lambda))_s$$
$$+\int_0^t \sin(\beta\tilde\theta_s(\lambda'))a(s)\{d(\operatorname{Im}M(\lambda))_s-d(\operatorname{Im}M(\lambda'))_s\}\,.$$

We discuss only the first term and denote it by $I(t,\omega,\lambda,\lambda')$. The second term can be treated more easily.

Sublemma. *Let $K\subset(0,\infty)$ be a compact set and $p>0$. Then we have*

$$\mathbf{E}\sup_{t\in[0,\infty)}|I(t,\omega,\lambda,\lambda')|^{2p}\leqq \mathrm{const}\,|\lambda-\lambda'|^{2\eta p}\,,$$

where the constant is independent of $\lambda,\lambda'\in K$, and $0<\eta\leqq\delta$ (δ is introduced in (2) *of Theorem).*

Proof. Fix $t_0>0$. We use moment inequality in the martingale theory (Ikeda and Watanabe [9]) and we have:

$$\mathbf{E}\sup_{s\in[t_0,t]}|I(s,\omega,\lambda,\lambda')-I(t_0,\omega,\lambda,\lambda')|^{2p}$$
$$\leqq \mathrm{const}\,\mathbf{E}\left\{\int_{t_0}^t|\sin(\beta\tilde\theta(s,\lambda))-\sin(\beta\tilde\theta(s,\lambda'))|^2a(s)^2d\langle\operatorname{Im}M(\lambda)\rangle_s\right\}^p$$
$$\leqq \mathrm{const}\,\mathbf{E}\sup_{s\in[t_0,t]}|\tilde\theta(s,\lambda)-\tilde\theta(s,\lambda')|^{2p}\left\{\int_{t_0}^t a(s)^2ds\right\}^p. \tag{2.12}$$

We estimate $\mathbf{E}\sup|\tilde\theta(s,\lambda)-\tilde\theta(s,\lambda')|^{2p}$. We rewrite (2.4) as follows:

$$\frac{d}{dt}\tilde\theta(t,\lambda)=-\tfrac12\lambda^{-1/2}\{1-\cos 2(\tilde\theta(t,\lambda)+\lambda^{1/2}t)\}a(t)F(X_t)\,.$$

We apply Lemma 2.1 and obtain:

$$\tilde\theta(t,\lambda)=\tilde\theta(t_0,\lambda)-\tfrac12\lambda^{-1/2}\int_{t_0}^t a(s)\,(dg_1-dM_1)_s$$
$$+\tfrac12\lambda^{-1/2}\operatorname{Re}\int_{t_0}^t \exp(2i\tilde\theta(s,\lambda))a(s)\,(dg-dM)_s$$
$$=\tilde\theta(t_0,\lambda)-\tfrac12\lambda^{-1/2}[a(s)g_1(s)]_{t_0}^t+\tfrac12\lambda^{-1/2}\int_{t_0}^t\{g_1(s)a'(s)ds+a(s)dM_1\}$$
$$+\tfrac12\lambda^{-1/2}\operatorname{Re}[\exp(2i\tilde\theta(s,\lambda))a(s)g(s)]_{t_0}^t$$
$$-\tfrac12\lambda^{-1/2}\operatorname{Re}\int_{t_0}^t 2i\tilde\theta'(s,\lambda)\exp(2i\tilde\theta(s,\lambda))a(s)g(s)ds$$
$$-\tfrac12\lambda^{-1/2}\operatorname{Re}\int_{t_0}^t \exp(2i\tilde\theta(s,\lambda))a'(s)g(s)ds$$
$$-\tfrac12\lambda^{-1/2}\operatorname{Re}\int_{t_0}^t \exp(2i\tilde\theta(s,\lambda))a(s)dM_s\,.$$

Here we have used $\bar{F}=\int_M F(x)\mu(dx)=0$. Set:

$$J(s)=\tilde{\theta}(s,\lambda)-\tilde{\theta}(s,\lambda').$$

We note that for any $0<\eta\leqq 1$, the following holds when $\lambda,\lambda'\in K$:

$$|\lambda^{-1/2}-\lambda'^{-1/2}|\leqq \mathrm{const}\,|\lambda-\lambda'|^{\eta}.$$

We also note that returning to the proof of Lemma 2.1, we have:

$$|g(s,\lambda)-g(s,\lambda')|\leqq \mathrm{const}\,|\lambda-\lambda'|^{\eta}(1+|s|^{\eta}),$$

and

$$d\langle \operatorname{Re}M(\lambda)-\operatorname{Re}M(\lambda')\rangle_s\leqq \mathrm{const}\,|\lambda-\lambda'|^{\eta}(1+|s|^{\eta})ds,$$

which is also correct for $\operatorname{Im}M$. From the above, it is easy to see that the following holds using moment inequality again:

$$\begin{aligned}\mathbf{E}\sup_{s\in[t_0,t]}|J(s)|^{2p}\leqq{}& \mathrm{const}\,\mathbf{E}|J(t_0)|^{2p}+\mathrm{const}\Big\{|\lambda-\lambda'|^{\eta}\int_{t_0}^{t}(1+|s|^{\eta})a(s)^2ds\Big\}^{2p}\\ &+\mathrm{const}\left\{|\lambda-\lambda'|^{2\eta}\int_{t_0}^{t}(1+|s|^{\eta})a(s)^2ds\right\}^{p}\\ &+\mathrm{const}\,\mathbf{E}\sup_{s\in[t_0,t]}|J(s)|^{2p}\left\{\int_{t_0}^{t}(1+|s|^{\eta})a(s)^2ds\right\}^{2p}\\ &+\mathrm{const}\,\mathbf{E}\sup_{s\in[t_0,t]}|J(s)|^{2p}\left\{\int_{t_0}^{t}(1+|s|^{\eta})a(s)^2ds\right\}^{p}.\end{aligned}$$

Taking t_0 sufficiently large such that the last two terms in the right-hand side are smaller than the left-hand side, and noting that for fixed $t_0>0$:

$$\mathbf{E}|J(t_0)|^{2p}\leqq \mathrm{const}\,|\lambda-\lambda'|^{2\eta p},$$

we have

$$\mathbf{E}\sup_{s\in[t_0,t]}|J(s)|^{2p}\leqq \mathrm{const}\,|\lambda-\lambda'|^{2\eta p}.$$

This together with (2.12) shows:

$$\mathbf{E}\sup_{s\in[t_0,t]}|I(s,\omega,\lambda,\lambda')-I(t_0,\omega,\lambda,\lambda')|^{2p}\leqq \mathrm{const}\,|\lambda-\lambda'|^{2\eta p}.$$

Since it is obvious that:

$$\mathbf{E}\sup_{s\in[0,t_0]}|I(s,\omega,\lambda,\lambda')|^{2p}\leqq \mathrm{const}\,|\lambda-\lambda'|^{2\eta p},$$

we come to the conclusion of the sublemma.

We return to the proof of (2) of Lemma 2.2. If we take $p>0$ sufficiently large such that $2\eta p>1$ holds, then the sublemma immediately shows that Kolmogorov's theorem holds. Therefore we can regard $\lambda\to I(\cdot,\omega,\lambda)\in C_b([0,\infty))$ as a continuous map and we have:

$$\sup_{\lambda\in K}\|I(\cdot,\omega,\lambda)\|_{\infty}=\sup_{\lambda\in K}\sup_{t\in[0,\infty)}|I(t,\omega,\lambda)|<\infty,$$

where $K \subset (0, \infty)$ is a compact set. To complete the proof, we must deal with the term of the form $m(\infty) - m(t)$. However, the argument is almost parallel to the previous one. As in (2.11), we set:

$$\tilde{I}(t, \omega, \lambda) = \int_t^\infty \sin(\beta \tilde{\theta}_s) a(s) d(\operatorname{Im} M)_s .$$

We regard $\tilde{I}$ as a Banach space-valued stochastic process:

$$\lambda \to \tilde{I}(\cdot, \omega, \lambda) \in C_0([0, \infty)),$$

where $C_0([0, \infty)) = \left\{ f \in C([0, \infty));\ \lim_{t \to \infty} f(t) = 0 \right\}$ equipped with the sup norm, and apply Kolmogorov's theorem. Then it is clear that:

$$\sup_{\lambda \in K} |\tilde{I}(t, \omega, \lambda)| = o(1), \quad \text{as} \quad |t| \to \infty$$

with probability one, where $K \subset (0, \infty)$ is a compact set. Now we have finished the proof of Lemma 2.2.

Proof of (2) *of Proposition 2.1.* Applying Lemma 2.2 to Eq. (2.4), we immediately obtain our result.

To prove (1) of Proposition 2.1, we need more arguments. From (2.4) we have:

$$A(t)^{-1} \log \tilde{r}_t = A(t)^{-1} \log \tilde{r}_0 + A(t)^{-1} \cdot \tfrac{1}{2} \lambda^{-1/2} \operatorname{Im} \int_0^t \exp(2i(\tilde{\theta}_s + \lambda^{1/2} s)) a(s) F(X_s) ds .$$

By Lemma 2.2, we see that the following holds:

$$\begin{aligned} \lim_{t \to \infty} A(t)^{-1} \log \tilde{r}_t &= -\frac{1}{4\lambda} \operatorname{Im} i \int_M F(x) (L + 2i\lambda^{1/2})^{-1} F(x) \mu(dx) \\ &= \gamma(\lambda) . \end{aligned}$$

We note that this holds with probability one as far as any initial phase $\tilde{\theta}_0$ is fixed and non-random. We take two independent vectors

$$r_j(t, \omega, \lambda)^t (\sin \theta_j(t, \omega, \lambda),\ \cos \theta_j(t, \omega, \lambda)), \qquad (j = 1, 2)$$

satisfying (2.4) and set:

$$U(t, \omega, \lambda) = \begin{pmatrix} r_1 \sin \theta_1 & r_2 \sin \theta_2 \\ r_1 \cos \theta_1 & r_2 \cos \theta_2 \end{pmatrix}.$$

We assume $U(0, \omega, \lambda) \in SL(2, \mathbf{R})$. Then it is easy to see that $U(t, \omega, \lambda) \in SL(2, \mathbf{R})$ holds for any $t > 0$. So far we have proved:

$$\lim_{t \to \infty} A(t)^{-1} \log \| U(t, \omega, \lambda) \| = \gamma(\lambda) \tag{2.13}$$

holds with probability one. Our aim is to seek the characteristic random initial phase described in (1) in Proposition 2.1. When the potential is not decaying, i.e., $A(t) = t$, it is a customary way to apply Osceledec' theorem to this end. Let us recall a deterministic version of Osceledec' theorem.

Osceledec' Pseudo-Theorem (see Ruelle [18]). *Let* $U(t) \in SL(2, \mathbf{R})$ *for* $t \geqq 0$. *Assume:*

$$\lim_{t \to \infty} A(t)^{-1} \sup_{s \in [0,1]} \log \| U(t+s) U(t)^{-1} \| = 0,$$

and

$$\lim_{t \to \infty} A(t)^{-1} \log \| U(t) \| = \gamma .$$

Then there exists a $\theta \in [0, \pi]$ *such that:*

$$\lim_{t \to \infty} A(t)^{-1} \log \| U(t) \cdot \theta \| = -\gamma ,$$

and

$$\lim_{t \to \infty} A(t)^{-1} \log \| U(t) \cdot \varphi \| = \gamma$$

hold for any $\varphi = \theta \bmod \pi$. *Here, for a* 2×2 *matrix* U *and* θ, *the operation* $U \cdot \theta$ *is defined as follows:*

$$U \cdot \theta = U^t (\sin \theta, \cos \theta) .$$

This theorem can be proved for the case $A(t) = t$. For general $A(t)$, we want to apply this theorem to each ω such that (2.13) holds. However, if $\exp\{-2\gamma A(t)\}$ is not integrable on $[0, \infty)$, then the above theorem is not generally true, although if $\exp\{-2\gamma A(t)\}$ is integrable on $[0, \infty)$, then we know the theorem is correct imitating Ruelle's argument. Therefore we want to show that the above Pseudo-Theorem holds with probability one (not a priori for each ω) in our case. Let:

$$|U(t, \omega, \lambda)| = \{ U(t, \omega, \lambda)^* U(t, \omega, \lambda) \}^{1/2} , \tag{2.14}$$

$0 < \lambda_- \leqq \lambda_+$ be two eigenvalues of $|U(t, \omega, \lambda)|$.

Then we have the following spectral decomposition:

$$|U(t, \omega, \lambda)| = \lambda_- P(t, \omega, \lambda) + \lambda_+ (I - P(t, \omega, \lambda)) , \tag{2.15}$$

where $P(t, \omega, \lambda)$ is a projection matrix.

Lemma 2.4. *Let* $\lambda > 0$ *be fixed. Under the assumption of* (1) *in the theorem, with probability one, there exists a projection matrix* $P(\infty, \omega, \lambda)$ *such that the following holds:*

$$\limsup_{t \to \infty} A(t)^{-1} \log \| P(t, \omega, \lambda) - P(\infty, \omega, \lambda) \| \leqq -2\gamma(\lambda) .$$

Proof. Owing to the relation (2.14) and (2.15), we can explicitly calculate every component of $P(t, \omega, \lambda)$ and obtain:

$$P(t, \omega, \lambda) = -\{ (r_1^2(t) + r_2^2(t))^2 - 4 \}^{-1/2} \begin{pmatrix} P_{11}(t, \omega, \lambda) & P_{12}(t, \omega, \lambda) \\ P_{21}(t, \omega, \lambda) & P_{22}(t, \omega, \lambda) \end{pmatrix} ,$$

where

$$P_{11} = \tfrac{1}{2} [r_1^2 - r_2^2 - \{ (r_1^2 + r_2^2)^2 - 4 \}^{1/2}] ,$$

$$P_{12} = P_{21} = r_1 r_2 \cos(\theta_1 - \theta_2) ,$$

$$P_{22} = \tfrac{1}{2} [r_2^2 - r_1^2 - \{ (r_1^2 + r_2^2)^2 - 4 \}^{1/2}] .$$

Set:

$$R(t,\omega,\lambda)=r_1(t)\cdot r_2(t)^{-1}.$$

Sublemma. *Let $\lambda>0$ be fixed. Under the assumption of* (1) *in the theorem, with probability one, $\lim_{t\to\infty} R(t,\omega,\lambda)=R(\infty,\omega,\lambda)$ exists and*

$$\limsup_{t\to\infty} A(t)^{-1}\log|R(t,\omega,\lambda)-R(\infty,\omega,\lambda)|\leqq -2\gamma(\lambda)$$

holds.

Proof. First we note that since $U(t,\omega,\lambda)\in SL(2,\mathbf{R})$, we have:

$$\det U(t,\omega,\lambda)=r_1(t)r_2(t)\sin(\theta_1(t)-\theta_2(t))=1,$$

and therefore from Proposition 2.1 we have:

$$\begin{aligned}\lim_{t\to\infty} A(t)^{-1}\log\{\sin(\theta_1(t)-\theta_2(t))\} &= -\lim_{t\to\infty} A(t)^{-1}\{\log r_1(t)+\log r_2(t)\}\\ &= -2\gamma(\lambda). \qquad (2.16)\end{aligned}$$

From (2.4) we have:

$$\begin{aligned}R(t,\omega,\lambda)&=R(0,\omega,\lambda)\exp\left[\tfrac{1}{2}\lambda^{-1/2}\int_0^t\{\sin 2(\theta_1(s)+\lambda^{1/2}s)\right.\\ &\qquad\left.-\sin 2(\theta_2(s)+\lambda^{1/2}s)\}a(s)F(X_s)ds\right]\\ &=R(0,\omega,\lambda)\exp\left\{\tfrac{1}{2}\lambda^{-1/2}\int_0^t\cos(\theta_1(s)+\theta_2(s)+2\lambda^{1/2}s)\right.\\ &\qquad\left.\times\sin(\theta_1(s)-\theta_2(s))a(s)F(X_s)ds\right\}.\end{aligned}$$

Set:

$$Q(t,\omega,\lambda)=\lambda^{-1/2}\int_0^t\cos\{\theta_1(s)+\theta_2(s)+2\lambda^{1/2}s\}\sin\{\theta_1(s)-\theta_2(s)\}a(s)F(X_s)ds.$$

Noting that, when x and y are in a compact set in $\mathbf{R}$:

$$|e^x-e^y|\leqq \mathrm{const}\,|x-y|,$$

we have only to prove that for fixed $\lambda>0$, $\lim_{t\to\infty} Q(t,\omega,\lambda)=Q(\infty,\omega,\lambda)$ exists and

$$\lim_{t\to\infty} A(t)^{-1}\log|Q(t,\omega,\lambda)-Q(\infty,\omega,\lambda)|\leqq -2\gamma(\lambda)$$

holds with probability one.

Applying Lemma 2.1, we have:

$$Q(t,\omega,\lambda)=\lambda^{-1/2}\,\mathrm{Re}\int_0^t\exp\{i(\theta_1(s)+\theta_2(s))\}\sin(\theta_1(s)-\theta_2(s))a(s)\,(dg-dM).$$

As we have often discussed in this paper, we deal only with the term of the form:

$$m(t,\omega,\lambda)=\lambda^{-1/2}\int_0^t \cos(\theta_1(s)+\theta_2(s))\sin(\theta_1(s)-\theta_2(s))a(s)d(\mathrm{Re}\,M)_s\,.$$

There is a one-dimensional Brownian motion B such that $m(t,\omega,\lambda)=B(\langle m\rangle_t)$ holds. On the other hand, from (2.16), for any $\varepsilon>0$, there exists a $t_0>0$ such that if $t\geqq t_0$, then

$$\sin(\theta_1(t)-\theta_2(t))\leqq\exp\{-2(\gamma(\lambda)-\varepsilon)A(t)\}\,.$$

Therefore if $t_0\leqq t<T$, then the following holds:

$$\begin{aligned}\langle m\rangle_T-\langle m\rangle_t&=\lambda^{-1}\int_t^T \cos^2(\theta_1(s)+\theta_2(s))\sin^2(\theta_1(s)-\theta_2(s))a(s)^2d\langle\mathrm{Re}\,M\rangle_s\\&\leqq\mathrm{const}\int_t^T\exp\{-4(\gamma(\lambda)-\varepsilon)A(s)\}a(s)^2ds\\&=\mathrm{const}\,|\exp\{-4(\gamma(\lambda)-\varepsilon)A(T)\}-\exp\{-4(\gamma(\lambda)-\varepsilon)A(t)\}|\,.\end{aligned}$$

From this it is clear that $\lim_{t\to\infty} m(t,\omega,\lambda)=m(\infty,\omega,\lambda)$ exists and

$$m(t,\omega,\lambda)-m(\infty,\omega,\lambda)=o(\exp\{-\alpha(\gamma(\lambda)-\varepsilon)A(t)\})$$

holds for any $\alpha<2$, using Lévy's Hölder continuity of Brownian motion (Ito and McKean [10]). Since $\varepsilon>0$ and $\alpha<2$ are arbitrary, we come to the conclusion of the sublemma.

We return to the proof of Lemma 2.4. After a short calculation, we know that:

$$\begin{aligned}\|P(T,\omega,\lambda)-P(t,\omega,\lambda)\|=O(&|R(T,\omega,\lambda)-R(t,\omega,\lambda)|+|R^{-1}(T,\omega,\lambda)-R^{-1}(t,\omega,\lambda)|\\&+|\exp\{-2\gamma(\lambda)A(T)\}-\exp\{-2\gamma(\lambda)A(t)\}|)\,.\end{aligned}$$

Since the sublemma holds also for $R^{-1}(t,\omega,\lambda)$, we have finished the proof of Lemma 2.4.

Now we can identify θ such that

$$\lim_{t\to\infty} A(t)^{-1}\log\|U(t,\omega,\lambda)\cdot\theta\|=-\gamma(\lambda) \tag{2.17}$$

holds. Let θ be an angle such that:

$${}^t(\sin\theta,\ \cos\theta)\in P(\infty,\omega,\lambda)\mathbf{R}^2$$

holds. Since $P(\infty,\omega,\lambda)\cdot\theta={}^t(\sin\theta,\ \cos\theta)$, by (2.15), Lemma 2.4 and the following relation:

$$\lim_{t\to\infty} A(t)^{-1}\log\lambda_\pm=\pm\gamma(\lambda),$$

we have:

$$\begin{aligned}\limsup_{t\to\infty} A(t)^{-1}\log\|U(t,\omega,\lambda)\cdot\theta\|&=\limsup_{t\to\infty} A(t)^{-1}\log\|\,|U(t,\omega,\lambda)\cdot\theta|\,\|\\&\leqq\limsup_{t\to\infty} A(t)^{-1}\log[\|\lambda_-P(t,\omega,\lambda)\cdot\theta\|\\&\qquad\vee\|\lambda_+\{P(\infty,\omega,\lambda)-P(t,\omega,\lambda)\}\cdot\theta\|]\\&\leqq-\gamma(\lambda)\,.\end{aligned}$$

On the other hand, it is easy to see that for $U(t) \in SL(2, \mathbf{R})$ satisfying

$$\lim_{t\to\infty} A(t)^{-1} \log \|U(t)\| = \gamma(\lambda),$$

$$\liminf_{t\to\infty} A(t)^{-1} \log \|U(t)\cdot\theta\| \geqq -\gamma(\lambda)$$

holds for any θ. Therefore we have (2.17). It is clear that for any $\varphi \neq \theta \bmod \pi$,

$$\lim_{t\to\infty} A(t)^{-1} \log \|U(t,\omega,\lambda)\cdot\varphi\| = \gamma(\lambda)$$

holds in view of (2.13). We have completely finished the proof of Proposition 2.1.

3. Some Notations and Results from the Spectral Theory of One-Dimensional Schrödinger Operators

Let:

$$H_q = -\frac{d^2}{dt^2} + q(t)$$

on $L^2(\mathbf{R}, dt)$, where q is a bounded function. Since both the end points $\pm\infty$ are of the limit point type in Weyl's classification, there exists a spectral 2×2 matrix measure $\Sigma(d\lambda)$. Let $\sigma = \operatorname{trace}\Sigma$, which is usually called the spectral measure. Since every component of Σ is absolutely continuous with respect to σ, we have only to investigate σ.

It is well known that there exist generalized real-valued solutions $f_j(t,\xi)$ $(j=1,2)$ for $\sigma(d\xi)$-a.e. ξ such that the Green function $g_\lambda(t,s;q)$ is written in the following form:

$$g_\lambda(t,s;q) = \int_{\mathbf{R}} \frac{f_1(t,\xi)f_1(s,\xi) + f_2(t,\xi)f_2(s,\xi)}{\xi - \lambda}\sigma(d\xi). \tag{3.1}$$

See Kotani [14]. f_1 and f_2 are dependent or not accordingly as the multiplicity of the spectrum is 1 or 2.

Lemma 3.1. *We have for $j=1$ and 2 the following:*

$$\sup_{t\in\mathbf{R}} \int_{\mathbf{R}} \frac{f_j(t,\xi)^2 + f_j'(t,\xi)^2}{1+\xi^2}\sigma(d\xi) < \infty.$$

Proof. In view of (3.1), we have only to prove the following:

$$\operatorname{Im} g_i(t,t;q) + \operatorname{Im}\frac{\partial^2}{\partial t\partial s} g_i(t,s;q)\Big|_{s=t} \leqq c < \infty,$$

where c is independent of $t\in\mathbf{R}$. From the well-known equation:

$$(H_q - i)^{-1} = (H_0 - i)^{-1} - (H_0 - i)^{-1} q (H_0 - i)^{-1} + (H_0 - i)^{-1} q (H_q - i)^{-1} q (H_0 - i)^{-1}$$

and the exponential decay of $g_i(t,s;q) = (H_q - i)^{-1}(t,s)$ as $|t-s|\to\infty$, it is not difficult to show the assertion. We omit the detail.

Let $V(t,\lambda)\in SL(2,\mathbf{R})$ be the solution of the following equation:

$$\begin{cases} \dfrac{d}{dt}V(t,\lambda)=\begin{pmatrix} 0 & 1 \\ q(t)-\lambda & 0\end{pmatrix}V(t,\lambda) \\ V(0,\lambda)=1\,. \end{cases}$$

Lemma 3.2 (Carmona [3, 4]). *$\sigma(d\lambda)$ can be written in the form:*

$$\sigma(d\lambda)=\operatorname*{weak-lim}_{t\to\infty}\frac{1}{\pi}\int_0^\pi d\theta\|V(t,\lambda)\cdot\theta\|^{-2}\sigma_-^\theta(d\lambda),$$

where

$$\sigma_-^\theta(d\lambda)=\operatorname*{weak-lim}_{s\to-\infty}\frac{1}{\pi}\|V(s,\lambda)\cdot\theta\|^{-2}d\lambda\,.$$

Moreover

$$\int_0^\pi d\theta\sigma_-^\theta(d\lambda)=d\lambda\,.$$

Lemma 3.3 (Carmona [4]). *Let $A\subset\mathbf{R}$ be a Borel set. If*

$$\sup_{t\in[0,\infty),\,\lambda\in A}\|V(t,\lambda)\|\leqq c<\infty\,,$$

then $\sigma(d\lambda)$ is equivalent to $d\lambda$ on A.

Proof of Lemma 3.3. Since $V(t,\lambda)\in SL(2,\mathbf{R})$, it is easy to see that

$$\|V(t,\lambda)\|^{-1}\leqq\|V(t,\lambda)\cdot\theta\|\leqq\|V(t,\lambda)\|$$

holds for any θ. Then Lemma 3.2 implies:

$$\frac{1}{\pi c^2}\int_0^\pi d\theta\sigma_-^\theta(d\lambda)\leqq\sigma(d\lambda)\leqq\frac{c^2}{\pi}\int_0^\pi d\theta\sigma_-^\theta(d\lambda),$$

and therefore we have:

$$\frac{1}{\pi c^2}d\lambda\leqq\sigma(d\lambda)\leqq\frac{c^2}{\pi}d\lambda\quad\text{on } A\,.$$

4. Proof of Theorem

We first deal with the case (2) of the theorem. Let $K\subset(0,\infty)$ be a compact set. Then (2) of Proposition 2.1 and Lemma 3.3 show that the spectrum of $H(\omega)$ consists only of absolutely continuous part on K with probability one. Since $K\subset(0,\infty)$ is arbitrary, we have proved the first part of (2). We take two independent solutions ${}^t(\tilde{x}_j(t),\tilde{y}_j(t))$ $(j=1,2)$ which satisfy (2.3), (2.4) and:

$$\lim_{t\to\infty}{}^t(\tilde{x}_1(t),\tilde{y}_1(t))={}^t(1,0)\,,\qquad \lim_{t\to\infty}{}^t(\tilde{x}_2(t),\tilde{y}_2(t))={}^t(0,1)\,.$$

Considering (2.2) we set:

$$\begin{pmatrix}\varphi_\lambda(t)\\ \lambda^{-1/2}\dfrac{d}{dt}\varphi_\lambda(t)\end{pmatrix}=\begin{pmatrix}\cos(\lambda^{1/2}t) & \sin(\lambda^{1/2}t)\\ -\sin(\lambda^{1/2}t) & \cos(\lambda^{1/2}t)\end{pmatrix}\begin{pmatrix}\tilde{x}_1(t)+i\tilde{x}_2(t)\\ \tilde{y}_1(t)+i\tilde{y}_2(t)\end{pmatrix}.$$

Then it is easy to see that this $\varphi_\lambda(t)$ satisfies the asymptotic estimate described in (2) of the theorem.

We next deal with the case (1) of the theorem.

Lemma 4.1 (Kotani [13]). *Let $A \subset \mathbf{R}$ be a compact set and $\mathscr{F}_\pm$ be σ-fields generated by $\{X_s; s \in \mathbf{R}_\pm\}$. Suppose that F is non-flattening. Then the conditional expectation $\mathbf{E}[\sigma(d\lambda, \omega)|\mathscr{F}_\pm]$ is absolutely continuous with respect to $d\lambda$ on A with probability one, where $\sigma(d\lambda, \omega)$ is a spectral measure of $H(\omega)$.*

We can prove this lemma using Hörmander's hypo-ellipticity theorem. We omit the detail.

We set:

$$S = \{(\omega, \lambda); \text{ there exists } \theta(\omega, \lambda) \text{ such that}$$
$$\lim_{t \to \infty} A(t)^{-1} \log \|U(t, \omega, \lambda) \cdot \theta(\omega, \lambda)\| = -\gamma(\lambda) \text{ and}$$
$$\lim_{t \to \infty} A(t)^{-1} \log \|U(t, \omega, \lambda)\| = \gamma(\lambda) \text{ hold.}\}.$$

We note that S is $\mathscr{F}_\pm \times \mathscr{B}(\mathbf{R})$-measurable, where $\mathscr{B}(\mathbf{R})$ is a Borel field of $\mathbf{R}$, because the statement in S is equivalent to the existence of the projection matrix $P(\infty, \omega, \lambda)$ and the validity:

$$\lim_{t \to \infty} A(t)^{-1} \log \|P(t, \omega, \lambda) - P(\infty, \omega, \lambda)\| = -\gamma(\lambda).$$

Proposition 2.1 asserts that for fixed $\lambda > 0$, the statement in S holds for almost every ω. Therefore by Lemma 4.1 we have for any compact set A:

$$\iint 1_A(\lambda) 1_{S^c}(\omega, \lambda) \sigma(d\lambda, \omega) \operatorname{Prob}(d\omega) = \mathbf{E}\left[\int_A 1_{S^c}(\omega, \lambda) \mathbf{E}[\sigma(d\lambda, \omega)|\mathscr{F}_\pm]\right] = 0,$$

where $1_{S^c}(\omega, \lambda)$ is an indicator function of a complement of S. We can select Ω_0 such that $\operatorname{Prob}(\Omega_0) = 1$ and for each $\omega \in \Omega_0$, the statement in S holds for $\sigma(d\lambda, \omega)$-a.e. λ. Owing to the relation $r_t = \tilde{r}_t$, we can take for each $\omega \in \Omega_0$ a solution $\psi_+(t, \omega, \lambda)$ of (1.3) such that:

$$\lim_{t \to \infty} A(t)^{-1} \log \{\psi_+(t, \omega, \lambda)^2 + \psi'_+(t, \omega, \lambda)^2\}^{1/2} = -\gamma(\lambda)$$

holds for $\sigma(d\lambda, \omega)$-a.e. λ. By the same argument on the negative axis, we can take Ω_1 satisfying $\operatorname{Prob}(\Omega_1) = 1$ and a solution $\psi_-(t, \omega, \lambda)$ of (1.3) such that for each $\omega \in \Omega_1$:

$$\lim_{t \to -\infty} A(t)^{-1} \log \{\psi_-(t, \omega, \lambda)^2 + \psi'_-(t, \omega, \lambda)^2\}^{1/2} = -\gamma(\lambda)$$

holds for $\sigma(d\lambda, \omega)$-a.e. λ. We fix $\omega \in \Omega_0 \cap \Omega_1$. We show that $\psi_+(t, \omega, \lambda)$ is equal to $\psi_-(t, \omega, \lambda)$ up to a constant by proving that they are linearly dependent on $f_j(t, \omega, \lambda)$ $(j = 1, 2)$ which are generalized solutions in the sense of Sect. 3.

Let $K \subset (0, \infty)$ be a compact set and define:

$$\gamma_0 = \inf_{\lambda \in K} \gamma(\lambda) > 0.$$

By the definition of λ_0 which is introduced in Sect. 1,

$$\exp\{-(\gamma_0 + 2\gamma(\lambda_0))A(t)\}$$

is integrable on $(0, \infty)$. Therefore by Lemma 3.1, we have:

$$\int_0^\infty \exp\{-(\gamma_0+2\gamma(\lambda_0))A(t)\}dt \int_K \{f_j(t,\omega,\lambda)^2+f_j'(t,\omega,\lambda)^2\}\sigma(d\lambda,\omega)<\infty .$$

From this we have for $\sigma(d\lambda,\omega)$-a.e. λ in K:

$$\int_0^\infty \exp\{-(\gamma_0+2\gamma(\lambda_0))A(t)\}\{f_j(t,\omega,\lambda)^2+f_j'(t,\omega,\lambda)^2\}dt<\infty . \tag{4.1}$$

We set the Wronskian of f_j and ψ_+ as follows:

$$w(\lambda)=f_j'(t,\omega,\lambda)\psi_+(t,\omega,\lambda)-f_j(t,\omega,\lambda)\psi'_+(t,\omega,\lambda).$$

Then we have for any $\varepsilon\in(0,\gamma_0/2)$:

$$\begin{aligned}&\int_0^\infty \exp\left\{\left(\frac{\gamma_0}{2}-2\gamma(\lambda_0)-\varepsilon\right)A(t)\right\}\{f_j'(t,\omega,\lambda)\psi_+(t,\omega,\lambda)-f_j(t,\omega,\lambda)\psi'_+(t,\omega,\lambda)\}dt\\&\quad=w(\lambda)\int_0^\infty \exp\left\{\left(\frac{\gamma_0}{2}-2\gamma(\lambda_0)-\varepsilon\right)A(t)\right\}dt.\end{aligned} \tag{4.2}$$

The square of the left-hand side is not larger than the following by the Schwartz's inequality:

$$\begin{aligned}&\text{const}\int_0^\infty \exp\{-(\gamma_0+2\gamma(\lambda_0))A(t)\}\{f_j(t,\omega,\lambda)^2+f_j'(t,\omega,\lambda)^2\}dt\\&\quad\times\int_0^\infty \exp\{(2\gamma_0-2\gamma(\lambda_0)-2\varepsilon)A(t)\}\{\psi_+(t,\omega,\lambda)^2+\psi'_+(t,\omega,\lambda)^2\}dt,\end{aligned}$$

which is finite by (4.1) and:

$$\lim_{t\to\infty} A(t)^{-1}\log\{\psi_+(t,\omega,\lambda)^2+\psi'_+(t,\omega,\lambda)^2\}^{1/2}=-\gamma(\lambda)\leqq-\gamma_0$$

for $\lambda\in K$. On the other hand, the right-hand side of (4.2) is infinite unless $w(\lambda)=0$. Therefore $w(\lambda)$ must be 0 and f_j and ψ_+ are dependent. The same argument for ψ_- shows that ψ_+, ψ_-, and f_j $(j=1,2)$ are all dependent.

Since $\exp\{-2\gamma(\lambda)A(t)\}$ is integrable when $\lambda\in(0,\lambda_0)$, it is easy to see that $(\psi_\pm^2+\psi_\pm^2)$ is also integrable. Therefore on $[0,\lambda_0]$, the spectrum of $H(\omega)$ consists only of point spectrum with probability one.

The absence of $\Sigma_p H(\omega)$ in (λ_0,∞) is proved as follows. For $\sigma(d\lambda,\omega)$-a.e. λ, we have:

$$\lim_{t\to\pm\infty} A(t)^{-1}\log\|U(t,\omega,\lambda)\|^{-1}=-\gamma(\lambda).$$

Since $\exp(-2\gamma(\lambda)A(t))$ is not integrable when $\lambda\in(\lambda_0,\infty)$, it is easy to see from the above that $\|U(t,\omega,\lambda)\|^{-1}$ is not square-integrable. On the other hand, since $U(t,\omega,\lambda)\in SL(2,\mathbf{R})$ we have for any θ:

$$\|U(t,\omega,\lambda)\cdot\theta\|\geqq\|U(t,\omega,\lambda)\|^{-1}.$$

Hence $\|U(t,\omega,\lambda)\cdot\theta\|$, equivalently, $(\psi_\pm^2+\psi_\pm'^2)^{1/2}$ are not square-integrable.

The absence of $\Sigma_{ac}H(\omega)$ in (λ_0,∞) is proved as follows. We repeat a similar argument in the proof of Ishii-Pastur's theorem simplified by Kotani [13]. Let

$H^\theta(\omega)$ be a unique self-adjoint extension in $L^2([0,\infty),dt)$ of $\left(-\frac{d^2}{dt^2}+a(t)F(X_t)^{\cdot}\right)$ with domain

$$\{u\in C_0^2([0,\infty));\ u(0)\cos\theta-u'(0)\sin\theta=0\}.$$

Let $\sigma^\theta(d\lambda,\omega)$ be the associated spectral measure.

Lemma 3.1. *For each ω, there exists a generalized real-valued solution $f^\theta(t,\omega,\xi)$ which satisfies $H^\theta(\omega)f^\theta(t,\omega,\xi)=\xi f^\theta(t,\omega,\xi)$ and:*

$$\sup_{t\in[0,\infty)}\int_{\mathbf{R}}\frac{f^\theta(t,\omega,\xi)^2+f^{\theta\prime}(t,\omega,\xi)^2}{1+\xi^2}\sigma^\theta(d\xi,\omega)<\infty.$$

We omit the proof.

We fix non-random θ. Then we know from (1) of Proposition 2.1 that for fixed $\lambda>0$, every solution $\psi^\theta(t,\omega,\lambda)$ satisfying $H(\omega)\psi^\theta=\lambda\psi^\theta$ and

$$\psi^\theta(0,\omega,\lambda)\cos\theta-\psi^{\theta\prime}(0,\omega,\lambda)\sin\theta=0$$

grows as follows with probability one:

$$\lim_{t\to\infty}A(t)^{-1}\log\{\psi^\theta(t,\omega,\lambda)^2+\psi^{\theta\prime}(t,\omega,\lambda)^2\}^{1/2}=\gamma(\lambda). \tag{4.3}$$

By Fubini's theorem, for almost every ω, (4.3) holds for almost every λ with respect to Lebesgue measure. If $\sigma^\theta(d\lambda,\omega)$ has non-trivial absolutely continuous part with respect to Lebesgue measure, then (4.3) holds for λ in a non-trivial set with respect to $\sigma^\theta(d\lambda,\omega)$. This contradicts Lemma 3.1′. Therefore $\varrho^\theta(d\lambda,\omega)$ is not absolutely continuous with respect to Lebesgue measure. One can discuss the same thing also on the negative half-axis. Observing that the difference of the resolvents of $H(\omega)$ and $\{H(\omega)$ with Dirichlet condition at $t=0\}$ is of finite rank, we can conclude that $\sigma(d\lambda,\omega)$ has no absolutely continuous part, which completes the proof of the theorem.

5. Remarks on the Lyapounov Exponent

It is remarkable that if we multiply a decaying term $a(t)$ in front of the stationary random potential, then the Lyapounov exponent $\gamma(\lambda)$ can be expressed in such a simple form as (1.2). Delyon suggested the following interpretation of $\gamma(\lambda)$. For any $\varepsilon\geqq 0$, let $\tilde\gamma(\lambda,\varepsilon)$ be the Lyapounov exponent of a random system:

$$\tilde H(\omega,\varepsilon)=-\frac{d^2}{dt^2}+\varepsilon F(X_t(\omega))^{\cdot}.$$

Namely if we define $U(t,\varepsilon)\in SL(2,\mathbf{R})$ by the solution of:

$$\frac{d}{dt}U(t,\varepsilon)=\begin{pmatrix}0 & 1\\ \varepsilon F(X_t(\omega))-\lambda & 0\end{pmatrix}U(t,\varepsilon),\qquad U(0,\varepsilon)=I,$$

then

$$\tilde\gamma(\lambda,\varepsilon)=\lim_{t\to\infty}\frac{1}{t}\log\|U(t,\varepsilon)\|.$$

Suppose $\{X_t(\omega)\}$ is a Brownian motion on a compact Riemannian manifold and F is a smooth function satisfying $\mathbf{E}F(X_t)=0$. Then Arnold-Papanicolaou-Whistutz [1] showed that for $\lambda>0$:

$$\tilde{\gamma}(\lambda,\varepsilon)=\frac{\pi\tau(2\lambda^{1/2})}{4\lambda}\varepsilon^2+O(\varepsilon^3) \tag{5.1}$$

as $\varepsilon\downarrow 0$, where $\tau(\lambda)$ is the power spectrum of the stationary process $\{F(X_t)\}$, namely:

$$\varrho(t-s)=\mathbf{E}\{F(X_t)F(X_s)\}=\int_{\mathbf{R}} e^{i\lambda(t-s)}\tau(\lambda)d\lambda\,.$$

In this case, $\tau(\lambda)$ can be computed as:

$$\tau(\lambda)=\frac{1}{\pi}\int_0^\infty \frac{\xi}{\xi^2+\lambda^2}\sigma_F(d\xi)$$

by using the spectral measure σ_F introduced in Sect. 1. Therefore $\gamma(\lambda)$ can be identified with the coefficient of ε^2 in the asymptotic expansion of $\tilde{\gamma}(\lambda,\varepsilon)$. The form (5.1) can be predicted also by the following naive argument. Let $\lambda\in\mathbf{C}_+$. Then:

$$\tilde{\gamma}(\lambda,\varepsilon)=-\operatorname{Re}\mathbf{E}\{h_+(\lambda,\omega,\varepsilon)\}\,. \tag{5.2}$$

Here $h_+(\lambda,\omega,\varepsilon)\in\mathbf{C}_+$ is defined by:

$$h_+(\lambda,\omega,\varepsilon)=\lim_{\substack{t>s>0\\ t\downarrow 0}}\frac{\partial^2}{\partial t\partial s}g_\lambda^+(t,s,\omega,\varepsilon)\,,$$

where g_λ^+ is the Green function of $\tilde{H}(\omega,\varepsilon)$ with Dirichlet boundary condition at $t=0$ (see Kotani [15]). On the other hand, $h_+(\lambda,\omega,\varepsilon)$ has an asymptotic form (Kotani [15]):

$$\begin{aligned}h_+(\lambda,\omega,\varepsilon)=i\lambda^{1/2}-\varepsilon\int_0^\infty g_\lambda(t)^2F(X_t)dt+\varepsilon^2\int_0^\infty\int_0^\infty g_\lambda^+(t,s)g_\lambda(t)g_\lambda(s)\\ \times F(X_t)F(X_s)dtds+O(\varepsilon^2)\,,\end{aligned}$$

where

$$g_\lambda(t)=e^{i\lambda^{1/2}t}\,,\qquad g_\lambda^+(t,s)=(2i\lambda^{1/2})^{-1}\{e^{i\lambda^{1/2}(t+s)}-e^{i\lambda^{1/2}|t-s|}\}\,.$$

Therefore, assuming $\mathbf{E}\{F(X_t)\}=0$, (5.2) shows:

$$\tilde{\gamma}(\lambda,\varepsilon)=\operatorname{Im}\lambda^{1/2}-\varepsilon^2\operatorname{Re}\int_0^\infty\int_0^\infty g_\lambda^+(t,s)g_\lambda(t)g_\lambda(s)\varrho(t-s)dtds+O(\varepsilon^3)\,,$$

for any fixed $\lambda\in\mathbf{C}_+$. It is not difficult to see the coefficient of ε^2 converges to $\pi\tau(2\lambda^{1/2})/4\lambda$ if λ tends to a positive real number. This observation may be useful to predict the Lyapounov exponent of the discrete system:

$$(H(\omega)u)_n=u_{n+1}+u_{n-1}+a_nq_n(\omega)u_n \tag{5.3}$$

in $l^2(\mathbf{Z})$, where $\{q_n(\omega)\}$ are i.i.d. random variables with mean zero and a_n is a decaying sequence satisfying $\sum_{n=1}^\infty a_n^2=\infty$. Let us now trace the above argument in the discrete system. Define a self-adjoint operator $H_+(\omega,\varepsilon)$ on $l^2(\mathbf{Z}_+)$ by:

$$(H_+(\omega,\varepsilon)u)_n=\begin{cases}u_{n+1}+u_{n-1}+\varepsilon q_n(\omega)u_n\,, & n\geqq 2\,,\\ u_2+\varepsilon q_1(\omega)u_1\,, & n=1\,.\end{cases}$$

Then for $\lambda \in \mathbf{C}_+$, by Simon [20] we have:

$$\tilde{\gamma}(\lambda, \varepsilon) = -\mathrm{Re}\,\mathbf{E}\{\log m_+(\lambda, \omega, \varepsilon)\},$$

where $m_+(\lambda, \omega, \varepsilon) = ((H_+(\omega, \varepsilon) - \lambda)^{-1}\delta_1, \delta_1)$. The asymptotic expansion of $m_+(\lambda, \omega, \varepsilon)$ shows

$$\tilde{\gamma}(\lambda, \varepsilon) = \mathrm{Im}\,\theta + \mathrm{Re}\frac{\sigma^2}{2(4-\lambda^2)}\varepsilon^2 + O(\varepsilon^3), \tag{5.4}$$

for $\lambda \in \mathbf{C}_+$, where $\theta \in \mathbf{C}_+$ is defined as a solution of $\lambda = 2\cos\theta$. In the continuous system we can pass to the real line in (5.4) as we saw in the above, however in the descrete system it is known that there exists an anomaly in $(-2, 2)$ if θ is rational modulo π. This was observed by Kappus and Wegner [11] and Derrida and Gardner [7], and mathematically by Bonvier and Klein [2]. Therefore in the system (5.3), it may be false that

$$\gamma(\lambda) = \frac{\sigma^2}{2(4-\lambda^2)}, \qquad \sigma^2 = \mathbf{E}q_0^2.$$

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