Commun. Math. Phys. 112, 691–719 (1987)

Communications in
Mathematical
Physics

Dynamical Entropy of C^* Algebras and von Neumann Algebras

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Abstract. The definition of the dynamical entropy is extended for automorphism groups of C^* algebras. As an example, the dynamical entropy of the shift of a lattice algebra is studied, and it is shown that in some cases it coincides with the entropy density.

Introduction

While in the 19$^{\text{th}}$ century the concept of entropy appeared in thermodynamics and its connection with statistics was realized, a satisfactory mathematical theory became available only through the Kolmogorov-Sinai entropy of automorphisms, which was a byproduct of progresses in information theory [1].

Not only did this new mathematical concept of a dynamical entropy for a measure invariant under a transformation clarify the mathematical set-up of thermodynamics, especially because it allowed a formulation of variational principles [2] without appealing again and again to a thermodynamic limit, it became also the key notion in ergodic theory. Through the work of Bowen and Ruelle the thermodynamic formulation has invaded the theory of smooth dynamical systems and appeared to be a crucial tool for problems such as the iteration of fractional transformations in C [3].

From the early beginning of the work of Kolmogorov and Sinai it was clear that a quantum or non-commutative analogue was required for both to be applicable in microphysics and to provide an important mathematical concept which in fact von Neumann and Sinai were asking for. For instance the work of Cuntz and Krieger on subshifts of finite type leads to a natural non-commutative C^*-algebra together with an automorphism for which entropy and variational inequalities would be very relevant. As far as quantum thermodynamics is concerned, there exists a definition of entropy density [4] but it refers in a crucial manner to a net of finite subsystems and thus has no a priori invariance properties as the KS entropy does. More precisely, in the classical context a corollary of the KS theorem shows that the entropy density computed on a limit of finite systems is equal to the KS entropy of space translation. The latter does not refer to any specific sequence of finite subsystems, thus is more conceptual and also has stronger invariance properties.

The quantum case, which is the relevant one in nature, is not in every aspect more complicated than the classical case. For instance, the KMS condition which ties up the non-commutativity of the algebra or, more precisely, the lack of trace property of a state with the one-parameter group of time evolution is a very relevant and powerful property of equilibrium states. Certainly it has its classical analogue [5] but even to state it one has to introduce auxiliary structures beyond the algebra, the state and the time evolution.

If we are looking for the quantum mechanical generalization of the dynamical entropy we first have to notice that in the classical case it is non-trivial only if the generator of the time evolution has a continuous spectrum. In complete analogy we will construct a dynamical entropy which is zero for a finite quantum system, i.e. an algebra of type I. Thus a dynamical entropy >0 requires an infinite system and thus a thermodynamic limit. There exist other definitions [6] of the entropy of the time evolution of a finite system: there the infinite system is split into a finite part and an infinite reservoir and the entropy measures how much the completely positive map of time evolution for the finite part fails to be unitary. This notion is different from ours; we do not refer to a division of the system into reservoir and observable subsystem but consider the – non-commutative – algebra observables in the thermodynamic limit.

The price of non-commutativity is twofold: even if one starts with a trace such that the state does not see the non-commutativity of the algebra, it is no longer true that a finite number of finite subsystems generate a finite subsystem. In fact, one can easily construct an example of two finite-dimensional algebras such that they generate an infinite-dimensional algebra. However, thanks to the progress in quantum statistical mechanics [7] it was possible to handle the tracial case [8]. The task was to find a quantity $H_\phi(N_1, \ldots, N_k)$ which measures the information contained in the subalgebras $N_1 \subset M$ if the total system M is in a state ϕ. We require

$$H_\phi(N_1, \ldots, N_k) \leqq H_\phi(M_1, \ldots, M_k) \quad \text{if} \quad N_i \subset M_i,$$

$$H_\phi(N_1, N_2, \ldots, N_k) \leqq \sum_{i=1}^{k} H_\phi(N_i),$$

$$H_\phi(N_1, N_2, \ldots, N_k) \leqq H_\phi\left(\bigvee_i N_i\right),$$

where $\bigvee_i N_i$ is the algebra generated by the N_i and for commuting algebras equality should hold iff ϕ factorizes. Correlations should reduce the gained information. In this paper we shall generalize, following [8], this construction of the H's for non-tracial states and make this notion applicable to the relevant cases of physical systems at finite temperature.

There have been other attempts [9] to extend the classical theory to type III algebras by restricting the state to a maximal abelian time invariant subalgebra. This might give insight into macroscopic properties but is too restrictive for a microscopic theory of systems without non-trivial invariant subalgebras. In contradistinction our dynamical entropy may be >0 for such systems. The main problem that arises in passing to the type III case is an incompatibility between a state and a finite-dimensional subalgebra in the following sense. In the commutative case any finite-dimensional subalgebra A of $L^\infty(x,\mu)$ yields a conditional expectation $E: L^\infty \to A$ such that $\mu = \mu|_A \circ E$. For non-tracial states ϕ a conditional expectation $E: M \to A$ such that $\phi = \phi|_A \circ E$ exists only if A is invariant under the modular automorphism. It is exactly the desired property of ergodicity which denies the existence of such A's $\neq \alpha 1$. Moreover, this incompatibility is also the cause of the lack of monotonicity in A of the usual entropy of $\phi|_A$. Thus already for $H_\phi(A)$ we cannot use $S(\phi|_A)$. It turned out that the authors (A.C. on the one hand and H.N. and W.T. on the other) arrived independently at the following formula

$$H_\phi(N) = \sup_{\Sigma \phi_i = \phi} \sum_i \left(S(\phi|_N, \phi_{i|N}) - \phi_i(1)\log\phi_i(1)\right).$$

In [10] this formula proved to be useful to deduce the various properties of the entropy. In [11] it was used to define the entropy of automorphisms in the non-tracial situation. In this paper we shall improve the results of [11] and get a more conceptual understanding of the function $H_\phi(N_1, \ldots, N_k)$ by extending it to completely positive maps with finite rank.

Roughly speaking, the motivation for our construction is that we try to map the non-abelian case as well as possible to an abelian situation where one knows how to compute the dynamical entropy. Thus we introduce the notion of an abelian model which is a map P from the algebra A with state ϕ onto an abelian algebra B with state μ such that $\phi = \mu \circ P$. Of course, P cannot be an algebraic isomorphism, but to preserve as much structure as possible we use for P completely positive maps. In general, the entropy of μ, $S(\mu)$, is in no way related to the $S(\phi)$ and to arrive at a definition which implies $S(\mu) = S(\phi)$ if A is abelian requires the following considerations.

An abelian model (B, P, μ) is equivalent to a decomposition $\phi = \sum \mu_i \hat{\phi}_i$, where for a basis e_i in B the map P is given by $P(A) = \sum e_i \hat{\phi}_i(A)\ \forall A \in B$ and $\mu_i = \mu(e_i)$. If ϕ is given by a density matrix ϱ the model corresponds to a decomposition of unity $\sum x_i = 1$, $x_i \in A^+$, such that $\mu_i = \phi(x_i)$ and $\hat{\phi}_i$ is given by the density matrix $\varrho_i = \sqrt{\varrho}\, x_i \sqrt{\varrho}/\mu_i$. Thus the model can be thought of as a measurement of the observables $\{x_i\}$. The information gained by the measurement will be $\varepsilon = S(\phi) - \sum \mu_i S(\hat{\phi}_i)$, the difference of the entropy of ϕ and its components weighted with their probabilities μ_i. It is zero iff $\hat{\phi}_i = \phi\ \forall i$, in which case no additional information is obtained. ε assumes its maximal value $S(\phi)$ iff all $\hat{\phi}_i$ are pure. This observation was the basis for considering the quantity $H_\phi(N)$.

The notion of an abelian model still contains the useless possibility that the $\hat{\phi}_i$ do not carry different information, for instance, they may all be equal, $\phi = \phi \sum \mu_i$. In this case $S(\mu) = -\sum \mu_i \log \mu_i$ may arbitrarily exceed $S(\phi)$. To discriminate against such models we introduce the notion of the entropy defect $s_\mu(P) = S(\mu) - \varepsilon$, the difference of the abelian entropy and the information gain. It attains its minimal value 0 iff $\hat{\phi}_i(\hat{\phi}_j) = 0$ for $i \neq j$. With the entropy defect we are in the position to define $H_\phi(N_1, \ldots, N_k)$ as the difference of the abelian entropy and the sum of the entropy defects for the N_i's.

The aim of this paper is to give detailed proofs of these results and thus to provide a starting point for further investigations.

I. Preliminaries on Relative Entropy

Since the proof by Lieb [12] of the Wigner-Yanase-Dyson conjecture, the main properties of the quantum relative entropy

$$S(\phi,\psi)=\mathrm{Trace}(\varrho_\psi(\log\varrho_\psi-\log\varrho_\phi))$$

[for states ϕ, ψ on a finite-dimensional C^* algebra A; $\phi(x)=\mathrm{Tr}(\varrho_\phi x)$, $\psi(x)=\mathrm{Tr}(\varrho_\psi x)\,\forall x\in A$] have been established. By the results of Araki [13] the relative entropy can be defined for arbitrary normal states on a von Neumann algebra. The main properties of $S(\phi,\psi)$ are its joint convexity and weak lower semicontinuity. They can be restated by writing S as a supremum of weakly continuous affine functionals and there is indeed an explicit way of doing so, due to Pusz, Woronowicz, and Kosaki:

Lemma I.1 [14]. *Let A be a unital C^* algebra; ϕ, ψ positive linear forms on A, then the relative entropy $S(\phi,\psi)$ is given by*

$$S(\phi,\psi)=\sup\int\limits_0^\infty\left[\frac{\psi(1)}{1+t}-\psi(y(t)^*y(t))-\frac{1}{t}\,\phi(x(t)x(t)^*)\right]\frac{dt}{t},$$

where $x(t)+y(t)=1$ and the sup *is taken over all step functions $x(t)$ with values in A which are equal to* 0 *in a neighbourhood of* 0.

This can be used as definition for arbitrary states over a C^* algebra. We shall now list for the convenience of the reader the known properties of $S(\phi,\psi)$ which will be used in the rest of this paper:

(2) *Scaling property:* $S(\lambda_1\phi,\lambda_2\psi)=\lambda_2 S(\phi,\psi)+\lambda_2\psi(1)\log(\lambda_2/\lambda_1)$, $\lambda_i\in R^+$.

(2) *Positivity:* $S(\phi,\psi)\geqq 0$ for states ϕ, ψ, $=0$ iff $\phi=\psi$.

(3) *Joint convexity:* For $\lambda_i\geqq 0$, $\sum\lambda_i=1$ one has

$$S(\sum\lambda_i\phi_i,\sum\lambda_j\psi_j)\leqq\sum_{i=1}^n\lambda_i S(\phi_i,\psi_i).$$

(4) *Monotone properties:*

a) Decrease in the first argument. $\phi_1\leqq\phi_2\Rightarrow S(\phi_1,\psi)\geqq S(\phi_2,\psi)$.

b) Superadditivity in the second argument. $S\left(\phi,\sum_i\psi_i\right)\geqq\sum_i S(\phi,\psi_i)$.

c) If $\gamma:A\to B$ is unital completely positive then

$$S(\phi\circ\gamma,\psi\circ\gamma)\leqq S(\phi,\psi).$$

(5) *Lower semicontinuity:* The map $(\phi,\psi)\in A_+^*\times A_+^*\to S(\phi,\psi)\in\,]-\infty,\infty]$ is weakly lower semicontinuous.

(6) *Martingale convergence:* Let $(\gamma_\nu)_{\nu\in N}$ be a sequence of completely positive unital maps $\gamma_\nu:A\to A$ which converges pointwise in norm to id_A, then:

$$S(\phi\circ\gamma_\nu,\psi\circ\gamma_\nu)\xrightarrow[\nu\to\infty]{}S(\phi,\psi).$$

(7) *Invariance:* Let $B \subset A$ be a subalgebra and $E: A \to B$ a conditional expectation. For any $\phi, \psi \in B^*_+$ one has:

$$S(\phi \circ E, \psi \circ E) = S(\phi, \psi).$$

Remarks I.2. 1. In the following completely positive unital maps will play a crucial role. For the convenience of the reader we shall first summarize the terminology. The word "unital" stands for containing or preserving the unit element and "normal" for commuting with sup. A completely positive unital map ϕ between two unital C^* algebras A and B is a positive unital map such that the map ϕ between $M_n(A)$ and $M_n(B)$ the $n \times n$ matrices with elements from A (respectively B),

$$(\phi(a))_{ij} = \phi(a_{ij})$$

is positive. For them positivity is strengthened to Schwarz positivity $\phi(a^*a) \geqq \phi(a^*)\phi(a)$. With respect to composition they form a semigroup which contains $*$-homomorphisms, in particular the natural inclusion if $A \subset B$. If A or B are abelian, any positive map is completely positive. In the linear space of linear maps the completely positive unital maps are a closed convex set. If $B \subset A$ a positive unital map with $\phi(b_1 a b_2) = b_1 \phi(a) b_2$, $b_i \in B$, $a \in A$, is called unital conditional expectation. It is automatically completely positive. Let ω be a faithful normal state on a von Neumann algebra A and let B be a unital subalgebra. Then there exists a conditional expectation ϕ with $\omega \circ \phi = \omega$ if and only if the modular automorphism group σ^ω satisfies $\sigma^\omega_t B \subset B$. In this case ω determines ϕ uniquely and we call ϕ canonically associated to ω.

2. We do not explicitly write to which algebra $S(\phi, \psi)$ refers and always understand this expressed by the states ϕ and ψ.

3. Since $S(\lambda\phi, \lambda\psi) = \lambda S(\phi, \psi)$ the convexity (3) is equivalent to the subadditivity in both arguments:

$$S(\sum \phi_i, \sum \psi_i) \leqq \sum_i S(\phi_i, \psi_i).$$

4) Since the natural inclusion is completely positive, (4b) says in particular the monotonicity for subalgebras $B \subset A$

$$S(\phi|_B, \psi|_B) \leqq S(\phi, \psi).$$

II. Entropy Defect of a Completely Positive Unital Map

Let A, B be unital C^* algebras and $P: A \to B$ a completely positive unital map. The transpose P^* maps the state space $\Sigma(B)$ to $\Sigma(A)$ by $P^*\mu = \mu \circ P$. When B is commutative, i.e. $B = C(X)$ for a compact space X, it is equivalent to give a completely positive unital map $P: A \to B$ or a weakly continuous map $x \to P^*_x$ of X to $\Sigma(A)$ [15].

Let then μ be a state on B, i.e. a probability measure $d\mu(x)$ on X. For each $x \in X$, P^*_x is the state on A defined by $\int d\mu(x) P^*_x(A) = \mu(P(A)) = P^*_\mu(A)$. The relative entropy $S(P^*\mu, P^*_x)$ is a well-defined positive real number, lower semicontinuous as a function of $x \in X$. We put

$$\varepsilon_\mu(P) = \int_X S(P^*\mu, P^*_x) d\mu(x). \tag{II.1}$$

From now on we shall assume the algebras to be finite dimensional. Then the entropy of a state ϕ over A, defined by [10]

$$S(\phi) \equiv \sup_{\phi = \mu \circ P} \varepsilon_\mu(P)$$

is finite and

$$\varepsilon_\mu(P) = S(P^*\mu) - \int d\mu(x) S(P_x^*). \tag{II.2}$$

$\varepsilon_\mu(P)$ enjoys the following properties which can be deduced from (I.1):

Proposition II.2. a) $\varepsilon_\mu(\lambda_1 P_1 + \lambda_2 P_2) \leqq \lambda_1 \varepsilon_\mu(P_1) + \lambda_2 \varepsilon_\mu(P_2)$ *for* $\lambda_1, \lambda_2 \geqq 0, \lambda_1 + \lambda_2 = 1$.

b) $\varepsilon_\mu(P) \leqq S(P^*\mu)$. Iff P_x^* *is pure* a.e. *then* $\varepsilon_\mu(P) = S(P^*\mu)$.

c) *Let* $\gamma : A_1 \to A$ *be a completely positive unital map, then* $\varepsilon_\mu(P \circ \gamma) \leqq \varepsilon_\mu(P)$. *Equality holds if* γ *is a conditional expectation.*

d) *One has* $\varepsilon_\mu(P) \leqq S(\mu)$ *for any* P.

e) $\varepsilon_{\lambda_1 \mu_1 + \lambda_2 \mu_2}(P) \geqq \lambda_1 \varepsilon_{\mu_1}(P) + \lambda_2 \varepsilon_{\mu_2}(P)$ *for* $\lambda_1, \lambda_2 \geqq 0, \lambda_1 + \lambda_2 = 1$.

Definition II.4. The *entropy defect* $s_\mu(P)$ of the completely positive unital map $P : A \to B$ is the positive real number

$$s_\mu(P) = S(\mu) - \varepsilon_\mu(P).$$

The prototype of a completely positive map with zero entropy defect is the conditional expectation E from the algebra A of $n \times n$ matrices to a maximal abelian subalgebra B of the centralizer of a given state ϕ on A with $\mu = \phi|_B$.

Proposition II.5. a) $s_\mu(\lambda_1 P_1 + \lambda_2 P_2) \geqq \lambda_1 s_\mu(P_1) + \lambda_2 s_\mu(P_2)$ *for* $\lambda_1, \lambda_2 \geqq 0, \lambda_1 + \lambda_2 = 1$.

b) *If* P *is an extreme point of completely positive unital maps, then* $s_\mu(P) = S(\mu) - S(\mu \circ P)$.

c) $s_\mu(P \circ \gamma) \geqq s_\mu(P)$ *with equality if* $\gamma : A_1 \to A$ *is a conditional expectation.*

d) *For each subalgebra* B_1 *of* B *let* P_{B_1} *be the composition with* P *of the conditional expectation from* B *to* B_1 *canonically associated to* μ, *then*

$$s_\mu(P_{B_1 \vee B_2}) \leqq s_\mu(P_{B_1}) + s_\mu(P_{B_2}),$$

where $B_1 \vee B_2$ *is the subalgebra generated by* B_1 *and* B_2.

Proof. a)–c) follow from Proposition II.2.

d) Let us label the minimal projectors of B_1 by $i \in \{1, \ldots, n_1\}$, of B_2 by $j \in \{1, \ldots, n_2\}$ and of $B_1 \vee B_2$ by (i, j). We can assume that $B = B_1 \vee B_2$, and let E_i be the conditional expectation of B onto B_i determined by μ. Let $\lambda_{i,j} = \mu(i, j)$ be the value of μ on the minimal projectors (i, j), then one has, with obvious notations:

$$P_{i,j}^*(k, l) = \delta_{ik}\delta_{jl}, \qquad P^*\mu = \sum_{i,j} \lambda_{i,j} P_{i,j}^*,$$

$$(E_1 \circ P)_i^* = \left(\sum_j \lambda_{i,j} P_{i,j}^*\right)\left(\sum_j \lambda_{i,j}\right)^{-1}, \qquad (E_2 \circ P)_j^* = \left(\sum_i \lambda_{i,j} P_{i,j}^*\right)\left(\sum_i \lambda_{i,j}\right)^{-1}.$$

If we let $\psi_{i,j} = \lambda_{i,j} P_{i,j}^*$, $\psi_i^1 = \sum_j \psi_{i,j}$, $\psi_j^2 = \sum_i \psi_{i,j}$, we get

$$s_\mu(P_B) = \sum_{i,j} (-\lambda_{i,j} \log \lambda_{i,j} - \lambda_{i,j} S(P^*\mu, P_{i,j}^*)) = -\sum_{i,j} S(\psi, \psi_{i,j}),$$

where $\psi=P^*\mu$. Similarly

$$s_\mu(P_{B_1})=-\sum_i S(\psi,\psi_i^1), \qquad s_\mu(P_{B_2})=-\sum_j S(\psi,\psi_j^2).$$

One has $\sum_i S(\psi,\psi_{i,j})-S(\psi_j^2,\psi_{i,j})=S(\psi,\psi_j^2)$ so that

$$\sum_{i,j} S(\psi,\psi_{i,j})-\sum_j S(\psi,\psi_j^2)=\sum_{i,j} S(\psi_j^2,\psi_{i,j})\geqq \sum_i S\left(\sum_j \psi_j^2,\sum_j \psi_{i,j}\right)=\sum_i S(\psi,\psi_i^1)$$

using the joint convexity of the relative entropy. □

III. The Function $H_\phi(\gamma_1,\dots,\gamma_n)$ for Completely Positive Maps

In this section we shall extend the function H_ϕ of [11], defined there for arbitrary finite dimensional subalgebras to a more general and more manageable situation, where the subalgebras are replaced by completely positive maps. Let A be a unital C^* algebra and recall [15] that there is a natural bijection between completely positive maps $\theta: M_n(C)\to A$ and positive elements of $M_n(A)$. The positive element associated to θ is the matrix $(x_{ij})=\theta(e_{ij})$, where e_{ij} are the matrix units in $M_n(C)$. The condition $\theta(1)=1$ means that $\sum x_{ii}=1$. This correspondence insures the existence of plenty such completely positive maps, while in general A does not contain any finite dimensional subalgebra. The following is based on the result of Choi and Effros [16]:

Proposition III.1. *Let A be a nuclear unital C^* algebra. There is a sequence $(\theta_\nu)_{\nu\in N}$ of completely positive unital maps $A\to A$ such that*

1. For each ν there exists a finite dimensional C^* algebra A_ν and unital completely positive maps $\sigma_\nu: A\to A_\nu$, $\tau_\nu: A_\nu\to A$ such that $\theta_\nu=\tau_\nu\circ\sigma_\nu$.

2. For any $x\in A$, $\lim_{\nu\to\infty}\|\theta_\nu(x)-x\|=0$.

The construction of the function H_ϕ is done by comparison with the abelian situation in which the ambient C^* algebra is commutative. Let us start from a C^* algebra A, a state ϕ and n completely positive unital maps $\gamma_1,\dots,\gamma_n$ from finite dimensional C^* algebras $A_1,\dots,A_n$ to A.

Definition III.2. An *abelian model* for $(A,\phi,\gamma_1,\dots,\gamma_n)$ is given by

1. An abelian finite dimensional C^* algebra B, a state μ on B and subalgebras $B_1,\dots,B_n$ of B.
2. A completely positive unital map $P: A\to B$ with $\phi=P^*\mu$.

For each $j=1,2,\dots,n$ let $E_j: B\to B_j$ be the canonical conditional expectation (associated to μ), then $\varrho_j=E_j\circ P\circ\gamma_j$ is a completely positive map from A_j to B_j.

We define the entropy of the abelian model as

$$S(\mu|_{\vee B_j})-\sum_j s_\mu(\varrho_j), \tag{III.3}$$

with $\varrho_j^*\mu=\gamma_j^*P^*\mu=\gamma_j^*\phi=\phi\circ\gamma_j$ we can express this quantity in terms of entropies as follows:

$$\begin{aligned}(\text{III.3})&=S(\mu|_{\vee B_j})-\sum_j S(\mu|_{B_j})+\sum_j\{S(\phi\circ\gamma_j)-\int d\mu|_{B_j}(x)S((\phi\circ\gamma_j)_x)\}\\ &\leqq \sum_{j=1}^n S(\phi\circ\gamma_j).\end{aligned}$$

The last inequality comes from the subadditivity $S(\mu|_{\vee B_j}) \leqq \sum_j S(\mu|_{B_j})$. Note that this bound does not depend on the abelian model.

Definition III.4. $H_\phi(\gamma_1, \ldots, \gamma_n)$ is the sup of the entropy of the abelian models for $(A, \phi, \{\gamma_j\})$.

By (III.3) one has $H_\phi(\gamma_1, \ldots, \gamma_n) \leqq \sum_{j=1}^{n} S(\gamma_j^* \phi)$.

Remarks III.5. 1. For any finite dimensional C^* algebra A there exists a matrix algebra $M_n(C)$ containing A as a subalgebra and a conditional expectation $M_n \xrightarrow{E} A$. Thus Proposition III.6 will show that to compute $H_\phi(\gamma_1, \ldots, \gamma_p)$ one can assume that all γ_j's are maps from a matrix algebra to A. Moreover, completely positive unital maps from $M_n(C)$ to A correspond exactly to elements a_{ik} of $M_n(A)^+$, $\sum_i a_{ii} = 1 \in A$. Letting $M_\infty(A)$ be the increasing union of the algebras $M_n(A)$, with maps

$$x \in M_n \to \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+1},$$

we see that we can consider H_ϕ as a function defined on all finite subsets X of

$$\left\{ x \in M_\infty(A)^+, \sum_i x_{ii} = 1 \right\}.$$

2. If A is abelian and the A_i are finite dimensional subalgebras of A, γ_i the inclusions we may take

$$B = \bigvee_{i=1}^{n} A_i, \qquad \varrho_i = \mathrm{id}|_{A_i} \quad \text{and} \quad \mu = \phi|_B.$$

Then $\varrho_i{}^*{}_x$ is pure, thus with Proposition II.2b,

$$\varepsilon_\mu(\varrho_i) = S(\varrho_i^* \mu) = S(\mu|_{B_i}).$$

Thus the entropy defects vanish and we have the classical definition

$$H_\phi(\gamma_1, \ldots, \gamma_n) = S(\phi|_{\vee A_i}).$$

3. Let A_i be subalgebras of A and γ_i the inclusion. An abelian model is characterized by the minimal projections Q_{i_j} in B_j, the minimal projections $Q_{i_1 \ldots i_n}$ in B and the map

$$P(A) = \sum \hat{\phi}_{i_1 \ldots i_n}(A) Q_{i_1 \ldots i_n} = \sum \hat{\phi}_{(i)}(A) Q_{(i)}.$$

P is a completely positive map iff $\hat{\phi}_{(i)}$ are states over A. $\phi = P^* \mu$ corresponds to

$$\phi(A) = \sum_{(i)} \hat{\phi}_{(i)}(A) \mu(Q_{(i)}).$$

Thus an abelian model is in this case in one-to-one correspondence with a decomposition of the state ϕ. In the GNS-construction such a decomposition can also be represented in terms of positive elements x_i, $\sum_i x_i = 1$ of the commutant $\pi(A')$. $\phi(A) = \sum_i \phi(x_i A)$. Thus the definition of H can then be written in the following equivalent ways:

$$\eta(x)\equiv -x\ln x\,,\qquad \sum_{\substack{i_1\ldots i_n\\ i_k\text{ fixed}}}\phi_{i_1\ldots i_n}\equiv\phi_{i_k}\,,\qquad \hat{\phi}\equiv\phi/\phi(1)\,,$$

$$\begin{aligned}
H_\phi(A_1,\ldots,A_n) &= \sup_{\sum_{(i)}\phi_{(i)}=\phi}\left[\sum_{(i)}\eta(\phi_{(i)}(1))+\sum_{k=1}^{n}\sum_{i_k}S(\phi_{|A_k}|\phi^{(k)}_{i_k|A_k})\right]\\
&= \sup_{\sum_{(i)}\phi_{(i)}=\phi}\left[\sum_{(i)}\eta(\phi_{(i)}(1))-\sum_{k=1}^{n}\sum_{i_k}\eta(\phi_{i_k}(1))\right.\\
&\quad\left.+\sum_{k=1}^{n}\sum_{i_k}S(\phi_{|A_k}|\hat{\phi}^{(k)}_{i_k|A_k})\phi^{(k)}_{i_k}(1)\right]\\
&= \sup_{\substack{\Sigma x_{(i)}=1\\ 0\leqq x_{(i)}\leqq 1\\ x_{(i)}\in\pi(A')}}\left(\sum_{(i)}\eta(\phi(x_{(i)})+\sum_{k=1}^{n}\sum_{i_k}S(\phi_{|A_k}|\phi(\cdot\,x^{(k)}_{i_k})_{|A_k})\right]\\
&= \sup_{\Sigma\phi_{(i)}=\phi}\left[\eta(\phi_{(i)}(1)-\sum\sum\eta(\phi_{i_k}(1))\right.\\
&\quad\left.+\sum_{k=1}^{n}S(\phi_{|A_k})-\sum\phi^{(k)}_{i_k}(1)S(\hat{\phi}^{(k)}_{i_k|A_k})\right].
\end{aligned}$$

The main properties of H_ϕ are the following:

Proposition III.6. a) *Let $\theta_j: A'_j\to A_j$ be completely positive unital maps, then*

$$H_\phi(\gamma_1\circ\theta_1,\ldots,\gamma_n\circ\theta_n)\leqq H_\phi(\gamma_1,\ldots,\gamma_n)\,.$$

Equality holds if $A_j\subset A'_j$ is a conditional expectation for all j.

b) *If $\theta: A\to A$ are completely positive unital maps with $\phi\circ\theta=\phi$, then*

$$H_\phi(\theta\circ\gamma_1,\ldots,\theta\circ\gamma_n)\leqq H_\phi(\gamma_1,\ldots,\gamma_n)\,.$$

Equality holds if the θ are automorphisms.

c) *$H_\phi(\gamma_1,\ldots,\gamma_n)$ depends only upon the set $\{\gamma_1,\ldots,\gamma_n\}=X$, that is $H(\gamma,\gamma)=H(\gamma)$ and so on.*

d) *With the notations of* c) *one has* $\max\{H_\phi(X),\,H_\phi(Y)\}\leqq H_\phi(X\cup Y)\leqq H_\phi(X)+H_\phi(Y)$.

e) *Concavity:*

$$\begin{aligned}
&\lambda H_{\phi_1}(\gamma_1,\ldots,\gamma_n)+(1-\lambda)H_{\phi_2}(\gamma_1,\ldots,\gamma_n)+(n-1)(\lambda\ln\lambda+(1-\lambda)\ln(1-\lambda))\\
&\leqq H_{\lambda\phi_1+(1-\lambda)\phi_2}(\gamma_1,\ldots,\gamma_n)\\
&\leqq \lambda H_{\phi_1}(\gamma_1,\ldots,\gamma_n)+(1-\lambda)H_{\phi_2}(\gamma_1,\ldots,\gamma_n)-\lambda\ln\lambda-(1-\lambda)\ln(1-\lambda)\,.
\end{aligned}$$

f) $H_\phi(\lambda\gamma_1+(1-\lambda)\gamma'_1,\ldots,\lambda\gamma_n+(1-\lambda)\gamma'_n)\leqq\lambda H_\phi(\gamma_1,\ldots,\gamma_n)+(1-\lambda)H_\phi(\gamma'_1,\ldots,\gamma'_n)$.

Proof. Note first that the notion of an abelian model just depends upon the pair (A,ϕ) and the integer n but not the γ_j's.

a) follows from Proposition II.5c.

b) $$H_\phi((\theta\circ\gamma_i))=\sup_{\mu\circ P=\phi}\left[S(\mu)-\sum_j s_\mu(E_j\circ P\circ\theta\circ\gamma_j)\right]$$
$$=\sup_{\mu\circ P\circ\theta=\phi}\left[S(\mu)-\sum_j s_\mu(E_j\circ P\circ\theta\circ\gamma_j)\right]$$
$$\leqq\sup_{\mu\circ P'=\phi}\left[S(\mu)-\sum_j s_\mu(E_j\circ P'\circ\gamma_j\right]=H_\phi((\gamma_j)).$$

c) One has to show that $H_\phi(\gamma_1,\gamma_2,\dots,\gamma_n,\gamma_n)=H_\phi(\gamma_1,\gamma_2,\dots,\gamma_n)$. Given an abelian model $(B,\mu,(B_j),P)$ for n we get one for $n+1$ taking $B_{n+1}=C$. Then

$$\bigvee_{j=1}^{n+1}B_j=\bigvee_{j=1}^{n}B_j,$$

and since any map to C has zero entropy defect we get the first inequality $\geqq$. To prove that $H_\phi(\gamma_1,\dots,\gamma_n,\gamma_n)\leqq H_\phi(\gamma_1,\dots,\gamma_n)$, let $(B,\mu,(B_j),P)$ be an abelian model for $n+1$. Put $B'_j=B_j$ for $j=1,\dots,n-1$ and $B'_n=B_n\vee B_{n+1}$, so that $(B,\mu,(B'_j),P)$ is an abelian model for n. One has

$$\bigvee_1^n B'_j=\bigvee_1^{n+1}B_j,$$

thus the answer follows from Proposition II.5d.

d) One has to show that

$$H_\phi(\gamma_1,\dots,\gamma_k,\dots,\gamma_{k+p})\leqq H_\phi(\gamma_1,\dots,\gamma_k)+H_\phi(\gamma_{k+1},\dots,\gamma_{k+p}).$$

But given an abelian model for $k+p$, $(B,\mu,(B_j),P)$ one gets one for k by taking $j\in\{1,\dots,k\}$ and for p by taking $j\in\{k+1,\dots,k+p\}$. The inequality then follows from

$$S(\mu_{|B_1\vee\dots\vee B_{k+p}})\leqq S(\mu_{|B_1\vee\dots\vee B_k})+S(\mu_{|B_{k+1},\dots,B_{k+p}}).$$

e) Concave side. Let

$$A_i\xrightarrow{\gamma_i}A\xrightarrow{P(1)}B^{(1)}\xrightarrow{E_i}B_i^{(1)}$$

optimize $H_{\phi_1}(\gamma_1,\dots,\gamma_n)$, respectively,

$$A_i\xrightarrow{\gamma_i}A\xrightarrow{P(2)}B^{(2)}\xrightarrow{E_i}B_i^{(2)}\qquad H_{\phi_2}(\gamma_1,\dots,\gamma_n).$$

Then we use for $H_{\lambda\phi_1+(1-\lambda)\phi_2}$ the abelian model $B=B^{(1)}+B^{(2)}$, $B_i=B_i^{(1)}+B_i^{(2)}$, $P=P^{(1)}+P^{(2)}$, and the state $\mu=\lambda\mu_1+(1-\lambda)\mu_2$. In this case we have

$$S(\mu)=\lambda S(\mu_1)+(1-\lambda)S(\mu_2)-\lambda\ln\lambda-(1-\lambda)\ln(1-\lambda),$$

and similarly for $S(\mu_{|B_i})$. Concavity of the entropy tells us

$$S((\lambda\phi_1+(1-\lambda)\phi_2)\circ\gamma_j)\geqq\lambda S(\phi_1\circ\gamma_j)+(1-\lambda)S(\phi_2\circ\gamma_2),$$

and finally

$$\int_X d\mu_{|B_j}(x)=\lambda\sum_{X_1}d\mu_{1|B_j^{(1)}}(x)+(1-\lambda)\int_{X_2}d\mu_{2|B_j^{(2)}}(x).$$

Using these relations in (III.3) we conclude

$$\begin{aligned}
H_{\lambda\phi_1+(1-\lambda)\phi_2}(\gamma_1,\dots,\gamma_n) &\geqq \lambda S(\mu_{1|\vee B_j^{(1)}})+(1-\lambda)S(\mu_{2|\vee B_j^{(2)}}) \\
&\quad -\lambda\ln\lambda-(1-\lambda)\ln(1-\lambda)-\sum_j\{\lambda S(\lambda_{1|B_j^{(1)}})+(1-\lambda)S(\mu_{2|B_j^{(2)}}) \\
&\quad -\lambda\ln\lambda-(1-\lambda)\ln(1-\lambda)\}+\sum_j\{\lambda S(\phi_1\circ\gamma_j)+(1-\lambda)S(\phi_2\circ\gamma_j) \\
&\quad -\lambda\int d\mu_{B_j^{(1)}}S((\phi_1\circ\gamma_j)_x)-(1-\lambda)\int d\mu_{B_j^{(2)}}S((\phi_2\circ\gamma_j)_x)\} \\
&=\lambda H_{\phi_1}(\gamma_1,\dots,\gamma_n)+(1-\lambda)H_{\phi_2}(\gamma_1,\dots,\gamma_n)+(n-1)(\lambda\ln\lambda+(1-\lambda)\ln(1-\lambda)).
\end{aligned}$$

Convex Side. Suppose B, P, μ are the optimal model (within ε) for $H_{\lambda\phi_1+(1-\lambda)\phi_2}(\gamma_1,\dots,\gamma_2)$. To P and μ there corresponds a decomposition

$$\phi=\lambda\phi_1+(1-\lambda)\phi_2=\sum_{(i)}\mu_{(i)}\hat{\phi}_{(i)}=\sum_{(i)}\phi(x_{(i)}\cdot),$$

where

$$x_{(i)}\in\pi_\phi(A)',\qquad x_{(i)}\geqq 0,\qquad \sum_{(i)}x_{(i)}=1\quad\text{(see III.5, 3)}.$$

To this decomposition we associated the decompositions

$$\phi_1(A)=\sum_{(i)}\phi_1(x_{(i)}A),\qquad \phi_2(A)=\sum_{(i)}\phi_2(x_{(i)}A),$$

which define abelian models $(B, P^{(1)}, \mu^{(1)})$ and $(B, P^{(2)}, \mu^{(2)})$, where $\mu_{(i)}^{(\alpha)}=\phi_\alpha(x_{(i)})$ such that $\mu=\lambda\mu^{(1)}+(1-\lambda)\mu^{(2)}$. According to (III.5, 3) we have

$$\begin{aligned}
H_{\lambda\phi_1+(1-\lambda)\phi_2}(\gamma_1,\dots,\gamma_n) &= S(\lambda\mu^{(1)}+(1-\lambda)\mu^{(2)}) \\
&\quad +\sum_{l=1}^{n}\sum_{i_l}S((\lambda\phi_1+(1-\lambda)\phi_2)\circ\gamma_l,(\lambda\phi_1+(1-\lambda)\phi_2)(x_{i_l}\cdot)\circ\gamma_l \\
&\leqq \lambda S(\mu^{(1)})+(1-\lambda)S(\mu^{(2)})-\lambda\ln\lambda-(1-\lambda)\ln(1-\lambda) \\
&\quad +\lambda\sum_{l=1}^{n}\sum_{i_l}S(\phi_1\circ\gamma_l,\phi_1(x_{i_l}\circ\gamma_l))+(1-\lambda)\sum_{l=1}^{n}\sum_{i_l}S(\phi_2\circ\gamma_l,\phi_2(x_{i_l}\circ\gamma_l)) \\
&\leqq \lambda H_{\phi_1}+(1-\lambda)H_{\phi_2}-\lambda\ln\lambda-(1-\lambda)\ln(1-\lambda).
\end{aligned}$$

Here we used the short hand notation $\phi_1(x_i\circ\gamma)$ for the state $(\phi_1 x_i\circ\gamma)(A)=\langle\phi_1|x_i\gamma(A)|\phi_1\rangle$, where ϕ_1 is the vector in the GNS-construction such that $\phi_1(A)=\langle\phi_1|A|\phi_1\rangle$ and similarly for ϕ_2.

f) Suppose B, P, μ are the optimal model for $H_\phi((\lambda\gamma_i+(1-\lambda)\gamma_i'))$. Then

$$\begin{aligned}
H_\phi((\lambda\gamma_i+(1-\lambda)\gamma_i')) &\cong S(\mu)-\sum S(\mu\circ E_i) \\
&\quad +\sum\int S((P\circ(\lambda\gamma_i+(1-\lambda)\gamma_i'))^*\mu,(P\circ(\lambda\gamma_i+(1-\lambda)\gamma_i'))_x^*)d\mu_x \\
&\leqq S(\mu)-\sum S(\mu\circ E_i)+\lambda\sum\int S((P\circ\gamma_i)^*\mu,(P\circ\gamma_i)_x^*)d\mu_x \\
&\quad +(1-\lambda)\sum\int S((P\circ\gamma_i')^*\mu,(P\circ\gamma_i')_x^*)d\mu_x \\
&\leqq \lambda H_\phi((\gamma_i))+(1-\lambda)H_\phi((\gamma_i')),
\end{aligned}$$

due to the convexity of the relative entropy. $\square$

IV. Continuity of Entropy in the Norm Topology

In this section we investigate the continuity of the function $H_\phi(\gamma_1, \ldots, \gamma_n)$ with respect to the norm topology on the γ's,

$$\|\gamma - \gamma'\| = \sup_{\|x\| \leqq 1} \|\gamma(x) - \gamma'(x)\|.$$

Lemma IV.1. *Let A be a finite dimensional C^* algebra of dimension d, and ϕ, ψ be states on A, $\varepsilon = \|\phi - \psi\|$. One has:*

$$|S(\phi) - S(\psi)| \leqq 3\varepsilon(\tfrac{1}{2} + \log(1 + d\varepsilon^{-1})).$$

Proof. One has, with $\eta(s) = -s\log s$, the equality

$$\eta(s) = \int_0^\infty \left(1 - \frac{t}{s+t} - \frac{s}{1+t}\right) \mathrm{d}t,$$

thus if we let Tr be the trace on A which is equal to 1 on any minimal projection and ϱ_ϕ, ϱ_ψ be the density matrices assigned to ϕ and ψ, we get:

$$S(\phi) - S(\psi) = \mathrm{Tr}(\eta(\varrho_\phi) - \eta(\varrho_\psi)) = \mathrm{Tr} \int_0^\infty \left[\frac{-t}{\varrho_\phi + t} + \frac{t}{\varrho_\psi + t} + \frac{\varrho_\psi - \varrho_\phi}{1+t}\right] \mathrm{d}t.$$

By construction the integrand is for each value of t an operator of norm less than one. Thus one has for any $\delta > 0$:

$$|S(\phi) - S(\psi)| \leqq \delta d + \int_\delta^\infty \|(\varrho_\phi + t)^{-1}[t(\varrho_\phi - \varrho_\psi) + t(\varrho_\psi^2 - \varrho_\phi^2) + \varrho_\phi(\varrho_\psi - \varrho_\phi)\varrho_\psi]$$
$$\times (\varrho_\psi + t)^{-1}\|_1 \frac{dt}{1+t},$$

whose $\| \ \|_1$ is the norm $T \to \mathrm{Tr}|T|$ on A. One gets

$$|S(\phi) - S(\psi)| \leqq \delta d + \left(\int_\delta^\infty \left(\frac{3t}{t^2} + \frac{1}{(1+t)^2}\right) \frac{dt}{1+t}\right) \varepsilon$$
$$\leqq \delta \mathrm{d} + \varepsilon\left(3\log\left(1 + \frac{1}{\delta}\right) + \frac{1}{2}\right).$$

Taking $\delta = \varepsilon/d$ yields the result. □

Lemma IV.2. *Let A, B be finite dimensional C^* algebras, with B abelian and $d = \dim A$. Let μ be a state on B. Then for any completely positive unital maps ϱ, $\varrho' : A \to B$ one has*

$$|s_\mu(\varrho) - s_\mu(\varrho')| \leqq 6\varepsilon(\tfrac{1}{2} + \log(1 + d\varepsilon^{-1})), \quad \textit{where} \quad \varepsilon = \|\varrho - \varrho'\|.$$

Proof. $|S_\mu(\varrho) - S_\mu(\varrho')| = |\varepsilon_\mu(\varrho) - \varepsilon_\mu(\varrho')|$, and one has

$$\varepsilon_\mu(\varrho) = \int S(\varrho^*\mu, \varrho_x^*) d\mu(x) = S(\varrho^*\mu) - \int S(\varrho_x^*) d\mu(x).$$

Furthermore $\|\varrho^*\mu - \varrho'^*\mu\| \leqq \varepsilon$, for any state μ and therefore also $\|\varrho_x^* - \varrho_x'^*\| \leqq \varepsilon$. Thus the result follows from Lemma IV.1. □

Proposition IV.3. *Let A be a C^* algebra, ϕ a state on A and A_j, $j=1,\ldots,n$, be finite dimensional C^* algebras, γ_j, γ'_j be completely positive unital maps from A_j to A. Let d be the* max *of the dimensions of the A_j's and* $\varepsilon = \max_j \|\gamma_j - \gamma'_j\|$, then

$$|H_\phi((\gamma_j)_{j=1,\ldots,n}) - H_\phi((\gamma'_j)_{j=1,\ldots,n})| \leqq 6n\varepsilon(\tfrac{1}{2} + \log(1+d\varepsilon^{-1}))\,.$$

Proof. With the notations of Definition III.2, $\varrho_j = E_j \circ P \circ \gamma_j$, $\varrho'_j = E_j \circ P \circ \gamma'_j$. As E_j and P are contractions, one has $\|\varrho_j - \varrho'_j\| \leqq \varepsilon$ for all j and hence:

$$|s_\mu(\varrho_j) - s_\mu(\varrho'_j)| \leqq 6\varepsilon(\tfrac{1}{2} + \log(1+d\varepsilon^{-1}))\,.$$

Thus the respective entropies of any abelian model differ by at most $6n\varepsilon(\frac{1}{2} + \log(1+d\varepsilon^{-1}))$. Taking the sup over all models one can find for each model for $H_\phi(\gamma')$ one for $H_\phi(\gamma')$ within the stated margin. $\square$

V. Entropy for Automorphisms of Nuclear C^* Algebras

Let A be a (unital) C^* algebra, ϕ a state on A and θ an automorphism of A which preserves ϕ.

For each completely positive unital map $\gamma : M \to A$ from a matrix algebra to A, the following limit exists by Proposition III.6b) and d):

$$h_{\phi,\theta}(\gamma) = \lim_{n\to\infty} \frac{1}{n} H_\phi(\gamma, \theta\circ\gamma, \ldots, \theta^{n-1}\circ\gamma)\,.$$

Definition V.1. The entropy $h_\phi(\theta)$ is the supremum of $h_{\phi,\theta}(\gamma)$ for all possible γ's.

Of course to be able to compute $h_\phi(\theta)$ it is necessary to have an analogue of the Kolmogorov-Sinai theorem for the ordinary entropy of automorphisms. This is achieved in the present context by the following use of Proposition III.1:

Theorem V.2. *Let τ_n be a sequence of completely positive unital maps $\tau_n : A_n \to A$ such that for suitable completely positive unital maps $\sigma_n : A \to A_n$ one has $\tau_n \circ \sigma_n \to \mathrm{id}_A$ in the pointwise norm topology. Then:*

$$\lim_{n\to\infty} h_{\phi,\theta}(\tau_n) = h_\phi(\theta)\,.$$

Proof. Let B be a finite dimensional C^* *algebra and* $\gamma : B \to A$ a completely positive unital map. We just have to show that $\underline{\lim}\, h_{\phi,\theta}(\tau_n) \geqq h_{\phi,\theta}(\gamma)$. Let $\gamma_n = \tau_n \circ \sigma_n \circ \gamma$, then $\lim_{n\to\infty} \|\gamma_n - \gamma\| = 0$, since the γ_n's are contractions which converge pointwise in norm to γ on the finite dimensional algebra B. Thus Proposition IV.3 shows that $\lim_{n\to\infty} |h_{\phi,\theta}(\gamma_n) - h_{\phi,\theta}(\gamma)| = 0$. Next by Proposition III.6a) one has $h_{\phi,\theta}(\gamma_n) \leqq h_{\phi,\theta}(\tau_n)$, so that $\underline{\lim}\, h_{\phi,\theta}(\tau_n) \geqq h_{\phi,\theta}(\gamma)$. $\square$

Remark V.3. Thanks to Proposition III.6a) it is irrelevant whether we restrict, in Definition V.1 and Theorem V.2, to matrix algebras instead of arbitrary finite dimensional C^* algebras.

The simplest example of a sequence (τ_n) to which Theorem V.2 applies is that of an AF algebra, i.e. a C^* algebra which is the norm closure of an increasing union

$\bigcup\limits_1^\infty A_n$ of finite dimensional unital subalgebras. In that case we let $\tau_n: A_n \to A$ be the homomorphism of inclusion, which we take unital. For $\sigma_n: A \to A_n$ we choose an arbitrary projection of norm one, the existence of such projections is clear since A_n is finite dimensional. Of course we cannot require that this projection preserves ϕ but this is not needed. Since $\sigma_{n|A_n} = \mathrm{id}_{A_n}$, one checks that for any such choice of σ's one has $\tau_n \circ \sigma_n(x) \to x$ in norm for any $x \in A$, and we can state:

Corollary V.4. *Let $A = \overline{\bigcup A_n}$ be an AF algebra, then for any state ϕ on A and automorphism $\theta \in \mathrm{Aut}(A, \phi)$, one has*

$$h_\phi(\theta) = \lim_{n \to \infty} h_{\phi,\theta}(A_n).$$

We always take the convention that a subalgebra $A_1 \subset A$ is standing for the (completely positive unital) inclusion $A_1 \to A$.

VI. Continuity of Entropy in the Strong Topology

In this section we sharpen the results of Sect. IV to the fact that the function $H_\phi(\gamma_1, \ldots, \gamma_n)$ is continuous even with respect to the following distance on the γ's:

$$\|\gamma - \gamma'\|_\phi = \sup_{\|x\| \leqq 1} \|\gamma(x) - \gamma'(x)\|_\phi,$$

where for any $x \in A$, $\|x\|_\phi = (\phi(x^* x))^{1/2}$.

We fix a matrix algebra M_d so that γ_i, $i = 1, \ldots, n$, is a completely positive unital map from M_d to A (III.5, 1 shows that this is not restricted). We define the *size* of an abelian model $(B, \mu, (B_j), P)$ for $(A, \phi, (\gamma_j))$ as $\sup\limits_{j=1,\ldots,n} \dim B_j$.

Given $\varepsilon > 0$, let $r(d, \varepsilon)$ be the minimum number of balls of radius $\varepsilon/2$ needed to cover the state space of $M_d(C)$.

Lemma VI.1. *There exists an abelian model with size smaller than $r(d, \varepsilon)$ and entropy larger than $H_\phi(\gamma_1, \ldots, \gamma_n) - n\varepsilon_1$, where $\varepsilon_1 = 3\varepsilon(\frac{1}{2} + \log(1 + d\varepsilon^{-1}))$.*

Proof. Let $(B, \mu, (B_j), P)$ be an abelian model for $(A, \phi, (\gamma_j))$. We can assume that $B = \bigvee\limits_1^n B_j$ so that if X_j is the spectrum of B_j then elements in $X = \mathrm{Sp}\, B$ are parametrized by $\prod\limits_1^n X_j$. Let $(U_i)_{i \in I}$ be a partition of the state space Σ of M_d in subsets with diameters less than ε. We can assume that I has $r(d, \varepsilon)$ elements. For each $j = 1, \ldots, n$, the map ϱ_j^*, where $\varrho_j = \mathrm{E}_j \circ P \circ \gamma_j$, yields a map from X_j to Σ and hence to I, which we call α_j. Thus, if r, $s \in X_j$,

$$\alpha_j(r) = \alpha_j(s) \Rightarrow \|\varrho_j^*(r) - \varrho_j^*(s)\| < \varepsilon.$$

The transpose of α_j maps $C(I)$ to $C(X_j) = B_j$, and we let $B_j' = \alpha_j^t C(I) \subset B_j$. Clearly $(B, \mu, (B_j'), P)$ is an abelian model with size $r(d, \varepsilon)$.

Let E_j' be the conditional expectation of B onto B_j'. Since $B_j' \subset B_j$ one has $E_j' = F_j \circ E_j$, where $F_j: B_j \to B_j'$ is the conditional expectation associated to the restriction of μ. Thus $\varrho_j' = E_j' \circ P \circ \gamma_j$ is equal to $F_j \circ \varrho_j$. To each $\alpha \in I$, there corresponds the minimal projector $\alpha_j^t(\alpha)$ in B_j', and therefore a state over B_j' which we denote by α. The state $(\varrho'_j)^*(\alpha)=\alpha\circ F_j\circ\varrho_j$ is the average of the states $\varrho_j^*(s), s\in X_j$, for the normalized measure $v_j=F_j^*(\alpha)$ which is supported by $\{s\in X_j, \alpha_j(s)=\alpha\}$. Thus one has:

$$\|\varrho_j'^*(\alpha)-\varrho_j^*(s)\|<\varepsilon, \qquad s\in X_j, \qquad \alpha_j(s)=\alpha,$$

and Lemma IV.1 shows that

$$|S(\varrho_j'^*(\alpha))-\int S(\varrho_j^*(s))dv_j|\leqq 3\varepsilon(\tfrac{1}{2}+\log(1+d\varepsilon^{-1}))=\varepsilon_1.$$

Since $\varrho_j^*\mu=\varrho_j'^*\mu$, one gets that $|\varepsilon_\mu(\varrho_j)-\varepsilon_\mu(\varrho'_j)|\leqq\varepsilon_1$ for $j=1,\dots,n$. Now the entropies of the two abelian models are:

$$S(\mu_{|B_1\vee\dots\vee B_n})-\sum_{j=1}^{n}S(\mu_{|B_j})+\sum_{j=1}^{n}\varepsilon_\mu(\varrho_j),$$

$$S(\mu_{|B'_1\vee\dots\vee B'_j})-\sum_{j=1}^{n}S(\mu_{|B_j})+\sum_{j=1}^{n}\varepsilon_\mu(\varrho'_j).$$

Thus it remains to show that:

$$S(\mu_{|\vee B'_j})-\sum S(\mu_{|B'_j})\geqq S(\mu_{|\vee B_j})-\sum S(\mu_{|B_j}).$$

Let $\bar\mu$ be the measure on ΠX_j which is the product of the $\mu_{|B_j}$ and F the conditional expectation $B_j\to B'_j$. Then the inequality is equivalent to

$$-S(\bar\mu\circ F,\mu\circ F)\geqq -S(\bar\mu,\mu),$$

which also holds according to (I.1.4c). □

Lemma VI.2. *Let B be a commutative algebra with* $\dim B=r$, ϱ, ϱ' *be completely positive maps from* $M_d(C)$ *to B and μ a state on B. Then, if* $\|\varrho-\varrho'\|_\mu<\varepsilon$ *one has*

$$|s_\mu(\varrho)-s_\mu(\varrho')|\leqq\delta(r,d,\varepsilon)\quad \textit{with}\quad \lim_{\varepsilon\to 0}\delta(r,d,\varepsilon)=0.$$

Proof. $s_\mu(\varrho)-s_\mu(\varrho')=-\varepsilon_\mu(\varrho)+\varepsilon_\mu(\varrho')$, and since $\|\mu\circ\varrho-\mu\circ\varrho'\|\leqq\varepsilon$, we just need to evaluate:

$$\int_X (S(\varrho_x)-S(\varrho_x'^*))d\mu(x), \quad \text{where} \quad X=\operatorname{Sp}B.$$

Let $\varepsilon_1>0$ and $X_1=\left\{x\in X, \mu(x)\geqq\frac{1}{r}\varepsilon_1\right\}$, then for $x\in X_1$ one has:

$$|\varrho(a)-\varrho'(a)|_x^2\leqq r\varepsilon_1^{-1}\|\varrho(a)-\varrho'(a)\|_\mu^2$$

for any $a\in M_d$. The hypothesis of VI.2 implies

$$\|\varrho_x^*-\varrho_x'^*\|\leqq(r\varepsilon_1^{-1})^{1/2}\varepsilon=\varepsilon_2\quad \forall x\in X_1.$$

By Lemma IV.1 we have

$$\int_{X_1} S(\varrho_x^*)-S(\varrho_x'^*)|\,d\mu(x)\leqq 3\varepsilon_2(\tfrac{1}{2}+\log(1+\varepsilon_2^{-1}d)).$$

Moreover

$$\int_{X_1^c}|S(\varrho_x^*)-S(\varrho_x'^*)|d\mu(x)\leqq 2\log d\mu(X_1^c)\leqq 2(\log d)\varepsilon_1.$$

Taking $\varepsilon_1 = \varepsilon^{2/3} r^{1/3}$, we get $\varepsilon_2 = \varepsilon^{2/3} r^{1/3}$, and hence:

$$\int_X |S(\varrho_x) - S(\varrho'_x)| d\mu(x) \leqq \varepsilon^{2/3} r^{1/3} (\tfrac{3}{2} + 3 \log(1 + d\varepsilon^{-2/3} r^{-1/3}) + 2 \log d) = \varepsilon_3 .$$

Thus finally

$$|\varepsilon_\mu(\gamma) - \varepsilon_\mu(\gamma')| \leqq \varepsilon \tfrac{3}{2} + 3 \log(1 + d\varepsilon^{-1})) + \varepsilon_3$$

and we can take

$$\begin{aligned}\delta(r, d, \varepsilon) = {} & \varepsilon^{2/3} r^{1/3} (\tfrac{3}{2} + 3 \log(1 + d\varepsilon^{-2/3} r^{-1/3}) + \log d) \\ & + \varepsilon(\tfrac{3}{2} + 3 \log(1 + d\varepsilon^{-1})) . \quad \square\end{aligned}$$

Theorem VI.3. *For any $d < \infty$ and $\alpha > 0$ there exists $\varepsilon > 0$ such that for any C^* algebra A with state ϕ, any n and completely positive unital maps $\gamma_j, \gamma'_j : M_d(C) \to A$ such that $\|\gamma_j - \gamma'_j\|_\phi \leqq \varepsilon$ $\forall j = 1, \ldots, n$, one has*

$$|H_\phi((\gamma_j)_{j=1,\ldots,n}) - H_\phi((\gamma'_j)_{j=1,\ldots,n})| \leqq n\alpha .$$

Proof. Let $(B, (B_j), \mu, P)$ be an abelian model, then note that with $\varrho_j = E_j \circ P \circ \gamma_j$, $\varrho'_j = E_j \circ P \circ \gamma'_j$, one has

$$\|\varrho_j - \varrho'_j\|_{\mu_j} \leqq \|\gamma_j - \gamma'_j\|_\phi .$$

This inequality follows from the Schwarz positivity of $E_j \circ P$, since $\mu_j \circ \mathrm{E}_j \circ \mathrm{P} = \mu \circ P = \phi$ and $(\varrho_j - \varrho'_j) = (E_j \circ P) \circ (\gamma_j - \gamma'_j)$. Now given $d < \infty$ and $\alpha > 0$, ε_0 such that $3\varepsilon_0(\frac{1}{2} + \log(1 + \varepsilon_0^{-1} d)) = \alpha/3$, and then choose $\varepsilon > 0$ such that with the notation of Lemma VI.2, one has $\delta(r(d, \varepsilon_0), d, \varepsilon) \leqq \alpha/3$. Assume $\|\gamma_j - \gamma'_j\|_\phi \leqq \varepsilon$ $\forall j = 1, \ldots, n$. By Lemma VI.1, to prove the inequality VI.3 we just have to show that for any abelian model of size r, the two entropies differ by at most $n\alpha/3$. But this is exactly the content of Lemma VI.2, since

$$\|\varrho_j - \varrho'_j\|_{\mu_j} \leqq \|\gamma_j - \gamma'_j\|_\phi \leqq \varepsilon . \quad \square$$

Corollary VI.4. *If the θ_i are completely positive unital maps with $\phi \circ \theta_i = \phi$, then again due to Schwarz positivity $\|\theta_i \circ \gamma_i - \theta_i \circ \gamma'_1\|_\phi \leqq \|\gamma_i - \gamma'_i\|_\phi$, and under the hypothesis of VI.3,*

$$|H_\phi((\theta_i \circ \gamma_i)) - H_\phi((\theta_i \circ \gamma'_i))| \leqq n\alpha .$$

VII. Entropy for Automorphisms of Hyperfinite von Neumann Algebras

In this section we shall use the estimates of Sect. VI to show that the entropy of an automorphism θ of a nuclear C^* algebra A with invariant state ϕ determines the entropy of the corresponding automorphism of the von Neumann algebra $\pi_\phi(A)'' = M$, the weak closure of A in the GNS construction of ϕ.

Let us start with a von Neumann algebra M and a *normal* state ϕ. When we deal with completely positive unital maps $M_d(C) \xrightarrow{\gamma_j} M$ there is no special assumption to make. Moreover note that for any abelian model $(B, \mu, (B_j), P)$ for $(M, \phi, (\gamma_j))$, with B finite dimensional, the map P satisfies $\mu \circ P = \phi$, so that P is necessarily normal if μ is faithful. We can always reduce B by the support of μ so that we may assume that μ is faithful. This shows that all properties of the function

$H_\phi(\gamma_1, \ldots, \gamma_n)$ apply in the von Neumann algebra context and involve only normal completely positive maps.

In particular, as in Sect. V, $h_{\phi,\theta}(\gamma)$ makes sense for any ϕ-preserving automorphism θ of M, and we take:

Definition VII.1. Let M be a hyperfinite von Neumann algebra, ϕ a normal state on M, and $\theta \in \mathrm{Aut}\, M$, $\phi \circ \theta = \phi$. Then the entropy $h_\phi(\theta)$ is the supremum of $h_{\phi,\theta}(\gamma)$ for all completely positive unital maps of a matrix algebra to M.

We shall shortly see that we just need to consider *subalgebras* of M instead of arbitrary completely positive maps. A need not contain finite dimensional subalgebras, but its entropy is determined by those of $M = \pi_\phi(A)''$. For the time note that we cannot apply Theorem V.2 since M is *not* nuclear as a C^* algebra. [For instance $\mathscr{L}(\mathscr{H})$, $\dim \mathscr{H} = \infty$, does not have the approximation property.]

Our first task is to compare the C^* algebra and von Neumann algebra definitions:

Theorem VII.2. *Let A be a nuclear C^* algebra; ϕ a state on A, $\theta \in \mathrm{Aut}\, A$, $\phi \circ \theta = \phi$. Let $M = \pi_\phi(A)''$, $\bar{\phi}$, $\bar{\theta}$ the natural extensions of ϕ, θ to M, then*

$$h_{\bar{\phi}}(\bar{\theta}) = h_\phi(\theta).$$

It should be remarked [16] *that A being nuclear, we have a hyperfinite $M = \pi_\phi(A)''$.*

Proof. Since any abelian model for (A, ϕ) gives one for $(M, \bar{\phi})$, it is clear that for any completely positive unital map $M_d \xrightarrow{\gamma} A$ one has $h_{\phi,\theta}(\gamma) = h_{\bar{\phi},\bar{\theta}}(\gamma)$. This shows that $h_\phi(\theta) \leqq h_{\bar{\phi}}(\bar{\theta})$.

Next, since (Remark III.5) completely positive maps $M_d \to A$ (respectively M) identify with $M_d(A)^\dagger$ (respectively $M_d(M)^\dagger$) it follows that for any completely positive unital map $M_d \xrightarrow{\gamma} M$ one can find a sequence (X_n) of completely positive unital maps: $M_d \xrightarrow{\gamma_n} A$ such that $\lim\limits_{n \to \infty} \|\gamma - \gamma_n\|_\phi = 0$. Then Theorem VI.3 shows that $h_{\bar{\phi},\bar{\theta}}(\gamma_n) \to h_{\bar{\phi},\bar{\theta}}(\gamma)$ when $n \to \infty$. $\square$

Our next step is to show that we have

Lemma VII.3. *Let M, ϕ, θ be as above, then* $\sup\limits_{\gamma} h_{\phi,\theta}(\gamma) = \sup\limits_{N} h_{\phi,\theta}(N)$ *where N runs through all finite dimensional subalgebras of M.*

Proof. There exists an increasing sequence of finite dimensional subalgebras $N_j \subset N_{j+1}$ of M and normal conditional expectations $E_j : M \to N_j$, $E_j E_{j+1} = E_j$, such that for any $x \in M$ one has $E_j(x) \to x$ strongly in the representation π_ϕ.

Now let $\tau_j : N_j \to M$ be the homomorphism of inclusion. We just have to show that for any γ, completely positive and unital, $\gamma : M_d \to M$ one has:

$$h_{\phi,\theta}(\gamma) \leqq \lim h_{\phi,\theta}(\tau_j).$$

Let then $\gamma_j = \tau_j \circ E_j \circ \gamma$. One has $\gamma_j(x) \to \gamma(x)$ strongly for any $x \in M_d$ and hence $\|\gamma_j - \gamma\|_\phi \to 0$ when $j \to \infty$. Thus, by Theorem VI.3 one gets $h_{\phi,\theta}(\gamma_j) \xrightarrow[j \to \infty]{} h_{\phi,\theta}(\gamma)$, but by Proposition III.6a) one has:

$$h_{\phi,\theta}(\gamma_j) \leqq h_{\phi,\theta}(\tau_j).$$

Since for finite dimensional subalgebras the present definition of the H_ϕ's coincides with the one of [11] (see III.5.3), so does $h_{\phi,\theta}(N)$ for von Neumann algebras.

It is clear that Lemma VII.3 contains implicitly a Kolmogorov-Sinai theorem, but we shall get a better such result, not invoking the conditional expectations E_j.

Theorem VII.4. *Let M be a hyperfinite von Neumann algebra, ϕ a normal state in M, $\theta \in \operatorname{Aut} M$, $\phi \circ \theta = \phi$. Let N_k be an ascending sequence of finite dimensional von Neumann algebras with $\bigcup_k N_k$ weakly dense in M, then:*

$$h_\phi(\theta) = \lim_{k\to\infty} h_{\phi,\theta}(N_k).$$

Proof. Let $N \subset M$ be a finite dimensional subalgebra. We have to show that $h_{\phi,\theta}(N) \leqq \lim_{k\to\infty} h_{\phi,\theta}(N_k)$.

For each k we shall construct a completely positive unital map $\bar{\gamma}_k : N \to N_k$ in such a way that for any $x \in N$ one has $\gamma_k(x) \to x$ in the strong topology where $\gamma_k = i_k \circ \bar{\gamma}_k$ and i_k is the inclusion $N_k \to M$. It then follows from Theorem VI.3 that

$$h_{\phi,\theta}(N) = \lim_{k\to\infty} h_{\phi,\theta}(\gamma_k).$$

By Proposition III.6a) $h_{\phi,\theta}(\gamma_k) \leqq h_{\phi,\theta}(i_k)$, thus the conclusion. To construct the γ_k's one starts from matrix units (e^q_{ij}) in N, where the label q comes from the center of N and for each q, the indices i, j vary from 1 to n_q. Now if ε is small enough and k so large that the unit ball of N is contained in the unit ball of N_k up to ε in the metric $\|x-y\|^\#_\phi = (\phi((x-y)^*(x-y) + x-y)(x-y)^*))^{1/2}$ one can (cf. [8]) construct a system of matrix units (f^q_{ij}) in N_k such that

$$\|e^q_{ij} - f^q_{ij}\|^\#_\phi \leqq \delta(\varepsilon) \quad \forall i, j, q,$$

where $\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$. Now it is not true in general that $\sum_{i,q} f^q_{ii} = 1$, but $Q = \sum_{i,q} f^q_{ii}$ is a projection, and thus we can define γ_k by:

$$\gamma_k(x) = \theta(x) + \phi(x)(1-Q) \quad \forall x \in N,$$

where θ is the homomorphism sending e^q_{ij} to f^q_{ij}. Now γ_k is completely positive, unital, and since $\phi(1-Q) \to 0$ when $k \to \infty$, one has $\gamma_k(x) \to x$ strongly for any $x \in N$. $\square$

Properties of the Entropy of an Automorphism VII.5. i) Covariance: $h_\phi(\theta) = h_{\phi\circ\sigma}(\sigma^{-1}\theta\sigma)$ for all automorphisms σ of A (respectively M).

ii) Additivity in $\theta : h_\phi(\theta^n) = |n| h(\theta)\ \forall n \in N$.

iii) Affinity in $\phi : h_{\lambda\phi_1 + (1-\lambda)\phi_2}(\theta) = \lambda h_{\phi_1}(\theta) + (1-\lambda) h_{\phi_2}(\theta)\ \forall 0 \leqq \lambda \leqq 1$.

Proof. i) Since $H_{\phi\circ\sigma}(\sigma^{-1}\gamma_1, \ldots, \sigma^{-1}\gamma_k) H_\phi(\gamma_1, \ldots, \gamma_k)$, we have

$$\frac{1}{k} H_{\phi\circ\sigma}(\sigma^{-1}\gamma, \sigma^{-1}\theta\sigma\sigma^{-1}\gamma, \ldots, (\sigma^{-1}\theta\sigma)^{k-1} \circ \sigma^{-1}\gamma) = \frac{1}{k} H_\phi(\gamma, \theta\circ\gamma, \ldots, \theta^{k-1}\circ\gamma).$$

Taking $k \to \infty$ and $\sup_\gamma$ gives i).

ii) From III.6d) we can infer the monotonicity

$$H(\gamma, \theta \circ \gamma, \ldots, \theta^n \circ \gamma, \ldots, \theta^{kn-1} \circ \gamma) \geqq H(\gamma, \theta^n \circ \gamma, \ldots, \theta^{n(k-1)} \circ \gamma).$$

Suppose γ gives, within some ε, the sup for $h(\theta^n)$, then

$$\begin{aligned} h_\phi(\theta) &\geqq \lim_{k \to \infty} \frac{1}{kn} H(\gamma, \theta \circ \gamma, \ldots, \theta^{kn-1} \circ \gamma) \\ &\geqq \lim_{k \to \infty} \frac{1}{kn} H(\gamma, \theta^n \circ \gamma, \ldots, \theta^{(k-1)n} \circ \gamma) = \frac{1}{n}(h_\phi(\theta^n) - \varepsilon). \end{aligned}$$

On the other hand, since M was supposed to be hyperfinite, we have seen in the Proof of VII.4 that $\forall \delta > 0$, $a \in N$, $i = 0, \ldots, n-1$, there exists a finite-dimensional B and $\gamma^i : \theta^i N \to B$ such that

$$\phi((\theta^i a^* - \gamma^i(a^*))(\theta^i a - \gamma^i(a))) < \delta,$$

and thus, because of the invariance of ϕ under θ,

$$\phi[(\theta^{i+k} a^* - \theta^k \gamma^i(a^*))(\theta^{i+k} a - \theta^k \gamma^i(a))] < \delta \quad \forall i, 0 \leqq i \leqq n-1, \quad \text{and} \quad \forall k.$$

These are the conditions under which VI.3 tells us

$$\begin{aligned} &H_\phi(N, \theta N, \ldots, \theta^n N, \ldots, \theta^{nk-1} N) \\ &\leqq H_\phi(\gamma^0, \ldots, \gamma^{n-1}, \theta^n \gamma^0, \ldots, \theta^n \gamma^{n-1}, \ldots, \theta^{(k-1)n} \gamma^{n-1}) + \varepsilon \\ &\leqq H_\phi(B, \theta^n B, \ldots, \theta^{(k-1)n} B) + \varepsilon. \end{aligned}$$

We have for N optimizing $h(\theta)$,

$$\begin{aligned} h(\theta) - \varepsilon &= \lim_{k \to \infty} \frac{1}{nk} H_\phi(N, \theta N, \ldots, \theta^{nk-1} N) - \varepsilon \\ &\leqq \lim_{k \to \infty} \frac{1}{nk} H_\phi(B, \theta^n B, \ldots, \theta^{(k-1)n} B) \leqq \frac{1}{n} h(\theta^n). \end{aligned}$$

iii) Using first ii), then III.6e), we have

$$\begin{aligned} &|h_{\lambda\phi_1 + (1-\lambda)\phi_2}(\theta) - \lambda h_{\phi_1}(\theta) - (1-\lambda) h_{\phi_2}(\theta)| \\ &= \frac{1}{n} |h_{\lambda\phi_1 + (1-\lambda)\phi_2}(\theta^n) - \lambda h_{\phi_1}(\theta^n) - (1-\lambda) h_{\phi_2}(\theta^n)| \\ &\leqq \frac{1}{n}(-\lambda \log \lambda - (1-\lambda)\log(1-\lambda)). \end{aligned}$$

Letting n go to infinity gives iii).

Remark VII.6. If θ is the modular automorphism of ϕ, so is $\sigma^{-1}\theta\sigma$ for $\phi \circ \sigma$:

$$\phi(A\theta_t B) = \phi(B\theta_{i-t} A) \Leftrightarrow \phi(\sigma A \sigma\sigma^{-1}\theta_t \sigma B) = \phi \circ \sigma(B\sigma^{-1}\theta_{i-t}\sigma A).$$

Hence the entropy coincides for conjugate modular automorphisms. Since for hyperfinite III_1-factors the entropy of a modular automorphism may assume any value $\geqq 0$, this illustrates the discontinuity of $\theta \to h_\phi(\theta)$, since they are all approximately conjugate [17].

VIII. Estimate of $H_\phi(N_1, \ldots, N_k)$

Combining Theorem VII.2 with Theorem VII.4 the difficulty in computing the entropy $h_\phi(\theta)$ of an automorphism of a nuclear C^* algebra is reduced to the estimation of $H_\phi(N_1, \ldots, N_k)$, where the N_j's are finite dimensional subalgebras of a hyperfinite von Neumann algebra M. A useful and straightforward upper bound is obtained by Proposition III.6a) and c):

Lemma VIII.1. *If $N_1, \ldots, N_k$ are subalgebras of a finite dimensional subalgebra $N \subset M$, then: $H_\phi(N_1, \ldots, N_k) \leqq S(\phi_{|N})$.*

Proof. $H_\phi(N_1, \ldots, N_k) \leqq H_\phi(N, \ldots, N) = H_\phi(N) \leqq S(\phi_{|N})$. □

When ϕ is a trace, one has the following lower bound for H_ϕ, which allows to compute it in many cases:

Proposition VIII.2 [8]. *Let A_i be pairwise commuting abelian subalgebras of M and A the algebra they generate, then, if ϕ is a trace*

$$H_\phi(A_1, \ldots, A_n) = S(\phi_{|A}).$$

This statement is no longer true, even for $n=1$, when ϕ is not a trace. For instance, if $M = M_2(C)$, ϕ is a pure state and A is an abelian subalgebra of M such that $\phi_{|A}$ is not pure, one has:

$$0 = H_\phi(A) < S(\phi_{|A}).$$

[One has $H_\phi(A) = 0$ because $H_\phi(M) = 0$.]

It is however tempting in the setup of Proposition VIII.2 (with ϕ non-tracial) to construct an abelian model (cf. Definition III.2) where $B_j = A_j$, B is the abelian algebra generated by the A_j's and μ is the restriction of ϕ to B. What is missing still is the completely positive map P from M to B. The inclusion $\gamma: B \to M$ is completely positive, but we need also a completely positive map from M to B. Given two von Neumann algebras M_1, M_2, with faithful normal states ϕ_1, ϕ_2 every unital completely positive map $\gamma: M_1 \to M_2$ such that $\phi_2 \circ \gamma = \phi_1$ has a canonical adjoint $\gamma^\dagger: M_2 \to M_1$, uniquely determined by the equality:

$$(\phi_2^{1/2} x \phi_2^{1/2}) \circ \gamma = \phi_1^{1/2} \gamma^\dagger(x) \phi_1^{1/2} \quad \forall x \in M_2.$$

Here we have used the following notation, given a von Neumann algebra M, a faithful normal state ϕ and an $x \in M$, we let $\psi_x = \phi^{1/2} x \phi^{1/2} \in M_*$ be the linear functional on M such that:

$$\psi_x(y) = \phi(y \sigma^\phi_{-i/2}(x)) = \phi(x \sigma^\phi_{-i/2}(y)) \quad \forall y \in M,$$

where σ_t^ϕ, $t \in \mathbb{R}$ is the modular automorphism group of ϕ. By [18] the map $x \to \phi^{1/2} x \phi^{1/2}$ is a completely positive bijection of M with the linear span of the face of ϕ in M_*. Thus:

Proposition VIII.3. a) *Let $\gamma: M_1 \to M_2$ be completely positive unital with $\phi_2 \circ \gamma = \phi_1$, then its adjoint $\gamma^\dagger: M_2 \to M_1$ is completely positive unital with $\phi_1 \circ \gamma^\dagger = \phi_2$.*

b) *One has $(\gamma \circ \gamma')^\dagger = \gamma'^\dagger \circ \gamma^\dagger$, $\gamma^{\dagger\dagger} = \gamma$.*

c) *If γ is the inclusion of a von Neumann subalgebra $M_1 \subset M_2$ and $\sigma_t^{\phi_2}(M_1) = M_1$ $\forall t \in \mathbb{R}$, then $\gamma^\dagger$ is the canonical conditional expectation of M_2 on M_1.*

Proof. Cf. [18]. □

Coming back to the above discussion we can now consider the completely positive map $P=\gamma^\dagger$ from M to B, where γ is the inclusion $\gamma: B\to M$. Then B, $\mu=\phi_{|B}$, B_j, $P=\gamma^\dagger$ is an abelian model. For each j, $E_j=i_j^\dagger$, where i_j is the inclusion $B_j\to B$. Thus $E_j\circ P=(i_j^\dagger\circ\gamma^\dagger)=(\gamma\circ i_j)^\dagger=\gamma_j^\dagger$ and $E_j\circ P\circ\gamma_j=\gamma_j^\dagger\circ\gamma_j$. Thus we get:

Proposition VIII.4. *Let A_i, $i=1,\dots,n$, be pairwise commuting abelian subalgebras of M and A the algebra they generate. Then*

$$H_\phi(A_1,\dots,A_n)\geqq S(\phi_{|A})-\sum_1^n s_{\phi_{|A_j}}(\gamma_j^\dagger\circ\gamma_j),$$

where $\gamma_j: A_j\to M$ is the inclusion.

Let us now compute $s_\mu(\gamma^\dagger\circ\gamma)$, where γ is the inclusion of the finite-dimensional abelian algebra A in M and $\mu=\phi_{|A}$. Let $e_1,\dots,e_k$ be the minimal projections of A.

Lemma VIII.5. *Let $\mu_{ij}=(\phi^{1/2}e_i\phi^{1/2})(e_j)$, $\mu_j=\phi(e_j)$. Then*

$$s_\mu(\gamma^\dagger\circ\gamma)=\sum_{i,j}\eta(\mu_{ij})-\sum_j\eta(\mu_j).$$

Proof. One has by VIII.3a) $\mu\circ\gamma^\dagger\circ\gamma=\mu$. Thus $s_\mu(\gamma^\dagger\circ\gamma)=\sum S(\psi_j)\mu_j$, where ψ_j is the composition of the pure state $\mu(e_j\cdot)/\mu_j$ with $\gamma^\dagger\circ\gamma$. Furthermore,

$$\mu_j\psi_j(e_i)=\mu(e_j\gamma^\dagger(e_i))=(\mu^{1/2}\gamma^\dagger(e_i)\mu^{1/2})(e_j)=(\phi^{1/2}e_i\phi^{1/2})(e_j)=\mu_{ij}.$$

Thus:

$$S(\psi_j)=\sum_i\eta(\mu_{ij}/\mu_j)\quad\text{and}\quad s_\mu(\gamma^\dagger\circ\gamma)=\sum_{i,j}\eta(\mu_{ij})-\sum_j\eta(\mu_j).\quad\square$$

Corollary VIII.6. a) *Let $N\subset M$ be a finite dimensional subalgebra which is invariant under the modular automorphism group σ_t^ϕ, $t\in\mathbb{R}$, then $H_\phi(N)=S(\phi_{|N})$.*

b) *Let $N\subset M$ be a finite dimensional subalgebra, let M_ϕ be the centralizer, i.e. the set of elements of M which are invariant under σ_t^ϕ, assume that $N\cap M_\phi$ contains an maximal abelian subalgebra of N, then*

$$H_\phi(N)=S(\phi_{|N}).$$

Proof. It is enough to prove b) since, if $\sigma_t^\phi(N)=N\ \forall t\in\mathbb{R}$, the centralizer of $\phi_{|N}$ is $M_\phi\cap N$. Let then $A\subset M_\phi\cap N$ be maximal abelian in N. One has $A\subset N_\psi$, where $\psi=\phi_{|N}$, and hence $S(\phi_{|N})=S(\phi_{|A})$. Thus $\mu_{ij}=\delta_{ij}\mu_j$ and by Lemma VIII.5 one has $s_\phi(\gamma^\dagger\circ\gamma)=0$, where $\gamma: A\subset M$ is the inclusion. The conclusion follows from VIII.4.

Remark VIII.7. Given a finite dimensional subalgebra N of M, the difference $\delta_\phi(N)=S(\phi_{|N})-H_\phi(N)$ is a positive convex function of ϕ. In particular, $\{\phi,\delta_\phi(N)=0\}$ is a convex set which contains all states for which the hypothesis of VIII.6b) holds.

Corollary VIII.8. *Let $N_1,\dots,N_k$ be finite dimensional subalgebras of M and assume that they contain abelian subalgebras $A_j\subset N_j\cap M_\phi$ pairwise commuting and such that $A=\bigvee A_j$ is maximal abelian in the algebra N generated by the N_j's. Then:*

$$H_\phi(N_1,\dots,N_k)=S(\phi_{|N}).$$

Proof. One has $S(\phi_{|N})=S(\phi_{|A})$, since $A\subset M_\phi$, thus Proposition VIII.4 applies, with $s_\phi(\gamma_j^\dagger\circ\gamma_j)=0$ since $A_j\subset M_\phi$. $\square$

In realistic situations however M_ϕ is often trivial and it becomes crucial to estimate $s_\phi(\gamma^\dagger \circ \gamma)$. By Lemma VIII.5 it is a measurement of the lack of commutativity between A and ϕ.

We shall now estimate $H_\phi(N_1, \ldots, N_k)$ in the following situation, the N_j's are pairwise commuting matrix algebras $N_j \subset M$ with the same dimension d, and each N_j contains a matrix subalgebra $M_j \subset N_j$, $\dim M_j = d'$ satisfying the following condition of commutation with ϕ:

$$\forall x \in M_j \quad \text{one has} \quad \|\sigma^\phi_{i/2}(\sigma^{\phi_j}_{-i/2}(x)) - x\| \leqq \varepsilon \|x\|, \quad \text{where} \quad \phi^j = \phi_{|N_j}.$$

Proposition VIII.9. *Under the above hypothesis, one has*

$$H_\phi(N_1, \ldots, N_k) \geqq S(\phi_{|A}) - 6k\varepsilon(\tfrac{1}{2} + \log(d\varepsilon^{-1})) - 2k\log(d/d'),$$

where $A = \bigvee A_j$ is the algebra generated by maximal abelian subalgebra A_j of the centralizer of ϕ_j in N_j.

Proof. First $H_\phi(N_1, \ldots, N_k) \geqq H_\phi(M_1, \ldots, M_k)$ and we just have to prove the inequality for the latter. We construct an abelian model as follows, $B = A$, $B_j = A_j$, $\mu = \phi_{|B}$ and the completely positive map P is the adjoint of the inclusion $\gamma_B : B \to M$. Now for each j the expectation $E_j : B \to B_j$ is the adjoint of the inclusion $B_j \subset B$, so that $E_j \circ P = (\gamma_{B_j})^\dagger$, where $\gamma_{B_j} : B_j \to M$ is the inclusion. Let $\gamma_{M_j} : M_j \to M$ be the inclusion, we need to evaluate the entropy defect of $(\gamma_{B_j})^\dagger . \gamma_{M_j} \equiv \alpha_j$, and then use:

$$H_\phi(M_1, \ldots, M_k) \geqq S(\phi_{|A}) - \sum s_\mu(\alpha_j).$$

Let γ'_{B_j}, γ'_{M_j} be the inclusions in N_j, so that:

$$\gamma_{B_j} = \gamma_{N_j} \circ \gamma'_{B_j}, \qquad \gamma_{M_j} = \gamma_{N_j} \circ \gamma'_{M_j},$$
$$\gamma^\dagger_{B_j} \circ \gamma_{M_j} = (\gamma'_{B_j})^\dagger \circ (\gamma_{N_j})^\dagger \circ \gamma_{N_j} \circ \gamma'_{M_j}.$$

We claim first that $\|(\gamma_{N_j})^\dagger \circ \gamma_{N_j} \circ \gamma'_{M_j} - \gamma'_{M_j}\| \leqq \varepsilon$. This follows from

Lemma VIII.10. *Let $N \subset M$ be a von Neumann subalgebra and $\psi = \phi_{|N}$, $\gamma : N \to M$ the inclusion. Let $x \in N$ be in the domain of $\sigma^\psi_{i/2}$ with $\sigma^\psi_{-i/2}(x)$ in the domain of $\sigma^\phi_{i/2}$, then:*

$$\|\gamma^\dagger \circ \gamma(x) - x\| \leqq \|\sigma^\phi_{i/2}(\sigma^\psi_{-i/2}(x)) - x\|.$$

Proof. For any $y \in N$ one has: $\psi^{1/2} y \psi^{1/2}(\gamma^\dagger \gamma(x)) = (\phi^{1/2} y \phi^{1/2})(x)$ and

$$(\psi^{1/2} y \psi^{1/2})(x) = \psi(y\sigma^\psi_{-i/2}(x)) = \phi(y\sigma^\psi_{-i/2}(x)) = (\phi^{1/2} y \phi^{1/2})(\sigma^\phi_{i/2}\sigma^\psi_{-i/2}(x)).$$

Thus for $y \geqq 0$ one gets:

$$|(\psi^{1/2} y \psi^{1/2})(\gamma^\dagger \circ \gamma(x) - x)| \leqq \psi(y)\, \|\sigma^\phi_{i/2}\sigma^\psi_{-i/2}(x) - x\|.$$

Thus by Lemma IV.2 one has:

$$|s_\mu(\alpha_j) - s_\mu((\gamma'_{B_j})^\dagger \circ \gamma'_{M_j})| < 6\varepsilon(\tfrac{1}{2} + \log(d\varepsilon^{-1})).$$

It remains to estimate $s_\mu(\gamma'^\dagger_{B_j} \circ \gamma'_{M_j})$, where now everything takes place inside N_j. This follows from:

Lemma VIII.11. *Let $P \subset N$ be matrix algebras, ϕ a state on N, $\gamma_P : P \to N$ the inclusion, A an abelian maximal subalgebra of N in the centralizer of ϕ. Then $s_\phi(\gamma^\dagger_A \circ \gamma_P) \leqq 2\log d''$, where $d'' = \dim N/\dim P$.*

Proof. Let $\mu = \phi_{|A}$, $\alpha = \gamma_A^\dagger \circ \gamma_P$. One has:

$$s_\mu(\alpha) = S(\mu) - S(\mu \circ \alpha) + \int S(\alpha_x) d\mu(x).$$

Since $A \subset N_\phi$, $\gamma_A^\dagger$ is the conditional expectation of N onto A. Thus

$$s_\mu(\alpha) = S(\phi) - S(\phi \circ \gamma_P) + \int S(\omega_x \circ \gamma_P) d\mu(x),$$

where ω_x is a pure state on N for each $x \in \mathrm{Sp}(A)$. Since for matrix algebras $P \subset N$ implies $N = P \otimes P'$, subadditivity [7] tells us $|S(\phi) - S(\phi_{|P})| \leqq S(\phi_{|P'}) \leqq \log d''$ and $S(\omega_{x|P}) = S(\omega_{x|P'}) \leqq \log d''$, because ω_x is pure for N in the optimal decomposition. □

IX. Entropy of Space Translations for the Gibbs State of a One-Dimensional Quantum System with Finite Range Interaction

We shall adopt the notations of [19]. Thus we represent the lattice points by Z and for each lattice point j we have a matrix algebra $(Aj) = M_q(C)$, where q is finite independent of j. To each subset $I \subset Z$ we associate the C^* algebra $A(I) = \bigotimes_{j \in I} A(j)$. We let $\tau_s(n)$, $n \in Z$, be the automorphism of $A(Z)$ given by the lattice translation by n. As in [19, 2.12] we let $\Phi \in A([0,r])$ be the contribution of one lattice site to the interaction, and r the range of the interaction. The Gibbs state ϕ_Φ for the infinite system is invariant under the generator $\theta = \tau_s(1)$ of space translations, and our aim is to evaluate the entropy $h_\phi(\theta)$.

Theorem IX.1. *Assume to every $I' = [0,n]$ and $\varepsilon > 0$ there exists an $I = [-N, n+N]$ such that, with ψ the restriction of ϕ to $A(I)$*

$$\|\sigma_{i/2}^\phi(\sigma_{-i/2}^\psi(Q)) - Q\| \leqq \varepsilon \|Q\| \quad \forall Q \in A(I')$$

and $\lim_{n\to\infty} N/n = 0$. Then $h_\phi(\theta)$ is equal to the mean entropy

$$h_\phi(\theta) = S(\phi) = \lim_{|I| \to \infty} \frac{1}{|I|} S(\phi_{|A(I)}).$$

The mean entropy is introduced in [4], where it is shown to converge and, in fact, to be an infimum. Examples where the above condition is satisfied will be discussed in the Appendix.

The inequality $h_\phi(\theta) \leqq S(\phi)$ is obvious, since by Corollary V.4 one has $h_\phi(\theta) = \lim_{n\to\infty} h_{\phi,\theta}(A[0,n])$ and the algebra generated by the $\theta^s A([0,n])$, $s = 0, 1, \ldots, k$, is equal to $A([0, n+k])$ so that by VIII.1:

$$\frac{1}{k} H_\phi(A([0,n]), \ldots, \theta^k A([0,n]), \leqq \frac{1}{k} S(\phi_{|A([0,n+k])}).$$

When $k \to \infty$, the right-hand side of the inequality converges to $S(\phi)$. To prove the other inequality we just have to find, given $\varepsilon > 0$, intervals I, J such that $h_{\phi,\theta}(A(I)) \geqq \frac{1}{|J|} S(\phi_{|A(J)}) - \varepsilon$. In order to exploit Proposition VIII.9, we choose I of the form: $[-N, n+N]$ for $n \in \mathbb{N}$, $N \in \mathbb{N}$. We let $I' = [0, n]$.

Lemma IX.2. *Let I, I' as above, ψ be the restriction of ϕ to $A(I)$ and $\varepsilon > 0$ such that:*

$$\|\sigma^{\phi}_{i/2}(\sigma^{\psi}_{-i/2}(Q)) - Q\| \leqq \varepsilon \|Q\| \quad \forall Q \in A(I').$$

Let $C_0 \subset A(I)$ be an abelian maximal subalgebra of the centralizer of ψ, let $m \geqq n + 2N$ and $C_k = \theta^{km} C_0$; C the abelian C^* algebra generated by the C_k, $k \geqq 0$, and ϕ_C the restriction of ϕ to C. Then

$$h_{\phi,\theta}(A(I)) \geqq \frac{1}{m} h_{\phi_C}(\theta^m) - O\left(\varepsilon, \frac{N}{m}\right).$$

Proof. For any k, let $N_k = \theta^{mk}(A(I))$ and $M_k = \theta^{mk}(A(I'))$. Then N_k, M_k satisfy the hypothesis of Proposition VIII.9. Moreover $\log(\dim N_k) = (n + 2N)\log q$ while $\log(\dim M_k) = n \log q$, thus with obvious notations:

$$H_\phi(N_0, \ldots, N_k) \geqq S(\phi_{|C[0,\ldots,k]}) - 6k\varepsilon(\tfrac{1}{2} + \log\varepsilon^{-1} + (n+2N)\log q) - 4kN \log q.$$

Now

$$\frac{1}{m} h_{\phi,\theta^m}(A(I)) = \lim_{k\to\infty} \frac{1}{mk} H_\phi(N_0, \ldots, N_k) \geqq \frac{1}{m}\left(\lim_{k\to\infty} \frac{1}{k} S(\phi_{|C[0,\ldots,k]})\right)$$
$$- 6\varepsilon\left(\frac{1/2 + \log(\varepsilon^{-1}) + (n+2N)\log q}{m}\right) - \frac{4N}{m}\log q.$$

We now have to evaluate the entropy $h_{\phi_C}(\theta^m)$ of the one-sided shift on the abelian C^* algebra C. Note that the entropy of the restriction of ϕ to C_0 is equal to $S(\phi_{|A(I)})$, since C_0 is an abelian maximal subalgebra of the centralizer of ϕ.

Lemma IX.3. *Let $1 > \eta > 0$ be such that, with $m \geqq n + 2N$ as above,*

$$|\phi(Q_1 Q_2) - \phi(Q_1)\phi(Q_2)| \leqq \eta \|Q_1\| \, \|Q_2\|$$

for any $Q_1 \in A(I')$, $Q_2 \in A\left(\bigcup_{k \geqq 1} (I + mk)\right)$. Then with the above notations one has:

$$\frac{1}{m} h_{\phi_C}(\theta^m) \geqq \frac{1}{m} S(\phi_{|A(I)}) - O(\eta).$$

Proof. Let X be the spectrum of the abelian C^* algebra $C_{[1,\infty[}$, i.e. the algebra generated by the $\theta^{mk}(C_0) = C_k$ for $k \geqq 1$. Then the disintegration of ϕ_C on $C = C_0 \times C'$ is of the form $\phi_C = \int (\omega_x \times \varepsilon_x) d\mu(x)$, where each ω_x is a state on C_0 which by the above inequality IX.3 is such that $\|\omega_x - \phi_{|C_0}\| \leqq \eta \forall x \in X$. The entropy of the shift is [1]

$$\int_X S(\omega_x) d\mu_x = S(\phi_{|C}) - S(\phi_{|C'})$$

and by Lemma IV.1 one has:

$$\left|S(\phi_{|C_0}) - \int_X S(\omega_x) d\mu(x)\right| \leqq 3\eta(\tfrac{1}{2} + \log(1 + d\eta^{-1})),$$

where $d = \dim C_0 = q^{n+2N}$. Thus:

$$\frac{1}{m} h_{\phi_C}(\theta^m) \geqq \frac{1}{m} S(\phi_{|A(I)}) - \frac{3\eta}{2m} - \frac{3\eta \log(\eta^{-1})}{m} - 3\eta \log(1+q).$$

Let us now show that the hypothesis of Lemma IX.3 is fulfilled for a given $\eta>0$ as soon as $m-(n+2N)$ is sufficiently large.

Lemma IX.4. *Given $\eta>0$, there exists $p\in\mathbb{N}$ such that*

$$|\phi(Q_1 Q_2)-\phi(Q_1)\phi(Q_2)|\leqq\eta\,\|Q_1\|\,\|Q_2\|$$

$$\text{for any } Q_1\in A(]-\infty,-p]),\ Q_2\in A([p,+\infty[).$$

Proof. ϕ satisfies the Gibbs condition, which tells us, that the interaction W between the right and the left can be removed by the factor

$$G=T\exp\left[\int_0^1 \sigma^{\phi}_{i\gamma}W d\gamma\right],$$

such that $Q\to\phi(GQ)$ is a product state for $A(]-\infty,0[)\times A([0,+\infty[)$. As shown in [19] G is a bounded essentially local operator. From the uniform clustering property of ϕ [19, (8.41)] it follows:

$$|\phi(Q)-\phi(GQ)|\leqq\eta\,\|Q\|\quad \forall Q\in A(Z\backslash]-p,p[).$$

For $Q_1\in A(]-\infty,-p])$, $Q_2\in A([p,+\infty[)$ one has with $\|Q_i\|\leqq 1$,

$$|\phi(GQ_1Q_2)-\phi(Q_1Q_2)|\leqq\eta,\qquad |\phi(GQ_j)-\phi(Q_j)|\leqq\eta,\qquad j=1,2,$$
$$\phi(GQ_1Q_2)=\phi(GQ_1)\phi(GQ_2),$$

thus:

$$|\phi(Q_1Q_2)-\phi(Q_1)\phi(Q_2)|\leqq 3\eta.\quad\square$$

Appendix

It remains to discuss in which physical situations the conditions of Theorem IX.1 are satisfied.

We have to estimate

$$\|\sigma^{\phi}_{i/2}(\sigma^{\psi}_{-i/2}(Q))-Q\|$$

for $Q\in A(I')$ and $\psi=\phi_{|A(I)}$.

$\sigma^{\psi}_{-i/2}Q$ is a bounded operator belonging to $A(I)$. If we are dealing with a one-dimensional lattice system with finite range interaction, it is shown in [19, 4.28], that all local operators are analytic with respect to time automorphism, therefore, the above norm is bounded, If we are considering a more than one-dimensional lattice system with short range interaction, a corresponding statement is only available for temperature above a critical one [20].

Lemma A.1. *Assume that $A=\bigotimes_{j\in Z}A_j$ and the time evolution is given by a $\Phi\in A([0,r])$. Let $H_I=\sum_{J\in I}\Phi(J)$ and $\bar{H}_I$ be the operator that implements the modular automorphism of $\psi=\phi_{|A(I)}$. Then*

$$W\equiv\exp[H_{I/2}]\exp[-\bar{H}_I]\exp[H_{I/2}],$$

and W is a bounded operator concentrated on the boundary of I, i.e. $W=W_1+W_2$, $W_1\in A[-N,0]\otimes A[n,n+N]$ and $\|W_2\|<\varepsilon(N)$.

Proof. This follows from combing the estimates in [19, 20]. If we can use the Campbell-Baker-Hausdorff formula, i.e. if the temperature is high enough so that the series converges, we can obtain

$$\bar{H}_I = H_I + V,$$

with V located on the boundary such that

$$\exp[H_{I/2}]\exp[-\bar{H}_{I/2}] = T\exp\left[\int_0^{1/2} \tau_\gamma V d\gamma\right]$$

gives the following desired bound:

Lemma A.2. *Under the assumptions of* (A.1) *there exists a critical temperature $\beta_c^{-1} < \infty$ such that for $\beta < \beta_c$, with ϕ a $1/\beta$* KMS *state, there exists $\forall \varepsilon > 0$, an N, such that with $\psi = \phi_{|A[-N, n+N]}$,*

$$\|\sigma^\phi_{i/2}(\sigma^\psi_{-i/2}(Q)) - Q\| < \varepsilon \|Q\| \quad \forall n, Q \in A[0, n].$$

For quasifree states it is easier to control $\bar{H}_I$ such that we obtain the estimate for all temperatures:

Lemma A.3. *Let ϕ be a quasifree interaction. Then $\beta_c = \infty$.*

Proof. Let a_x be a fermionic destruction operator at a lattice point $x \in Z$, $\{a_x, a^*_{x'}\} = \delta_{x,x'}$ and $a(f) = \sum_x a_x f_x$, $f \in l^2(Z)$. The time evolution is governed by $h \in B(l^2(Z))$,

$$\tau^t a(f) = a(e^{iht} f).$$

We have for the KMS state ω (with temperature $= 1$)

$$\omega(a(f)^* a(g)) = \left(f \left| \frac{1}{1+e^h} \right| g\right).$$

Denote by θ_I (respectively $\theta_{I^c} = 1 - \theta_I) \in B(l^2(Z))$ the projection operators onto $I \subset Z$ (respectively its complement I^c). The modular automorphism τ_I of $\omega_{|A_I}$ is generated by h_I

$$\tau^t a(f) = a(e^{ih_I t} f), \qquad f \in l^2(I), \tag{A.4}$$

which is determined by

$$\theta_I \frac{1}{1+e^{h_I}} = \theta_I \frac{1}{1+e^h} \theta_I, \qquad \theta_{I^c} \frac{1}{1+e^{h_{I^c}}} = \theta_{I^c} \frac{1}{1+e^h} \theta_{I^c},$$

with $h_I = \theta_I h_I \theta_I$, $h_{I^c} = \theta_{I^c} h_{I^c} \theta_{I^c}$. Thus

$$\begin{aligned} \theta_I(1+e^{h_I})\theta_I &= \theta_I \frac{1}{\theta_I \frac{1}{1+e^h} \theta_I} \theta_I \\ &= \theta_I(1+e^h)\theta_I - \theta_I e^h \theta_{I^c}[\theta_{I^c}(1+e^h)\theta_{I^c}]^{-1} \theta_{I^c} e^h \theta_I, \end{aligned}$$

and similarly for I^c. Together these relations say

$$\begin{aligned} e^{h_I + h_{I^c}} = e^h &- \theta_I e^h \theta_{I^c} - \theta_{I^c} e^h \theta_I - \theta_I e^h \theta_{I^c}[\theta_{I^c}(1+e^h)\theta_{I^c}]^{-1} \theta_{I^c} e^h \theta_I \\ &- \theta_{I^c} e^h \theta_I [\theta_I(1+e^h)\theta_I]^{-1} \theta_I e^h \theta_{I^c}. \end{aligned} \tag{A.5}$$

We want to estimate $\|\sigma^{\phi}_{i/2}(\sigma^{\psi}_{-i/2}(Q))-Q\|<\varepsilon\|Q\|\ \forall Q\in A_{I'}$. We are looking for an operator that implements $\sigma^{\phi}_{i/2}\sigma^{\psi}_{-i/2}Q$ for $Q\in A$, thus an operator that satisfies

$$Ua(f)U^{-1}=a(e^{v}f)-a(e^{v}e^{w}f),$$

where $e^{w}=(1-\theta_{I'})e^{-v}(1-\theta_{I'})+\theta_{I'}$ and therefore $e^{w}f=e^{-w^*}f=f$ for all f with $\operatorname{supp} f\in I'$.

We write $e^{v}e^{w}=e^{v_1}=e^{is}e^{\varrho}$ in its polar decomposition, i.e. $s=s^*$, $\varrho=\varrho^*$. Then U can be written in the form

$$\exp(i\sum a^*sa)\exp(\sum a^*\varrho a).$$

Thus

$$\|U-1\|\leqq\|\exp(\sum a^*\varrho a)-1\|+\|\exp(i\sum a^*sa)-1\|\leqq e^{\|\varrho\|_1}-1+e^{\|s\|_1}-1.$$

We observe

$$\|e^{\varrho}-1\|_1\leqq\|e^{v}-1\|_1,$$
$$\|e^{is}-1\|_1\leqq\|e^{\varrho}-1\|_1+\|e^{v_1}-1\|_1\leqq 2\|e^{v_1}-1\|.$$

If $\|e^{Q}-1\|_1<\varepsilon<\varepsilon_0$, this implies $\|Q\|_1<2\varepsilon$ if ε_0 is sufficiently small, and equivalently if $\|Q\|_1<\varepsilon<\varepsilon_0$, then $\|e^{Q}-1\|_1<2\varepsilon$. Further

$$\|\exp(\sum a^*\varrho a)-1\|\leqq\exp(\|\sum a^*\varrho a\|=1=e^{\|\varrho\|_1}-1,$$

thus we only have to estimate $\|v_1\|_1$.

With $e^{v_1}=1+D$, we have

$$\|v_1\|_1=\|\log(1+D)\|_1=\left\|-\sum_{n=1}^{\infty}\frac{(-D)^n}{n}\right\|_1\leqq\|D\|_1\frac{1}{\|D\|}\log\frac{1}{1-\|D\|}.$$

So what remains to show is that $\|D\|_1$, and therefore $\|D\|$ go to zero if I and I' are sufficiently separated. For this purpose we have first to construct

$$e^{v}=e^{-h/2}e^{(h_I+h_{I^c})/2}$$

to use VIII.10.

To extract its square root we employ the integral representation

$$\sqrt{a+b}=\sqrt{a}+\int_0^{\infty}d\alpha\frac{\alpha^2}{\alpha^2+a+b}b\frac{1}{\alpha^2+a},$$

and hence

$$\exp(-h/s)\exp(\tfrac{1}{2}(h_I+h_{I^c}))-1=\int_0^{\infty}d\alpha\exp(-h/2)\frac{\alpha^2}{\alpha^2+e^h+b}b\frac{1}{\alpha^2+e^h}$$

with

$$-b=\theta_I e^h\theta_{I^c}+\theta_I e^h\theta_{I^c}[\theta_{I^c}(1+e^h)\theta_{I^c}]^{-1}\theta_{I^c}e^h\theta_I+I\leftrightarrow I^c.$$

This gives the bound

$$\|D\|_1=\|(e^{v}-1)\theta_{I'}\|_1\leqq\int_0^{\infty}d\alpha\left\|e^{-h/2}\frac{\alpha^2}{\alpha^2+e^h+b}\right\|\left\|b\frac{1}{\alpha^2+e^h}\theta_{I'}\right\|_1.$$

Observing $e^h+b=\exp(h_I+h_{I^c})>0$ and h bounded from below, we note

$$\left\| e^{-h}\frac{\alpha^2}{\alpha^2+e^h+b}\right\| < c .$$

Furthermore b is of the form $b=c_1\theta_{I^c}+c_2\theta_{I^c}e^h\theta_I$ with c_1 and c_2 bounded operators. What remains is an estimate of

$$\left\|\theta_{I^c}\frac{1}{\alpha^2+e^h}\theta_{I'}\right\|_1 \quad \text{and} \quad \left\|\theta_{I^c}e^h\theta_I\frac{1}{\alpha^2+e^h}\theta_{I'}\right\|_1 .$$

For these trace norms we use the inequality

$$\|M\| \leqq \sum_j \sqrt{\sum_i \langle e_j|Me_i\rangle^2},$$

where the e_i are any orthonormal basis. We shall use as basis the vectors $|x\rangle$ which are 1 at the site x and zero otherwise. If h is in Fourier space multiplication with $\varepsilon(p)\in C^2([0,2\pi))$, then in this basis we have the matrix elements

$$g_\alpha(x-y)\equiv \left\langle y\left|\frac{1}{\alpha^2+e^h}\right|x\right\rangle = \int_0^{2\pi} dp\,\frac{e^{ip(x-y)}}{\alpha^2+e^{\varepsilon(p)}} = \frac{1}{|x-y|^2}\int_0^{2\pi} dp\, e^{ip(x-y)}$$
$$\times\left[\frac{\varepsilon''e^\varepsilon}{(\alpha^2+e^\varepsilon)^2}+\varepsilon'^2\left(\frac{e^\varepsilon}{(e^\varepsilon+\alpha^2)^2}-\frac{2e^\varepsilon}{(e^\varepsilon+\alpha^2)^3}\right)\right]$$

for $x\neq y$ and $\leqq 2\pi/(1+\alpha^2)$ for $x=y$. Altogether this shows

$$|g_\alpha(x-y)| \begin{cases} \leqq \dfrac{c(\alpha)}{|x-y|^2} & \text{for } x\neq y \\ \leqq c(\alpha) & \text{for } x=y \end{cases}$$

with $\int_0^\infty d\alpha\, c(\alpha)=c<\infty$. Now if $I^c=\{-\infty<x\leqq -N\}$ and $I'=\{0\leqq x\leqq n\}$, then

$$\int_0^\infty d\alpha\left\|I^c\frac{1}{\alpha^2+e^h}Q\right\|_1 \leqq \int_0^\infty d\alpha\sum_{x=1}^{n}\left(\sum_{y=-\infty}^{-N}|g_\alpha(x-y)|^2\right)^{1/2}$$
$$\leqq c\int_{x=1}^{n}\frac{1}{(x+N)^{3/2}} \leqq c/N^{1/2} .$$

With our assumption on h we also have $\langle x|e^h|y\rangle \leqq c/(|x-y|^2)$, and a similar bound can be worked out for the other term in b.

Combining these estimates, (A.3) follows. □

References

1. Kolmogorov, A.N.: Dokl. Akad. Nauk **119**, 861 (1958)
   Sinai, Yu.: Dokl. Akad. Nauk **124**, 768 (1959)
2. Ruelle, D.: Thermodynamic formalism. Reading, Ma: Addison-Wesley 1978
3. Ruelle, D., Bowen, L.: Invent. Math. **29**, 181 (1975)
4. Lanford, O.E., Robinson, D.W.: Mean entropy states in quantum statistical mechanics. J. Math. Phys. **9**, 120 (1968)

5. Aizenman, M., Goldstein, S., Gruber, C., Lebowitz, J.L., Martin, P.: On the equivalence between KMS-states and equilibrium states for classical systems. Commun. Math. Phys. **53**, 209 (1977)
6. Lindblad, G.: Quantum ergodicity and chaos. In: Fundamental aspects of quantum theory, p. 199. Gorini, V., Frigerio, A. (eds.). New York: Plenum Press 1986
7. Lieb, E.H., Ruskai, M.B.: Proof of the strong subadditivity of quantum-mechanical entropy. J. Math. Phys. **14**, 1938 (1973)
8. Connes, A., Störmer, E.: Acat Math. **134**, 289 (1975)
9. Emch, G.: Acta Phys. Austr. [Suppl.] **XV**, 79 (1976)
   Moore, S.M.: Rev. Colomb. Mat. **X**, 57 (1976)
10. Narnhofer, H., Thirring, W.: Fizika **17**, 257 (1985)
11. Connes, A.: C.R. Acad. Sci. Paris t **301** I, 1 (1985)
12. Lieb, E.H.: Adv. Math. **11**, 267 (1973)
13. Araki, H.: Pub. Res. Inst. Math. Sci. **9**, 165 (1973)
14. Kosaki, H.: Interpolation theory and the Wigner-Yanase-Dyson-Lieb concavity. Commun. Math. Phys. **87**, 315 (1982)
    Pusz, W., Woronowicz, S.: Form convex functions and the WYDL and other inequalities. Lett. Math. Phys. **2**, 505 (1978)
15. Choi, M.D., Effros, E.G.: Ann. Math. **104**, 585 (1976)
16. Choi, M.D., Effros, E.G.: J. Funct. Anal. **24**, 156 (1977)
17. Connes, A., Störmer, E.: J. Funct. Anal. **28**, 187 (1978)
18. Accardi, L., Cecchini, C.: Conditional expectations in von Neumann algebras and a theorem of Takesaki. J. Funct. Anal. **45**, 245–273 (1982)
19. Araki, H.: Gibbs states of a one dimensional quantum lattice. Commun. Math. Phys. **14**, 120 (1969)
20. Narnhofer, H.: Thermodynamical phases and surface effects. Acta Phys. Austr. **54**, 221 (1982)

Communicated by H. Araki

Received February 25, 1987; in revised form April 21, 1987