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Specifications and Martin Boundaries for $\mathscr{P}(\Phi)_2$-Random Fields

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Abstract. It is shown that $\mathscr{P}(\Phi)_2$-Gibbs states in the sense of Guerra, Rosen and Simon are given by a specification. The construction of the specification is based on finding a proper version of the interaction density given by the polynomial $\mathscr{P}$. The existence of this version follows from the fact that all powers of the solution of a Dirichlet problem for an open bounded set U with boundary data given by a distribution are integrable on U. As a consequence the Martin boundary theory for specifications can be applied to $\mathscr{P}(\Phi)_2$-random fields. It follows that any $\mathscr{P}(\Phi)_2$-Gibbs state can be represented in terms of extreme Gibbs states. In certain cases the extreme Gibbs states are characterized in terms of harmonic functions. It follows, in particular, that for any given boundary condition introduced so far the associated cutoff $\mathscr{P}(\Phi)_2$-measure has a representation as an integral over harmonic functions.

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1. Introduction

The aim of this paper is to construct "specifications" (cf. [F, P1]) for $\mathscr{P}(\Phi)_2$-random fields. These are known as basic non-trivial continuous models in Euclidean quantum field theory. Here $\mathscr{P}$ is a semibounded polynomial of one variable, Φ is the "field" and the index indicates 2 dimensions (cf. below and [S, Gl/J] for the precise definition). These models, which are usually realized as probability measures on a space of distributions like $\mathscr{D}'$ or $\mathscr{S}'$, have been studied extensively (see the references in [Gl/J]). They are constructed starting from an underlying (mean zero) Gaussian measure P_0, called the free field, which is associated with the differential operator $\Delta - m^2$, $m \in \mathbb{R}^2$, on $\mathbb{R}^2$ (cf. Sect. 2 below). In [G/R/S 1,2] Guerra, Rosen and Simon study $\mathscr{P}(\Phi)_2$-theory within the framework of statistical mechanics (cf. also [Fr/S]) and define a $\mathscr{P}(\Phi)_2$-random field to be a Gibbs state P in the following sense:

P is a probability measure on $(\mathscr{D}'; \mathscr{B})$, $\mathscr{B}$ being the σ-field generated by the coordinate mappings $\Phi(g)$, $g \in \mathscr{D}$, defined on $\mathscr{D}'$. P is locally absolutely continuous with respect to P_0 (cf. 5.5) and it is a local Markov field (cf. 5.2) with conditional probabilities given by

$$E_P(X \mid \sigma(\partial U)) = \frac{E_{P_0}(X \exp(-\lambda \int_U :\mathscr{P}(\Phi)(x): dx) \mid \sigma(\partial U))}{E_{P_0}(\exp(-\lambda \int_U :\mathscr{P}(\Phi)(x): dx) \mid \sigma(\partial U))}, \quad P\text{-a.s.}, \tag{1.1}$$

for every bounded, $\sigma(\bar{U})$-measurable function X on $\mathscr{D}'$ and every bounded open subset U of $\mathbb{R}^2$. Here ∂U and $\bar{U}$ means topological boundary respectively closure of U; $\sigma(V)$ for $V \subset \mathbb{R}^2$, V open, denotes the σ-field generated by $\Phi(g)$, $g \in \mathscr{D}$, $\operatorname{supp} g \subset V$, and we set for an arbitrary $A \subset \mathbb{R}^2$,

$$\sigma(A) \equiv \bigcap_{\substack{A \subset V \\ V \text{ open}}} \sigma(V). \tag{1.2}$$

Furthermore : : means Wick ordering with respect to P_0 (cf. Sect. 3) and $\lambda \geqq 0$ is a coupling constant.

(1.1) is referred to as the "DLR-equations" for P (cf. [*D*, *L*/*R*]). Since (1.1) depends on local P_0-zero sets, it only makes sense for measures which are locally absolutely continuous with respect to P_0.

The aim of this paper is to formulate DLR-equations and to define Gibbs states independently of P_0-zero sets. This is done by constructing an appropriate family $(\pi_U^\lambda)_U$ of probability kernels, called a "specification," to replace the right-hand side of (1.1). Before we give the definition of a specification, let us recall that a probability kernel $\pi(\Psi, A)$, $\Psi \in \mathscr{D}'$, $A \in \mathscr{B}$ on $(\mathscr{D}'; \mathscr{B})$ determines an operator acting on bounded, $\mathscr{B}$-measurable functions X and an operator acting on probability measures P on $(\mathscr{D}'; \mathscr{B})$. We denote the images under these operators by πX respectively $P\pi$. Furthermore, for $A \subset \mathbb{R}^2$ we set $A^c \equiv \mathbb{R}^2 \setminus A$.

Definition. Let $\mathscr{U}_1$ be a family of open subsets of $\mathbb{R}^2$ and for each $U \in \mathscr{U}_1$, let π_U be a probability kernel on $(\mathscr{D}'; \mathscr{B})$. The family $(\pi_U)_{U \in \mathscr{U}_1}$ is called a $(\sigma(U^c))_{U \in \mathscr{U}_1}$-*specification* if for every $U \in \mathscr{U}_1$,

$$\pi_U X \text{ is } \sigma(U^c)\text{-measurable for every bounded, } \mathscr{B}\text{-measurable function } X. \tag{1.3}$$

$$(\text{“Consistency”}) \text{ For each } V \subset \mathscr{U}_1 \text{ with } U \subset V,\ \pi_V(Z \pi_U X) = \pi_V(ZX) \tag{1.4}$$

for every bounded $\mathscr{B}$-measurable function X and every bounded $\sigma(U^c)$-measurable function Z.

If $\mathscr{U}_1$ consists of bounded sets, then $(\pi_U)_{U \in \mathscr{U}_1}$ is also called a family of *local characteristics* (cf. [F]). (1.4) is the analogue of the Chapman/Kolmogorovequations for the transition function of a Markov process. It is equivalent to saying that for each $\Psi \in \mathscr{D}'$ the expectation with respect to the probability measure $\pi_V(\Psi, \cdot)$ conditioned by $\sigma(U^c)$ is given by π_U. Indeed, specifications for $\mathscr{P}(\Phi)_2$-random fields have the same advantages as transition functions in the theory of Markov processes. We will describe the consequences below. We start with a summary of the construction:

We recall that it is possible to construct new specifications from a given one using additive functionals (cf. e.g. [P2]). The proper definition in our situation is the following:

Definition. Given a $(\sigma(U^c))_{U \in \mathscr{U}_1}$-specification $(\pi_U)_{U \in \mathscr{U}_1}$ a family $(a_U)_{U \in \mathscr{U}_1}$ of $\mathscr{B}$-measurable functionals on $\mathscr{D}'$ is called an *additive functional for* $(\pi_U)_{U \in \mathscr{U}_1}$ if given $U \in \mathscr{U}_1$:

For every $V \in \mathscr{U}_1$ with $U \subset V$ there exists a $\sigma(U^c)$-measurable function a_V^U such that for each $\Psi \in \mathscr{D}'$,

$$a_V = a_U + a_V^U \quad \pi_V(\Psi, \cdot)\text{-a.s.}, \tag{1.5}$$

$$\pi_U(e^{-\lambda a_U})(\Psi) < \infty \quad \text{for all } \lambda \in [0, \infty[\text{ and each } \Psi \in \mathscr{D}'. \tag{1.6}$$

Given $(\pi_U)_{U \in \mathscr{U}_1}$ and $(a_U)_{U \in \mathscr{U}_1}$ as above, then the family $(\pi_U^\lambda)_{U \in \mathscr{U}_1}$ of probability kernels defined by

$$\pi_U^\lambda(\Psi, A) \equiv \frac{\pi_U(1_A e^{-\lambda a_U})(\Psi)}{\pi_U(e^{-\lambda a_U})(\Psi)}, \quad \Psi \in \mathscr{D}', \quad A \in \mathscr{B}, \tag{1.7}$$

is again a $(\sigma(U^c))_{U \in \mathscr{U}_1}$-specification (with coupling constant λ).

The $\mathscr{P}(\Phi)_2$-specifications will be constructed according to this scheme.

We start with a specification $(\pi_U)_U$ associated with the differential operator $\Delta - m^2$, $m \in \mathbb{R}$. It is a special case of those already constructed in [R3] and it is called *free specification.* For $\Psi \in \mathscr{D}'$ the measure $\pi_U(\Psi, \cdot)$ is the translation of the free measure P_U with Dirichlet conditions on U^c. The translation is given by a distribution $\bar{H}_U(\Psi)$ which is the solution of a Dirichlet problem for U given Ψ as boundary data. In Sect. 2 we recall all basic notions and results from [R3].

In Sect. 3 and 4 we construct the additive functional which is formally given by

$$a_U(\Phi) = \text{“}\int_U :\mathscr{P}(\Phi)(x): dx\text{”}, \quad \Phi \in \mathscr{D}'.$$

Of course, the integrand makes no sense in general. But by approximating Φ by functions it is possible to define a_U as an L^2-limit with respect to P_0 or P_U (cf. [S], [Gl/J]). The Wick ordering : : and the fact that we are in a two dimensional situation, are essential to prove this convergence. In view of (1.7), however this definition of a_U up to P_0- or P_U-zero sets is not sufficient for our purpose. In fact, it can be proven that the measure $\pi_U(\Psi, \cdot)$, $\Psi \in \mathscr{D}'$, is in general not absolutely continuous with respect to P_0 and is either equal or singular to P_U even when restricted to $\sigma(U)$ (cf. 7.2). So we have to construct a proper version of the “limit” a_U. Since the Wick ordering : : is taken with respect to the free measure P_0 (i.e. we consider the so-called “Half-Dirichlet” case, cf. Sect. 3) we need to restrict ourselves to the class $\mathbb{L}$ of all open bounded sets satisfying condition (2.5). This condition which is essentially "log-normality" in the sense of [S, §VII 3] holds for all sets with "reasonable" boundaries.

The key fact for the construction is the following result (cf. 2.5):

For every $U \in \mathbb{L}$ the Dirichlet solution $\bar{H}_U(\Psi)$ is p-integrable on U for every $p \geqq 1$ and "sufficiently many" $\Psi \in \mathscr{D}'$.

Indeed, $\bar{H}_U(\Psi)$ will play a major role throughout this paper and methods from potential theory will be applied. In the conclusion of Sect. 4 we will see that the $\mathscr{P}(\Phi)_2$-specification $(\pi_U^\lambda)_{U \in \mathbb{L}}$ constructed this way can be interpreted as a perturbation of the usual "half-Dirichlet boundary conditioned cut-off $\mathscr{P}(\Phi)_2$-measures" (cf. [S, Sect. VII. 3]).

In Sect. 5 we study the Gibbs states for $(\pi_U^\lambda)_{U \in \mathbb{L}}$ which are defined as follows: Let $U \in \mathbb{L}$, $\lambda \geqq 0$ and P a probability measure on $(\mathscr{D}'; \mathscr{B})$ such that

$$E_P(X \mid \sigma(U^c)) = \pi_U^\lambda(X) \quad P\text{-a.s.} \tag{1.8}$$

for every bounded, $\mathscr{B}$-measurable function X. Then we set $P \in G_\lambda(U)$ and call P a *(local) U-Gibbs state.* Furthermore, we call

$$G_\lambda \equiv \bigcap_{U \in \mathbb{L}} G_\lambda(U), \tag{1.9}$$

the set of *(global) Gibbs states for* $(\pi_U^\lambda)_{U \in \mathbb{L}}$. We denote the extreme point of these two convex sets by $\partial G_\lambda(U)$, ∂G_λ respectively.

Equation (1.8) is the reformulation of the DLR-equations (1.1) that we mentioned earlier. It follows from the special form of $(\pi_U^\lambda)_{U \in \mathbb{L}}$ that every measure in G_λ is locally Markovian (cf. 5.7), but the following result is more subtle (cf. 5.6):

Every $P \in G_\lambda$ is locally equivalent to P_0.

This implies that the set of (global) Gibbs states in the sense of GRS is equal to G_λ (cf. 5.4), i.e. the GRS-Gibbs states are defined by the reformulated DLR-equations (1.8) and all a priori restrictions can be dropped.

It is well known that for every $\lambda \geqq 0$ there exist global GRS-Gibbs states (cf. [Fr/S, Theorem 7.2]) and therefore we have that $G_\lambda \neq \phi$. In Sect. 6 we give a new proof of this fact. Instead of the Bochner/Minlos theorem for nuclear spaces we apply (directly) a version of Kolmogorov's existence theorem generalized to inductive limits of standard Borel spaces (cf. [Pa, Theorem 4.2, p. 143]). The method is due to Preston (cf. [P2]) from whom we also learned the suitable compactness condition (cf. 6.3). This condition is satisfied in our situation because of inequality (6.2) due to Fröhlich and Simon (cf. [Fr/S]).

In Sect. 7 we study Martin boundaries for $(\pi_U^\lambda)_{U \in \mathbb{L}}$. The Martin boundary theory of random fields was developed by Dynkin [Dy 1, 2] and Föllmer [F] for an arbitrary specification on a standard Borel space with a non-empty set of (global) Gibbs states. (For a complete presentation see [P1, Chap. 2] and also [Bl/Pf and Ng/Z] for special cases). Using our specification we can apply this theory to $\mathscr{P}(\Phi)_2$-random fields and obtain the following results (cf. 7.4 and 7.5):

Any global GRS-Gibbs state can be represented in terms of extreme Gibbs states. More precisely, there exists a measurable space $(\mathscr{M}; \mathscr{A})$ called "Martin boundary" for $(\pi_U^\lambda)_{U \in \mathbb{L}}$, and a bijection between G_λ and the probability measures on

$(\mathscr{M};\mathscr{A})$ which maps ∂G_λ bijectively to the point masses on $(\mathscr{M};\mathscr{A})$. Any $P\in G_\lambda$ can be represented as an integral over $\mathscr{M}$.

"Locally" it is even possible to characterize the extreme Gibbs states. The particular form of $(\pi_U^\lambda)_{U\in\mathbb{L}}$ implies a one to one correspondence between the measures in $\partial G_\lambda(U)$ restricted to $\sigma(U)$ and $\mathscr{H}_0(U)$, where for $U\subset\mathbb{R}^2$, U open,

$$\begin{aligned}\mathscr{H}_0(U)=\{h:U\to\mathbb{R}:h \text{ is harmonic on } U \text{ and } p\text{-integrable}\\ \text{on } U \text{ (with respect to } dx) \text{ for every } p\geqq 1\}.\end{aligned}\tag{1.10}$$

As an immediate consequence we obtain (cf. 7.3, 7.7 (ii), and also (7.9)):

All boundary conditioned cutoff $\mathscr{P}(\Phi)_2$-measures in the sense of [G/R/S 2] (like e.g. "Dirichlet," "Neumann," "periodic," "free" boundary conditions) on U can be represented as an integral over $\mathscr{H}_0(U)$.

"Globally" the characterization of the extreme Gibbs states is a more difficult problem. We have a complete solution only in the case $\lambda=0$ (cf. 7.8):

The global Martin boundary for $(\pi_U)_{U\in\mathbb{L}}$ is equal to the set of all harmonic functions on $\mathbb{R}^2$.

In the general case $\lambda\geqq 0$ we prove the following partial result (cf. 7.10):

Let $P\in\partial G_\lambda$, then there exists a unique harmonic function h on $\mathbb{R}^2$ such that

$$P\{\Phi\in\mathscr{D}':\bar{H}_{\mathbb{R}^2}(\Phi)=h\}=1,\tag{1.11}$$

where the map $\Phi\mapsto\bar{H}_{\mathbb{R}^2}(\Phi)$, $\Phi\in\mathscr{D}'$, is defined to be the locally uniform limit of $\bar{H}_U(\Phi)$, $U\in\mathbb{L}$, as $U\nearrow\mathbb{R}^2$ (cf. Sect. 2). Furthermore, given a harmonic function h on $\mathbb{R}^2$, any $P\in G_\lambda$ satisfying (1.11) can be represented in terms of elements in ∂G_λ satisfying (1.11).

It is not clear yet whether for some $\lambda>0$ the correspondence between ∂G_λ and harmonic functions on $\mathbb{R}^2$ is one to one. One might hope that this is true for small λ. This would solve an open problem in $\mathscr{P}(\Phi)_2$-quantum field theory, namely: which "condition at infinity" implies that there is a unique global Gibbs state satisfying this condition (cf. [G/R/S 1] and [Fr/S]). This problem will be the subject of further study.

The results of this paper confirm that it is useful to consider $\mathscr{P}(\Phi)_2$-fields from a potential theoretic point of view which was already done earlier by Alveverio/Hoegh–Krohn in [A/H-K.], Dynkin in [Dy 3, 4] and the author in [R 1, 2, 3].

Apart from Sect. 6 all results of this paper can immediately be extended to more general symmetric, second order elliptic differential operators, essentially those for which the associated harmonic structure is a self-adjoint harmonic space in the sense of Maeda [M 1, 2]. Also most arguments in Sect. 6 remain valid, except inequality (6.2). In its proof Euclidean invariance of $\Delta-m^2$, $m\in\mathbb{R}$, has been used. This invariance is anyway necessary to construct the physically relevant, Lorentz-invariant Wightman field theories from $\mathscr{P}(\Phi)_2$-fields. It is therefore important to investigate invariance properties of our $\mathscr{P}(\Phi)_2$-specifications and their Gibbs states. Because of the length of this paper we do not include these considerations here. They will be presented elsewhere.

2. The Free Specification $(\pi_U)_{U \in \mathbb{L}}$

Let us fix some basic notations.

Let $\mathscr{D}'$ be the space of distributions on $\mathbb{R}^2$, i.e. the topological dual of $\mathscr{D} \equiv C_0^\infty(\mathbb{R}^2)$, the space of all infinitely differentiable functions on $\mathbb{R}^2$ with compact support equipped with the usual inductive topology (cf. [R3, Sect. 3]). Let $\langle , \rangle: \mathscr{D}' \times \mathscr{D} \to \mathbb{R}$ be its dualization. For $g \in \mathscr{D}$ we denote by $\Phi(g)$ the evaluation map on $\mathscr{D}'$ given by $g \mapsto \langle \Phi, g \rangle$, $\Phi \in \mathscr{D}'$.

For notational convenience we will not distinguish between $\Phi(g)$ and $\langle \Phi, g \rangle$ if there is no confusion possible. The σ-field $\mathscr{B}$ generated by $\{\Phi(g): g \in \mathscr{D}\}$ is equal to the Borel σ-field associated with the topology generated by $\Phi(g)$, $g \in \mathscr{D}$, on $\mathscr{D}'$. Given a sub σ-field $\mathscr{F}$ of $\mathscr{B}$ and given an $\mathscr{F}$-measurable function X on $\mathscr{D}'$ we also write $X \in \mathscr{F}$ and $X \in \mathscr{F}_b$ if it is in addition bounded.

Given an arbitrary measurable space $(\mathscr{X}; \mathscr{A})$ we denote the set of all probability measures on $(\mathscr{X}; \mathscr{A})$ by $\mathbb{P}(\mathscr{X}; \mathscr{A})$. Given $P \in \mathbb{P}(\mathscr{D}'; \mathscr{B})$ and $X \in \mathscr{B}_b$, let $E_P(X) \equiv P(X) \equiv \int X(\Phi)\, P(d\Phi)$ be the expectation of X under P and for a sub σ-field $\mathscr{F}$ of $\mathscr{B}$ let $E_P(\cdot \mid \mathscr{F})$ denote the conditional expectation with respect to P given $\mathscr{F}$.

We denote the system of all open, respectively relatively compact open subsets of $\mathbb{R}^2$ by $\mathscr{U}$, respectively $\mathscr{U}_c$.

Consider the differential operator $L \equiv \Delta - m^2$, $m \in \mathbb{R}$, on $\mathbb{R}^2$. For $U \in \mathscr{U}$ let G_U denote the (Dirichlet-) Green function of L on U extended to $\mathbb{R}^2 \times \mathbb{R}^2$ by zero. There is a family $(P_U)_{U \in \mathscr{U}}$ of Gaussian, mean zero measures on $(\mathscr{D}'; \mathscr{B})$ associated with $(G_U)_{U \in \mathscr{U}}$, their covariances being defined by

$$\int \Phi(g)\Phi(f) P_U(d\Phi) = \iint G_U(x,y) f(x) g(y)\, dx\, dy, \tag{2.1}$$

$U \in \mathscr{U}, f, g \in \mathscr{D}$. Here dx denotes the Lebesgue measure on $\mathbb{R}^2$ and "Gaussian" means that each $\Phi(g)$, $g \in \mathscr{D}$, has a Gaussian distribution under P_U. If $U = \mathbb{R}^2$, we set

$$G \equiv G_{\mathbb{R}^2} \text{ and } P_0 \equiv P_{\mathbb{R}^2};$$

P_0 is called the *free field of mass m*. By simple transformation arguments we may restrict ourselves to the case $m = 1$. From now on all potential theoretic notions are meant with respect to the harmonic space given by $\Delta - 1$. Given $U \in \mathscr{U}$ we define for $f, g \in \mathscr{D}$,

$$\langle f, g \rangle_{E,U} \equiv \iint G_U(x,y) f(x) g(y)\, dx\, dy \quad \text{and} \quad \| f \|_{E,U} \equiv \langle f, f \rangle_{E,U}^{1/2}.$$

We set $\langle , \rangle_E \equiv \langle , \rangle_{E,\mathbb{R}^2}$ and $\| \quad \|_E \equiv \| \quad \|_{E,\mathbb{R}^2}$.

The definition of the free specification is based on the solution $\bar{H}_U(\Psi)$ of a Dirichlet problem for $U \in \mathscr{U}$ with boundary data given by a distribution $\Psi \in \mathscr{D}'$. We recall its main properties (cf. [R3, Sect. 6]).

2.1 Theorem. *Let $U \in \mathscr{U}$. Then there exists a linear subspace $\Omega(U)$ of $\mathscr{D}'$ such that*

(i) $\Omega(U) \in \sigma(U^c)$.

(ii) $P_V(\Omega(U)) = 1$ *for every $V \in \mathscr{U}$, $U \subset V$, and a linear map $\bar{H}_U: \Omega(U) \to \mathscr{D}'$ such that for each $\Psi \in \Omega(U)$,*

(iii) $\bar{H}_U(\Psi)$ *is represented on U by a harmonic function.*

(iv) $\bar{H}_U(\Psi) = \Psi$ *on* $\operatorname{int}(U^c)$.

(v) *If U is bounded, regular and Ψ is represented by a continuous function, then $\Psi \in \Omega(U)$ and $\bar{H}_U(\Psi)$ is the ordinary solution of the Dirichlet problem with boundary data Ψ.*

(vi) *If $\Psi \in \mathscr{D}'$ is represented by a harmonic function on U, then $\Psi \in \Omega(U)$ and $\bar{H}_U(\Psi) = \Psi$. In particular,*

$$\bar{H}_U(\bar{H}_V(\Psi)) = \bar{H}_V(\Psi),$$

if $V \in \mathscr{U}$, $U \subset V$, $\Psi \in \Omega(V)$.

(vii) *For every $g \in \mathscr{D}$ the map $\Psi \mapsto \langle \bar{H}_U(\Psi), g \rangle$ is $\sigma(U^c)$-measurable if $\bar{H}_U(\Psi) \equiv 0$ for $\Psi \in \mathscr{D}' \backslash \Omega(U)$ and*

$$\langle \bar{H}_U(\Psi), g \rangle = E_{P_V}(\Phi(g) | \sigma(U^c))(\Psi),$$

P_V-a.s. *in $\Psi \in \mathscr{D}'$ for every $V \in \mathscr{U}$, $U \subset V$.*

2.2 Remark. (i) Because of the properties listed in 2.1 $\bar{H}_U(\Psi)$ can be interpreted as the balayage of the distribution Ψ on U^c.

(ii) For a different approach to construct a solution for a Dirichlet problem for U with boundary data given by a distribution if U is a rectangle, see [*D/Mi*].

(iii) Note that, in particular, $\bar{H}_{\mathbb{R}^2}$ is a map from $\Omega(\mathbb{R}^2)$ onto all harmonic functions on $\mathbb{R}^2$.

For $U \in \mathscr{U}_c$, let μ_x^U, $x \in U$, denote the associated harmonic measures (cf. [C/Co]). As in [R3] we denote for $\Psi \in \Omega(U)$ the harmonic function representing $\bar{H}_U(\Psi)$ on U by $x \mapsto \mu_x^U(\Psi)$, and set $\mu_x^U(\Psi) \equiv 0$, if $x \in \mathbb{R}^2 \backslash U$ or $\Psi \in \mathscr{D}' \backslash \Omega(U)$.

Given a sub σ-field $\mathscr{F}$ of $\mathscr{B}$ and $F \in \mathscr{B}$, we denote by $\sigma(\mathscr{F}, F)$ the σ-field generated by $\mathscr{F}$ and F. The following has been proven in [R3, Sect. 4]:

2.3 Proposition. *Let $U \in \mathscr{U}_c$ and $x \in U$. Then the function $\Phi \mapsto \mu_x^U(\Phi)$, $\Phi \in \mathscr{D}'$, is $\sigma(\sigma(\partial U), \Omega(U))$-measurable and*

$$\int |\mu_x^U(\Phi)|^2 dP_0(\Phi) = G\mu_x^U(x) = \int G\mu_x^U(z) d\mu_x^U(z),$$

where $G\mu_x^U(z) \equiv \int G(z, y)\mu_x^U(dy)$.

For the purposes of this paper we need a slightly modified version of the specification constructed in [R3] in the case of $\Delta - 1$. The reason is that the boundary behaviour of $\mu^U_{\cdot}(\Psi)$ is essential for the construction of the additive functionals. In fact, integrability properties turn out to be sufficient. Hence we define the linear space

$$\Omega_0(U) \equiv \{\Phi \in \Omega(U) : \int_U |\mu_x^U(\Phi)|^p dx < \infty \text{ for every } p \geqq 1\}. \tag{2.2}$$

We shall prove that $\Omega_0(U)$ is sufficiently large. We know by 2.1 (i) and 2.3 that

$$\Omega_0(U) \in \sigma(\sigma(\partial U), \quad \Omega(U)) \subset \sigma(U^c). \tag{2.3}$$

For $U \in \mathscr{U}$ define $\mathbb{M}(U)$ to be the set of all $P \in \mathbb{P}(\mathscr{D}'; \mathscr{B})$ such that for every $p \geqq 1$ there exists a constant $c_p > 0$ such that

$$\left(\int |\Phi(g)|^p dP\right)^{1/p} \leqq c_p \left(\int |\Phi(g)|^2 dP_0\right)^{1/2} (= \| g \|_E) \tag{2.4}$$

for every $g \in \mathscr{D}(U)$. Here for $U \in \mathscr{U}$ we set $\mathscr{D}(U) \equiv \{g \in \mathscr{D} : \operatorname{supp} g \subset U\}$.

Furthermore, define $\mathbb{L}$ to be the set of all $U \in \mathscr{U}_c$ such that

$$\int_U (G\mu_x^U(x))^p dx < \infty. \tag{2.5}$$

for every $p \geqq 1$.

2.4 Remark. (i) Let $U \in \mathscr{U}$ and $P \in \mathbb{P}(\mathscr{D}'; \mathscr{B})$ such that $\Phi(g)$, $g \in \mathscr{D}(U)$, has a Gaussian distribution under P and that the function $M(g) \equiv \int \Phi(g) dP$, $g \in \mathscr{D}(U)$, is continuous with respect to $\| \quad \|_E$ on $\mathscr{D}(U)$. Assume furthermore that there exists a function c on $\mathbb{R}^+$ bounded in a neighbourhood of 0 such that

$$\left(\int (\Phi(g) - M(g))^2 dP\right)^{1/2} \leqq c(\| g \|_E)$$

for every $g \in \mathscr{D}(U)$. Then standard arguments (using Laplace transforms) imply that $P \in \mathbb{M}(U)$. In particular, $P_V \in \mathbb{M}(U)$, for all $V \in \mathscr{U}$, $U \subset V$.

(ii) Condition (2.5) is satisfied by all sets in $\mathscr{U}_c$ with reasonable boundaries. The following considerations show that it is always satisfied by the "log-normal" sets introduced in [S, §VII 3], in particular by circles or rectangles. (Recall that "log-normality" is needed in order to define half-Dirichlet states; cf. [S] and Sect. 3. Note also that

$$G\mu_x^U(x) = \lim_{y \to x} (G(x, y) - G_U(x, y)), \quad x \in U;$$

cf. e.g. [R1, Sect. 2]): Let g: $\mathbb{R} \to \mathbb{R}^+$ such that

$$g(|x - y|) = G(x, y); \quad x, y \in \mathbb{R}^2,$$

where $| \quad |$ denotes the Euclidean norm on $\mathbb{R}^2$. Then

$$G\mu_x^U(x) = \int G(x, z) d\mu_x^U(z) \leqq g(d(x, \partial U)),$$

where $d(x, \partial U) \equiv \inf_{z \in \partial U} |x - z|$. We recall that g has only a logarithmic singularity at 0.

Now we are prepared to prove that $\Omega_0(U)$ is "sufficiently large," if $U \in \mathbb{L}$. This will be the key fact for the construction of our polynomial additive functionals in Sects. 3, 4:

2.5 Theorem. *Let $U \in \mathbb{L}$ and $P \in \mathbb{M}(U)$ such that $P(\Omega(U)) = 1$. Then $P(\Omega_0(U)) = 1$.*

Proof. Let $p \geqq 1$. Since $x \mapsto \mu_x^U(\Phi)$ is continuous on U for every $\Phi \in \Omega(U)$ we can use Fubini's theorem to obtain that

$$\int \int_U |\mu_x^U(\Phi)|^p dx \, dP(\Phi) = \int_U \int |\mu_x^U(\Phi)|^p dP(\Phi) dx.$$

Since $\mu_x^U(\Phi)$ is by construction a limit of coordinate functions $\Phi(g_n)$, $g_n \in \mathscr{D}(U)$, (2.4) and 2.3 imply that

$$\int |\mu_x^U(\Phi)|^p dP(\Phi) \leqq c(G\mu_x^U(x))^{p/2}, \quad x \in U,$$

for some constant c independent of x. The fact that $U \in \mathbb{L}$ now implies that

$$\int \int_U |\mu_x^U(\Phi)|^p dx dP(\Phi) < \infty,$$

hence $P(\Omega_0(U)) = 1$. □

2.6 Remark. Let $U \in \mathbb{L}$ and $P \in \mathbb{M}(U)$. Assume that there exists a constant $c > 0$ such that for every $f \in \mathscr{D}$

$$(\int \Phi(f)^2 dP)^{1/2} \leqq c(\int \Phi(f)^2 dP_0)^{1/2}.$$

Then $P(\Omega(U)) = 1$. This is a consequence of Minlos' theorem and the construction of $\Omega(U)$. Therefore, $P(\Omega_0(U)) = 1$ by 2.5.

Let us define for $U \in \mathscr{U}_c$,

$$\tilde{\mathscr{H}}_0(U) \equiv \{\Phi \in \mathscr{D}' : \Phi \text{ is represented on } U \text{ by a harmonic function which is } p\text{-integrable on } U \text{ for every } p \geqq 1\}. \tag{2.6}$$

For $\Psi \in \mathscr{D}'$ let $T_\Psi : \mathscr{D}' \to \mathscr{D}'$ be defined by $T_\Psi(\Phi) = \Phi + \Psi$, $\Phi \in \mathscr{D}'$; and let $T_\Psi(P)$ denote the image measure of $P \in \mathbb{P}(\mathscr{D}'; \mathscr{B})$ under T_Ψ. Now 2.1, (2.3), 2.4 i) and 2.5 imply:

2.7 Corollary. *Let* $U \in \mathbb{L}$.

(i) *If* $\Psi \in \tilde{\mathscr{H}}_0(U)$, *then* $T_\Psi(\Omega_0(U)) = \Omega_0(U)$.

(ii) *Theorem 2.1 (modified in the obvious way) remains true if* $\Omega(U)$ *and* $\mathscr{U}$ *is replaced by* $\Omega_0(U)$, $\mathbb{L}$ *respectively.*

Now we are prepared to give the definition of the free specification $(\pi_U)_{U \in \mathbb{L}}$ and summarize its main properties.

We change the definition of $\bar{H}_U$ outside $\Omega_0(U)$ for $U \in \mathbb{L}$ by setting

$$\bar{H}_U(\Psi) = 0, \quad \text{if} \quad \Psi \in \mathscr{D}' \setminus \Omega_0(U).$$

For $U \in \mathbb{L}$ define the probability kernel π_U on $(\mathscr{D}'; \mathscr{B})$ by

$$\pi_U(\Psi, \cdot) \equiv T_{\bar{H}_U(\Psi)}(P_U), \quad \Psi \in \mathscr{D}'. \tag{2.7}$$

Clearly, for each $\Psi \in \mathscr{D}'$ the measure $\pi_U(\Psi, \cdot)$ is Gaussian with Fourier transform

$$\pi_U(\exp i\Phi(g))(\Psi) = \exp(i\langle \bar{H}_U(\Psi), g\rangle - \tfrac{1}{2}\|g\|_{E,U}^2), \quad g \in \mathscr{D}.$$

$\pi_U(\Psi, \cdot)$ is not an element of $\mathbb{M}(U)$ in general, but because of 2.7 we have that

$$\pi_U(\Psi, \Omega_0(U)) = 1.$$

We should mention that the definition of π_U differs from that given in [R3], since $\Omega_0(U)$ is replacing $\Omega(U)$. But 2.7 implies that 7.4 in [R3] remains true:

2.8 Theorem. $(\pi_U)_{U \in \mathbb{L}}$ *defined by (2.7) is a* $(\sigma(U^c))_{U \in \mathbb{L}}$*-specification such that* $T_h(P_0)$ *is a Gibbs state for* $(\pi_U)_{U \in \mathbb{L}}$ *for every harmonic function* h *on* $\mathbb{R}^2$. *It is Markovian in the following sense:*

$$\text{If } U \in \mathbb{L} \text{ and } Z \in \sigma(U)_b, \text{ then } \pi_U(Z) \in \sigma(\sigma(\partial U), \Omega_0(U))_b. \tag{2.8}$$

Furthermore, it is "local" in the following sense:

$$\pi_U(\pi_U(Z)X) = \pi_U(Z)\pi_U(X) \text{ for all } Z, X \in \mathscr{B}_b. \tag{2.9}$$

Because of 2.8 we call $(\pi_U)_{U \in \mathbb{L}}$ the *free specification.*

3. A Perturbation Formula

At the beginning of this section we want to recall briefly the definition and basic properties of Wick monomials of generalized random fields (cf. e.g. [S, §V.1] and [Gl/J, Chap. 8]).

Throughout this section let $n \in \mathbb{N}$ be fixed. The n^{th} Hermite polynomial $H_n(t)$, $t \in \mathbb{R}$, is defined by

$$H_n(t) \equiv \sum_{m=0}^{[n/2]} (-1)^m \alpha_{nm} t^{n-2m}, \tag{3.1}$$

where

$$\alpha_{nm} \equiv \frac{n!}{(n-2m)!\,2^m m!}.$$

We define the following regularization for a distribution $\Phi \in \mathscr{D}'$: Let $d \in \mathscr{D}$, $d \geqq 0$, $\int d(x)\, dx = 1$ and $d(x) = d(-x)$ for every $x \in \mathbb{R}^2$. Define for $k \in \mathbb{N}$,

$$d_{k,x}(y) \equiv 2^{2k} d(2^k(x-y));\, x, y \in \mathbb{R}^2. \tag{3.2}$$

Then, set

$$\Phi_k(x) \equiv \langle \Phi, d_{k,x} \rangle; \quad x \in \mathbb{R}^2, \quad k \in \mathbb{N}.$$

For $U \in \mathscr{U}$ the n^{th} *Wick power of* $\Phi_k(x)$, $x \in \mathbb{R}^2$, $k \in \mathbb{N}$, with respect to P_U is defined by

$$:\Phi_k^n(x):_U \equiv (c_{U,k}(x))^{n/2} H_n(c_{U,k}(x)^{-1/2} \Phi_k(x)),$$

where

$$c_{U,k}(x) \equiv \int (\Phi_k(x))^2 P_U(d\Phi).$$

In the case of $U = \mathbb{R}^2$ we simply write $:\ :$ instead of $:\ :_{\mathbb{R}^2}$.

For $g \in L^p(\mathbb{R}^2, dx)$, $p > 1$, with compact support define for $U \in \mathscr{U}$,

$$:\Phi_k^n(g):_U \equiv \int :\Phi_k^n(x):_U g(x) dx, \quad k \in \mathbb{N}. \tag{3.3}$$

Then $(:\Phi_k^n(g):_U)_{k \in \mathbb{N}}$ converges in $L^2(\mathscr{D}', P_U)$ (cf. e.g. [Gl/J, Proposition 8.5.1]). Here $L^q(\mathscr{D}', P_U)$, $q > 0$, denotes the space of (classes of) q-integrable functions on $\mathscr{D}'$ with respect to P_U. The limit is P_U-a.s. linear in g.

The following theorem summarizes the main results on the "change of Wick ordering" (cf. [S, §VII. 3] and [Gl/J, Sect. 8.6]). This theorem is relevant for us since we want to construct our additive functionals $(a_U)_{U \in \mathbb{L}}$ corresponding to the so-called "Half Dirichlet" case, i.e. the Wick order is taken with respect to P_0.

We need the following notations: For $g \in L^p(\mathbb{R}^2, dx)$, $p > 0$, we set as usual

$$\|g\|_p \equiv \left(\int |g|^p dx\right)^{1/p},$$

and for $U \in \mathscr{U}$ and $X \in L^q(\mathscr{D}', P_U)$, $q > 0$,

$$\|X\|_{U,q} \equiv \left(\int |X|^q dP_U\right)^{1/q}.$$

Define for $U \in \mathscr{U}$, $c_U(x) \equiv G\mu_X^U(x)$, $x \in U$.

Furthermore, every function defined on a subset A of $\mathbb{R}^2$ is considered as a function on $\mathbb{R}^2$ which is zero outside A. Every locally (dx-) integrable function on $\mathbb{R}^2$ is identified with its associated element in $\mathscr{D}'$.

3.1 Theorem. *Let $p > 1$, $q \geqq 1$, and $U \in \mathbb{L}$. Then $(:\Phi_k^n(g):)_{k\in\mathbb{N}}$ converges in $L^q(\mathscr{D}', P_U)$ for every $g \in L^p(\mathbb{R}^2, dx)$ with compact support. For the limit $:\Phi^n(g):$ we have that*

$$:\Phi^n(g):=\sum_{m=0}^{[n/2]} \alpha_{nm}:\Phi^{n-2m}((-c_U)^m g):_U \quad P_U\text{-a.s.}. \tag{3.4}$$

In particular, $g \mapsto :\Phi^n(g):$ is P_U-a.s. linear.

Furthermore, if $K \subset \mathbb{R}^2$, K compact, then there exist constants α, $\gamma \geqq 0$ and $k_0 \in \mathbb{N}$ such that for every $g \in L^p(\mathbb{R}^2, dx)$ with support in K

$$\| :\Phi^n(g): \|_{U,q} \leqq \gamma \| g \|_p, \tag{3.5}$$

and for all $k \geqq k_0$,

$$\| :\Phi^n(g): - :\Phi_k^n(g): \|_{U,q} \leqq \gamma 2^{-k\alpha} \| g \|_p. \tag{3.6}$$

Proof. It is easily checked that (3.4) is true for $:\Phi_k^n(g):$ if c_U is replaced by $c_{\mathbb{R}^2,k} - c_{U,k}$ (cf. e.g. [S, I. 20b]). Since $U \in \mathbb{L}$, elementary arguments show that $(c_{\mathbb{R}^2,k} - c_{U,k})_{k\in\mathbb{N}}$ converges to c_U in $L^p(U, dx)$ for every $p \geqq 1$. Now one uses the fact that there exist α, $\gamma \geqq 0$ and $k_0 \in \mathbb{N}$ such that for every $g \in L^p(\mathbb{R}^2, dx)$ with support in K

$$\| :\Phi^n(g):_U \|_{U,q} \leqq \gamma \| g \|_p \quad \text{(cf. [}G/R/S2\text{, Lemma III.7)},$$

and for all $k \geqq k_0$,

$$\| :\Phi^n(g):_U - :\Phi_k^n(g):_U \|_{U,q} \leqq \gamma 2^{-k\alpha} \| g \|_p \quad \text{(cf. [Gl/J,}$$

Theorem 8.5.3]), and the assertion is a consequence of Hölder's inequality. □

3.2 Remark. In the preceding theorem $:\Phi^n(g):$ as an element of $L^q(\mathscr{D}', P_U)$, depends on U. But (3.6) and the Borel–Cantelli Lemma imply that the function $\Phi \mapsto \limsup_{k\to\infty} \Phi_k^n(g):$ is a version of: $\Phi^n(g):$ for every $U \in \mathbb{L}$. From now on $:\Phi^n(g):$ shall denote this particular version. So we can avoid to express the U-dependence in the notation.

The main step to construct our $\mathscr{P}(\Phi)_2$-additive functionals is the following theorem:

3.3 Theorem. *Let $U \in \mathbb{L}$, $p > 1$ and $g \in L^p(U, dx)$. Then there exists a version $a_{U,g}^{(n)}$ of $:\Phi^n(g):$ such that $a_{U,g}^{(n)}$ is $\sigma(U \cap \operatorname{supp} g)$-measurable and for every $h \in \mathscr{H}_0(U)$,*

$$a_{U,g}^{(n)}(\Phi + h) = \sum_{m=0}^{n} \binom{n}{m} :\Phi^m(h^{n-m}g):, \quad P_U\text{-a.s. in } \Phi \in \mathscr{D}'. \tag{3.7}$$

Remark. Because (3.7) says that for every $h \in \mathscr{H}_0(U)$

$$a_{U,g}^{(n)}(\Phi + h) = a_{U,g}^{(n)}(\Phi) + \sum_{m=0}^{n-1} \binom{n}{m} :\Phi^m(h^{n-m}g):, \quad P_U\text{-a.s. in } \Phi \in \mathscr{D}',$$

it can be interpreted as a *perturbation formula*.

The rest of this section is devoted to the proof of 3.3. It is based on a series of lemmas. We start with the following two facts: The first is a simple formula about Hermite polynomials and the second is a well-known theorem about harmonic functions.

3.4 Lemma. *For t, $s \in \mathbb{R}$,*

$$H_n(s+t) = \sum_{m=0}^{n} \binom{n}{m} H_m(s) t^{n-m}.$$

Proof. This is easily seen using Rodrigue's formula for Hermite polynomials. □

Theorem 3.5 *Every set of uniformly bounded harmonic functions on an open set is equicontinuous.*

Proof. [C/Co, Theorem 11.1.1]. □

Let $(K_j)_{j \in \mathbb{N}}$ be an exhaustion of U by compact sets, i.e.

$$K_j \subset \operatorname{int}(K_{j+1}), \quad j \in \mathbb{N}$$

and

$$U = \bigcup_{j \geqq 1} K_j,$$

such that for every $j \in \mathbb{N}$,

$$\int 1_{U \setminus K_j} dx \leqq 2^{-j}. \tag{3.8}$$

For $k \in \mathbb{N}$ and $h \in \mathscr{H}_0(U)$, set

$$h_k(x) \equiv \int h(y) d_{k,x}(y) dy, \quad x \in \mathbb{R}^2.$$

Furthermore, we define for $g \in L^2(U, dx)$ as usual

$$\operatorname{supp} g = \overline{\{g \neq 0\}},$$

where the closure is taken with respect to the Euclidean topology on $\mathbb{R}^2$.

3.6 Lemma *There exists a subsequence $(k_l)_{l \in \mathbb{N}}$ of $(k)_{k \in \mathbb{N}}$ such that for each $j \in \mathbb{N}$ there exists $l(j) \in \mathbb{N}$ such that for each $m \in \mathbb{N}$ and every harmonic function h on U,*

$$\sup_{x \in K_j} |h^m(x) - h_{k_l}^m(x)| \leqq m 2^{-l} \sup_{x \in K_{j+2}} |h(x)|^m$$

for all $l \geqq l(j)$.

Proof. It is, of course, enough to show the existence of this subsequence for a fixed $j \in \mathbb{N}$, because then, the usual "diagonal argument" yields the assertion. So, fix $j \in \mathbb{N}$. Define

$$\mathscr{H}_1 \equiv \{h: U \to \mathbb{R}: h \text{ is harmonic on } U \text{ and } |h| \leqq 1 \text{ on } K_{j+2}\}.$$

By 3.5 for every $l \in \mathbb{N}$ there exists a $\delta_l > 0$ such that whenever x, $y \in K_{j+1}$ with $|x - y| \leqq \delta_l$, then $|h(x) - h(y)| \leqq 2^{-l}$ for each $h \in \mathscr{H}_1$.

Thus we can select a subsequence $(k_l)_{l\in\mathbb{N}}$ of $(k)_{k\in\mathbb{N}}$ such that for each $x\in K_j$, each $h\in\mathscr{H}_1$ and $l\in\mathbb{N}$,

$$|h(x)-h_{k_l}(x)\leqq\int|h(x)-h(x+y)|d_{k_l,0}(y)dy\leqq 2^{-l},$$

and thus for every $m\in\mathbb{N}$, $|h^m(x)-h^m_{k_l}(x)|\leqq m2^{-l}$. Now the assertion follows by homogeneity. □

From now on we consider the subsequence $(k_l)_{l\in\mathbb{N}}$ of 3.6. Changing (3.2) we write again $(d_{k,x})_{k\in\mathbb{N}}$ instead of $(d_{k_l,x})_{l\in\mathbb{N}}$, $x\in\mathbb{R}^2$, and do the same with $:\Phi^n_k(x):$, $c_{U,k}$ etc.

Now we can define the proper version of $:\Phi^n(g):$. Given $g\in L^p(U,dx)$, $\operatorname{supp} g\subset U$, define

$$a^{(n)}_{U,g}(\Phi)\equiv\limsup_{k\to\infty}:\Phi^n_k(g):,\quad \Phi\in\mathscr{D}', \tag{3.9}$$

and then for arbitrary $g\in L^p(U,dx)$,

$$a^{(n)}_{U,g}(\Phi)=\limsup_{j\to\infty} a^{(n)}_{U,g_j}(\Phi),\quad \Phi\in\mathscr{D}', \tag{3.10}$$

where $g_j\equiv 1_{K_j}g$. Since $(K_j)_{j\in\mathbb{N}}$ is an exhaustion there is no ambiguity in the definition of $a^{(n)}_{U,g}$.

Remark. (3.5), (3.6), (3.8) and the Borel–Cantelli Lemma assures us that the "limsup" in (3.9) and (3.10) is a limit P_U-a.s. in $\Phi\in\mathscr{D}'$ and that $a^{(n)}_{U,g}$ is a version of $:\Phi^n(g):$.

It is clear that $a^{(n)}_{U,g}$ is $\sigma(U\cap\operatorname{supp} g)$-measurable; so it remains to show (3.7), which will be a consequence of the following lemma.

3.7 Lemma. *Let $j,m\in\mathbb{N}$, $q\geqq 1$. Then there exist constants $\alpha,\gamma>0$ and $k_0\in\mathbb{N}$ such that for every $h\in\mathscr{H}_0(U)$ and every $g\in L^p(U,.dx)$ with $\operatorname{supp} g\subset K_j$,*

$$\|:\Phi^n(h^m g):-:\Phi^n_k(h^m_k g):\|_{U,q}\leqq\gamma m2^{-k\alpha}\|g\|_p\sup_{x\in K_{j+2}}|h(x)|^m,$$

for every $k\geqq k_0$.

Proof. By (3.5) and (3.6) we can find $\alpha,\gamma>0$ and $k_0\in\mathbb{N}$ such that for all $k\geqq k_0$, $h\in\mathscr{H}_0(U)$ and $g\in L^p(K_j,dx)$,

$$\begin{aligned}\|:\Phi^n(h^m g):-:\Phi^n_k(h^m_k g):\|_{U,q}&\leqq\|:\Phi^n((h^m-h^m_k)g):\|_{U,q}\\&\quad+\|:\Phi^n(h^m_k g):-:\Phi^n_k(h^m_k g):\|_{U,q}\\&\leqq\gamma\|(h^m-h^m_k)g\|_p+\gamma 2^{-k\alpha}\|h^m_k g\|_p\\&\leqq(\gamma m2^{-k}+\gamma 2^{-k\alpha})\|g\|_p\sup_{x\in K_{j+2}}|h(x)|^m,\end{aligned}$$

where the last inequality follows by 3.6. □

Now let $g\in L^p(U,dx)$, $\operatorname{supp} g\subset U$. By 3.7 and the Borel–Cantelli Lemma it follows that for $h\in\mathscr{H}_0(U)$,

$$\lim_{k\to\infty}\sum_{m=0}^n\binom{n}{m}:\Phi^m_k(h^{n-m}_k g):=\sum_{m=0}^n\binom{n}{m}:\Phi^m(h^{n-m}g):$$

P_U-a.s. in $\Phi \in \mathscr{D}'$. Furthermore, by (3.3) and 3.4 we have that for every $\Phi \in \mathscr{D}'$,

$$:(\Phi + h)_k^n(g): = \sum_{m=0}^{n} \binom{n}{m} :\Phi_k^m(h_k^{n-m} g):.$$

Hence (3.7) follows for such g because of definition (3.9). Because of definition (3.10) the case of an arbitrary $g \in L^p(U, dx)$ can easily be reduced to the one treated above using (3.5), (3.8) and Hölder's inequality. Thus the proof of Theorem 3.3 is completed.

4. Polynomial Additive Functionals for $(\pi_U)_{U \in \mathbb{L}}$ and the $\mathscr{P}(\Phi)_2$-Specifications $(\pi_U^\lambda)_{U \in \mathbb{L}}$

From now on let $\mathscr{P}$ be a fixed lower bounded polynomial,

$$\mathscr{P}(t) \equiv \sum_{n=0}^{N} b_n t^n, \quad t \in \mathbb{R}, \quad b_n \in \mathbb{R}.$$

Using the results of Sect. 3 we can now define the additive functional for $(\pi_U)_{U \in \mathbb{L}}$ associated with the polynomial $\mathscr{P}$.

Given $U \in \mathbb{L}$, define for $g \in L^p(U, dx)$, $p > 1$,

$$a_{U,g}(\Phi) \equiv a_{U,g}^{\mathscr{P}}(\Phi) \equiv \sum_{n=0}^{N} b_n a_{U,g}^{(n)}(\Phi), \quad \Phi \in \mathscr{D}', \tag{4.1}$$

where $a_{U,g}^{(n)}$ is the version of 3.3.

If $g = 1_W$, W a Borel subset of U, we set $a_{U,W} \equiv a_{U,1_W}$ and

$$a_U \equiv a_{U,U}. \tag{4.2}$$

Symbolically, $a_U(\Phi) = $ "$\int_U :\mathscr{P}(\Phi)(x): dx$".

4.1 Theorem. *$(a_U)_{U \in \mathbb{L}}$ defined by (4.2) is an additive functional for $(\pi_U)_{U \in \mathbb{L}}$. Furthermore, $a_U \in \sigma(U)$ for every $U \in \mathbb{L}$.*

Proof. At first we show (1.5): Let U, $V \in \mathbb{L}$ with $U \subset V$. By definition of a_V we have that for all Φ, $\Psi \in \mathscr{D}'$,

$$a_V(\Phi + \bar{H}_V(\Psi)) = a_V(\Phi + \mu_\cdot^V(\Psi)).$$

Hence it follows by (3.7) that for each $\Psi \in \mathscr{D}'$,

$$\begin{aligned} a_V(\Phi + \bar{H}_V(\Psi)) &= \sum_{n=0}^{N} b_n \left(\sum_{m=0}^{n} \binom{n}{m} :\Phi^m((\mu_\cdot^V(\Psi))^{n-m} 1_U): \right) \\ &\quad + \sum_{n=0}^{N} b_n \left(\sum_{m=0}^{n} \binom{n}{m} :\Phi^m((\mu_\cdot^V(\Psi))^{n-m} 1_{V \setminus U}): \right) \\ &= a_U(\Phi + \bar{H}_V(\Psi)) + a_{V, V \setminus U}(\Phi + \bar{H}_V(\Psi)), \end{aligned}$$

P_V-a.s. in Φ (cf. also 3.2). Hence (1.5) follows by the first part of 3.3, which also implies the last part of the assertion. Property (1.6) is again a consequence of (3.7) by [Gl/J, Theorem 8.6.2]. □

Now we define $(\pi_U^\lambda)_{U\in\mathbb{L}}$, $\lambda \geqq 0$ by (1.7) using the additive functional $(a_U)_{U\in\mathbb{L}}$ constructed above for our given polynomial $\mathscr{P}$, i.e.

$$\pi_U^\lambda(\Psi, A) \equiv \frac{\pi_U(1_A e^{-\lambda a_U})(\Psi)}{\pi_U(e^{-\lambda a_U})(\Psi)}, \quad \Psi\in\mathscr{D}', \quad A\in\mathscr{B}. \tag{4.3}$$

4.2 Remark. Let $\Psi\in\mathscr{D}'$. By (2.9)

$$\pi_U^\lambda(\Psi, A) = \pi_U(1_A e^{-\lambda a_U}(\pi_U(e^{-\lambda a_U}))^{-1})(\Psi), \quad A\in\mathscr{B}.$$

As pointed out in the Introduction, 2.8 and 4.1 imply the following theorem:

4.3 Theorem. $(\pi_U^\lambda)_{U\in\mathbb{L}}$ *is a* $(\sigma(U^c))_{U\in\mathbb{L}}$*-specification. It is Markovian in the following sense*:

$$\text{If } U\in\mathbb{L} \text{ and } Z\in\sigma(U)_b, \text{ then } \pi_U^\lambda(Z)\in\sigma(\sigma(\partial U), \Omega_0(U)). \tag{4.4}$$

Furthermore, $(\pi_U^\lambda)_{U\in\mathbb{L}}$ *is "local" in the following sense*:

$$\pi_U^\lambda(\pi_U^\lambda(Z)X) = \pi_U^\lambda(Z)\pi_U^\lambda(X) \text{ for all } Z, X\in\mathscr{B}_b. \tag{4.5}$$

(4.5) follows by (2.9) and (4.4) is an immediate consequence of (2.8), since $a_U\in\sigma(U)$, $U\in\mathbb{L}$. We call $(\pi_U^\lambda)_{U\in\mathbb{L}}$, $\lambda \geqq 0$, *$\mathscr{P}(\Phi)_2$-specification.*

If we assume the "Continuum Hypothesis," polynomial additive functionals for $(\pi_U)_{U\in\mathbb{L}}$ can also be constructed using Mokobodzki's "medial limits" (cf. [De/Me]). But it turns out that in order to prove the required "convergence in measure" for all relevant measures one has to use similar arguments as those in this and the preceding section. Thus the construction is not essentially shorter. In addition, we can avoid the "Continuum Hypothesis."

We conclude this section with another consequence of our perturbation formula (3.7):

4.4 Remark. Let $U\in\mathbb{L}$, $\lambda \geqq 0$ and $g\in\mathscr{D}$. Then, if $\Psi\in\mathscr{D}'$,

$$\pi_U^\lambda(e^{i\Phi(g)})(\Psi) = e^{i\langle H_U(\Psi), g\rangle} \frac{P_U(e^{i\Phi(g)} e^{-\lambda Q_U(\mu_\cdot^U(\Psi))} e^{-\lambda a_U})}{P_U(e^{-\lambda Q_U(\mu_\cdot^U(\Psi))} e^{-\lambda a_U})},$$

where

$$Q_U(\mu_\cdot^U(\Psi))(\Phi) = \sum_{n=0}^{N} b_n \left(\sum_{m=0}^{n-1} \binom{n}{m} : \Phi^m((\mu_\cdot^U(\Psi))^{n-m} 1_U) : \right).$$

$Q_U(\mu_\cdot^U(\Psi))$ is a "polynomial in Φ" of one degree less than $\mathscr{P}$. Therefore, by the formula above $\pi_U^\lambda(\Psi, \cdot)$ can be treated as a *perturbation of the "half-Dirichlet boundary conditioned cut-off $\mathscr{P}(\Phi)_2$-measure"*, i.e. the measure $e^{-\lambda a_U} P_U / P_U(e^{-\lambda a_U})$.

5. The Gibbs States for $(\pi_U^\lambda)_{U\in\mathbb{L}}$

Fix $\lambda \geqq 0$. Let $(\pi_U^\lambda)_{U\in\mathbb{L}}$ be defined by (4.3). In this section we want to study the relation between the associated Gibbs states $G_\lambda(U)$, $U\in\mathbb{L}$, G_λ (now defined by (1.8) and (1.9)) and the local respectively global Gibbs states in the sense of Guerra, Rosen and Simon (cf. [G/R/S 1, 2]). We start with a proposition that summarizes useful properties of

$G_\lambda(U)$ and G_λ. The proof is an immediate consequence of the definitions and the "consistency property" (1.4) (cf. also [R3, 7.5]).

5.1 Proposition. *Let $U\in\mathbb{L}$ and $P\in\mathbb{P}(\mathscr{D}';\mathscr{B})$. Then:*

(i) *$P\pi_U^\lambda\in G_\lambda(U)$.*

(ii) *$P\in G_\lambda(U)$, if and only if $P=P\pi_U^\lambda$.*

(iii) *If $V\in\mathbb{L}$ with $U\subset V$, then $G_\lambda(V)\subset G_\lambda(U)$. In particular, if $(U_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{L}$ such that for every $V\in\mathbb{L}$ there exists $n\in\mathbb{N}$ with $V\subset U_n$, then*

$$G_\lambda=\bigcap_{n\geqq 1}G_\lambda(U_n).$$

(iv) *If $P\in G_\lambda(U)$, then $P(\Omega_0(U))=1$.*

Now we want to recall the definition of GRS-Gibbs states (cf. [G/R/S 1, p. 240 and also Theorem VII.2]).

5.2 Definition. Let $U\in\mathscr{U}_c$. An element $P\in\mathbb{P}(\mathscr{D}';\mathscr{B})$ is called *Markovian with respect* to U if

$$E_P(X\,|\,\sigma(U^c))=E_P(X\,|\,\sigma(\partial U))\quad P\text{-a.s.}$$

for every $X\in\sigma(\bar{U})_b$. P is a *local Markov field* or has the *local Markov property*, if P is Markovian with respect to each $U\in\mathscr{U}_c$.

For more details on Markov properties for generalized random fields see [A/H-K, D3 and R1, 2].

5.3 Definition. Let $U\in\mathscr{U}_c$ and $P\in\mathbb{P}(\mathscr{D}';\mathscr{B})$. Then P is called *U-Gibbsian in the sense of* GRS, if

(i) $P_{\restriction\sigma(\bar{U})}$ (i.e. the restriction of P to $\sigma(\bar{U})$) is absolutely continuous with respect to $P_{0\restriction\sigma(\bar{U})}$.

(ii) P has the local Markov property.

(iii) For every $X\in\sigma(\bar{U})_b$,

$$E_P(X\,|\,\sigma(\partial U))=\frac{E_{P_0}(Xe^{-\lambda a_U}|\,\sigma(\partial U))}{E_{P_0}(e^{-\lambda a_U}|\,\sigma(\partial U))},\quad P\text{-a.s.}. \tag{5.1}$$

P is called a GRS-*Gibbs state,* if it is U-Gibbsian for every $U\in\mathscr{U}_c$.

Remark. In [G/R/S 1] the authors consider in (5.1) an arbitrary version of the $L^2(P_0)$-limit of $(\int_U:\mathscr{P}(\Phi_k(x)):dx)_{k\in\mathbb{N}}$, whereas we take the particular version a_U constructed in Sect. 4. To be precise we should mention that in the original definition GRS use closed set instead of open sets. But by (3.5) we can always replace U by $\bar{U}$ as long as ∂U has Lebesgue measure zero. Furthermore, since every $U\in\mathscr{U}_c$ can be exhausted by a sequence in $\mathbb{L}$ and because of the martingale convergence theorem, we may replace $\mathscr{U}_c$ by $\mathbb{L}$ in the second part of 5.2 and 5.3 as well (cf. also [G/R/S 1 VII. 3]).

As mentioned in the introduction (5.1) is referred to as the "DLR-equations for $\mathscr{P}(\Phi)_2$-random fields." But in this formulation the local Markov property has already been incorporated, and since the right-hand side of (5.1) is determined only up to P_0-zero sets, one has to assume 5.3 (i). Our definition of Gibbs states based on the "reformulated DLR-equations" (1.8) is independent of P_0-zero sets, so the a priori assumptions (i) and (ii) can be avoided. An interesting fact is that because of the special form of our specification $(\pi_U^\lambda)_{U\in\mathbb{L}}$, every element in G_λ satisfies (i) and (ii) automatically. Therefore, we will be able to prove the following theorem:

5.4 Theorem. *Let $P\in\mathbb{P}(\mathscr{D}';\mathscr{B})$. Then P is a (global)* GRS-*Gibbs state if and only if $P\in G_\lambda$.*

For the proof we introduce the following notion:

5.5 *Definition.* Let $U\in\mathscr{U}$ and P_1, $P_2\in\mathbb{P}(\mathscr{D}';\mathscr{B})$. P_1 is called *locally absolutely continuous* with respect to P_2 on U if for every $K\subset U$, K compact, $P_{1\restriction_{\sigma(K)}}$ is absolutely continuous with respect to $P_{2\restriction_{\sigma(K)}}$. P_1 and P_2 are *locally equivalent* on U if, in addition, also P_2 is locally absolutely continuous with respect to P_1 on U.

The proof of 5.4 is an immediate consequence of the following three theorems to be proven below.

5.6 Theorem. *Let $U\in\mathbb{L}$ and $P\in G_\lambda(U)$. Then P, P_0 and P_U are locally equivalent on U. In particular, every $P\in G_\lambda$ is locally equivalent to P_0 on $\mathbb{R}^2$.*

5.7 Theorem. *Every $P\in G_\lambda$ is a local Markov field.*

5.8 Theorem. *Let $U\in\mathbb{L}$ and $P\in\mathbb{P}(\mathscr{D}';\mathscr{B})$. If P is a local Markov field and locally absolutely continuous with respect to P_0 on $\mathbb{R}^2$, then (5.1) and (1.8) are equivalent.*

In order to prove 5.6 we need some preparations. For the rest of this section fix $U\in\mathbb{L}$.

5.9 Lemma. *Let f, $g\in\mathscr{D}$. Then*

$$E_{P_U}(\exp(i\Phi(g)+\Phi(f)))=\exp(\tfrac{1}{2}(\|f\|_{E,U}^2-\|g\|_{E,U}^2)+i\langle g,f\rangle_{E,U}).$$

Proof. We may assume that $\|f\|_{E,U}\neq 0$. Set

$$g_1:=\frac{\langle f,g\rangle_{E,U}}{\|f\|_{E,U}^2}\quad\text{and}\quad g_2:=g-g_1.$$

Then $\langle g_1,g_2\rangle_{E,U}=\langle g_2,f\rangle_{E,U}=0$, and thus $\Phi(g_2)$ is independent of $\Phi(g_1)$ and $\Phi(f)$ (since $\langle\,,\rangle_{E,U}$ is the covariance of P_U and P_U is Gaussian). It follows that

$$\begin{aligned}E_{P_U}(\exp(i\Phi(g)+\Phi(f)))&=E_{P_U}(\exp i\Phi(g_2))E_{P_U}\left(\exp\left(\left(i\frac{\langle f,g\rangle_{E,U}}{\|f\|_{E,U}^2}+1\right)\Phi(f)\right)\right)\\&=\exp[\tfrac{1}{2}(\|f\|_{E,U}^2-\|g\|_{E,U}^2)+i\langle f,g\rangle_{E,U}].\end{aligned}$$ □

Let H_{-1} be the Sobolev space of order -1, i.e. the space of distributions obtained by completing $\mathscr{D}$ with respect to $\|\quad\|_E$. For an element $f\in H_{-1}$ and a closed set $A\subset\mathbb{R}^2$, define f^A to be the balayage of f on A, i.e. the projection of f onto the closed subspace of all elements in H_{-1} with support in A. We recall that the map G defined on $\mathscr{D}$ by $Gf(x)\equiv\int G(x,y)f(y)\,dy$, $f\in\mathscr{D}$, extends to an isometry between H_{-1} and the Dirichlet space given by $\Delta-1$, i.e. the space of all $u\in L^2(\mathbb{R}^2,dx)$ such that their distributional derivatives $\partial u/\partial x_i$, $i=1,2$, are again in $L^2(\mathbb{R}^2,dx)$ (see [Fu] for more details). We also recall that every element in this Dirichlet space has a quasi-continuous version $\tilde{u}$. We refer to [Fu, Chap. 3] for the precise definitions.

5.10 Proposition. *Let $V \in \mathscr{U}$ with $\bar{V} \subset U$ and $\Psi \in \mathscr{D}'$ such that there exists $f \in \mathscr{D}$ such that for every $g \in \mathscr{D}(V)$,*

$$\int \widetilde{G(f - f^{U^c})}\, g\, dx = \langle \Psi, g \rangle.$$

Then $T_\Psi(P_U)\restriction_{\sigma(V)}$ and $P_U\restriction_{\sigma(V)}$ are equivalent.

Proof. Define $X: \mathscr{D}' \to \mathbb{R}$ by

$$X(\Phi) \equiv \exp(\Phi(f) - \tfrac{1}{2}\| f \|^2_{E,U}), \quad \Phi \in \mathscr{D}'.$$

Then for every $g \in \mathscr{D}(V)$, we have by 5.9 that

$$\begin{aligned} E_{P_U}(\exp(i\Phi(g))X) &= \exp(-\tfrac{1}{2}\| g \|^2_{E,U} + i\langle g, f\rangle_{E,U}) \\ &= \exp(-\tfrac{1}{2}\| g \|^2_{E,U} + i\langle g, f - f^{U^c}\rangle_E) \quad \text{(cf. [R3, 5.2])} \\ &= \exp(-\tfrac{1}{2}\| g \|^2_{E,U} + i\int \widetilde{G(f - f^{U^c})}\, g\, dx) \\ &= \exp(-\tfrac{1}{2}\| g \|^2_{E,U} + i\langle \Psi, g\rangle) \\ &= \int \exp(i\Phi(g))\, dT_\Psi(P_U). \end{aligned}$$

Similarly it follows that if

$$Y(\Phi) \equiv \exp(-\Phi(f) + \langle \Psi, f\rangle - \tfrac{1}{2}\| f \|^2_{E,U}), \quad \Phi \in \mathscr{D}',$$

then

$$\int \exp(i\Phi(g))\, Y\, dT_\Psi(P_U) = E_{P_U}(\exp i\Phi(g)) \text{ for every } g \in \mathscr{D}(V). \qquad \square$$

The following corollary is the key point for the proof of Theorem 5.6. It is based on 5.10 and the fact that a harmonic function can locally be represented as the difference of two potentials. Since we restrict ourselves to the case of $L \equiv \Delta - 1$, we can do this in an explicit way (cf. also [H/St] for a special case).

5.11 Corollary. *Let $\Psi \in \mathscr{D}'$. Then P_U and $\pi_U^\lambda(\Psi, \cdot)$ are locally equivalent on U.*

Proof. By (4.3) we may assume that $\lambda = 0$. Let K be a compact subset of U. Let $\kappa \in \mathscr{D}(U)$ such that $\kappa \equiv 1$ on an open neighbourhood V of K. Define $f \equiv -L(\kappa \bar{H}_U(\Psi))$. Note that since $\bar{H}_U(\Psi)$ is represented by a harmonic function on U we have that $f \in \mathscr{D}(U)$.

Because of 5.10 it remains to show that for every $g \in \mathscr{D}(V)$,

$$\int \widetilde{G(f - f^{U^c})}\, g\, dx = \langle \bar{H}_U(\Psi), g \rangle. \tag{5.2}$$

So let $g \in \mathscr{D}(V)$. Since for an arbitrary $f_1 \in \mathscr{D}$, we have that

$$\widetilde{G(-Lf_1)} = f_1 \quad dx\text{-a.e.},$$

we conclude that

$$\int Gf g\, dx = \int \bar{H}_U(\Psi) g\, dx,$$

and furthermore, that

$$\int \widetilde{G(f^{U^c})}\, g\, dx = \int Gf\, dg^{U^c} = \int \kappa \bar{H}_U(\Psi)\, dg^{U^c} = 0.$$

This proves (5.2). $\square$

5.12 Remark. We will prove below (cf. 7.2) that on $\sigma(U)$, however, the measures P_U and $\pi_U^\lambda(\Psi,\cdot)$ are either singular or equal. Furthermore, $\pi_U^\lambda(\Psi,\cdot)$ is in general not absolutely continuous with respect to P_0 on $\sigma(U)$ (cf. 7.2). By 5.8 it follows that $G_\lambda(U)$ is strictly bigger than the set of U-Gibbsian states in the sense of GRS.

Now we are prepared to prove 5.6.

Proof of 5.6. Let K be a compact subset of U and $N\in\sigma(K)$. Assume $P_U(N)=0$. Then by 5.11 for every $\Psi\in\mathscr{D}'$, $\pi_U^\lambda(\Psi,N)=0$, hence

$$P(N)=\int \pi_U^\lambda(\Psi,N)P(d\Psi)=0.$$

Conversely, if $P(N)=0$, then there exists a $\Psi\in\mathscr{D}'$ such that $\pi_U^\lambda(\Psi,N)=0$. Hence, by 5.11, $P_U(N)=0$.

By 2.8 we know that $P_0\in G_0(U)$. Hence, the first part implies that P_0 is locally equivalent to P_U on U and everything is proven. □

Remark. The fact that P_0 and P_U are locally equivalent on U is, of course, well known (cf. e.g. [S, Theorem VIII 2]). The proofs in the literature so far were done by analytical methods. The new proof given above is of probabilistic nature.

Proof of 5.7. Let $X\in\mathscr{B}_b$. Assume first that $X\in\sigma(U)_b$. We know that

$$E_P(X\,|\,\sigma(U^c))=\pi_U^\lambda(X),\quad P\text{-a.s.}.$$

By (4.4) we have that $\pi_U^\lambda(X)\in\sigma(\sigma(\partial U),\Omega_0(U))$, and by 5.1 (iv) it follows that $P(\Omega_0(U))=1$. Thus

$$E_P(X\,|\,\sigma(U^c))=E_P(X\,|\,\sigma(\partial U)),\quad P\text{-a.s.}.$$

Now a simple argument using monotone class theorems extends this to $X\in\sigma(\sigma(U),\sigma(\partial U))_b$.

Let K be a compact set in $\mathbb{R}^2$ such that $U\subset\operatorname{int}K$. For a sub σ-field $\mathscr{A}$ of $\mathscr{B}$ and a probability measure P' on $(\mathscr{D}';\mathscr{B})$ we define $\mathscr{A}^{K,P'}$ to be the σ-field generated by $\mathscr{A}$ and $\{N\in\sigma(K): P'(N)=0\}$. By a slight modification of [R3, 5.4 and 5.5], it is easy to see that

$$\sigma(\sigma(U),\sigma(\partial U))^{K,P_0}=\sigma(\bar U)^{K,P_0},$$

and hence by 5.6,

$$\sigma(\sigma(U),\sigma(\partial U))^{K,P}=\sigma(\bar U)^{K,P}.$$

Thus,

$$E_P(X\,|\,\sigma(U^c))=E_P(X\,|\,\sigma(\partial U))\quad P\text{-a.s.}$$

for every $X\in\sigma(\bar U)_b$. □

Before we prove 5.8 we introduce the following algebra of sets. Let

$$\mathscr{L}\equiv\bigcup_{V\in\mathscr{U}_c}\sigma(V). \tag{5.3}$$

$\mathscr{L}$ is called the algebra of *local observables* and an element $X\in\mathscr{L}$ is called *locally measurable.*

We note that though we assume the local Markov property for P, the equivalence of (5.1) and (1.8) is not clear. One problem is to show that up to P-zero sets $\mathscr{B}=\sigma(\sigma(\bar{U}),\sigma(U^c))$. Since we are dealing with a measure space of distributions such a "sharp" decomposition is always difficult to prove. The proof of 5.8 is somewhat technical and based on the following three lemmas. Their proofs are all direct consequences of the construction of $\bar{H}_U$ in [R3]. We only indicate the main ideas and refer the reader to the corresponding details in [R3]. The arguments are so complicated since $\Omega_0(U)$ is not locally measurable.

5.13 Lemma. *There exists a countable dense subset $\mathscr{D}_1$ of $\mathscr{D}$ such that for each $g\in\mathscr{D}_1$, one can find a real function $Z_g\in\sigma(\bar{U})$ such that $\langle \Phi - \bar{H}_U(\Phi), g\rangle = Z_g(\Phi)$ for each $\Phi\in\Omega(U)$.*

Proof. This is a direct consequence of the construction of $\bar{H}_U$ (cf. Sect. 6 in [R3]). The main reason is that

$$\bar{H}_U(\Phi)=\Phi \quad \text{on} \quad \mathscr{D}(\operatorname{int} U^c)$$

(cf. 2.1 (iv)). □

5.14 Lemma. *Let $P\in\mathbb{P}(\mathscr{D}';\mathscr{B})$ which is locally absolutely continuous with respect to P_0. Then $P(\Omega_0(U))=1$.*

Proof. Since for every $x\in U$ there exists a map $\tilde{\mu}_x^U:\mathscr{D}'\to\mathbb{R}$ which is $\sigma(\partial U)$-measurable such that for every $\Phi\in\mathscr{D}'$ the map $x\to\tilde{\mu}_x^U(\Phi)$ is harmonic and $\tilde{\mu}_x^U(\Phi)=\mu_x^U(\Phi)$ for every $\Phi\in\Omega(U)$, $x\in U$ (cf. [R3, 4.8 (v)]), it suffices to show $P(\Omega(U))=1$. This follows from the construction of $\Omega(U)$. □

5.15 Lemma. *Let $X\in\mathscr{L}$, $X\geqq 0$. Then there exists $Z\in\mathscr{L}$, $Z\geqq 0$, such that $\pi_U(X)(\Psi) = Z(\Psi)$ for each $\Psi\in\Omega_0(U)$.*

Proof. Applying the usual monotone class theorems, we may assume that $X=1_F$ with

$$F=\{\Phi\in\mathscr{D}':\Phi(g_1)\in B_1,\dots,\Phi(g_n)\in B_n\},$$

where $n\in\mathbb{N}$; $g_1,\dots,g_n\in\mathscr{D}$ and $B_1,\dots,B_n$ are Borel subsets of $\mathbb{R}$. For every $\Psi\in\Omega_0(U)$, we have by definition that

$$\pi_U(1_F)(\Psi)=\int\prod_{i=1}^{n}1_{B_i}(\Phi(g_i)+\langle\bar{H}_U(\Psi),g_i\rangle)P_U(d\Phi).$$

But by construction of $\bar{H}_U$ and $\Omega_0(U)$ there exist $Z_i\in\mathscr{L}$, $1\leqq i\leqq n$, such that $\langle\bar{H}_U(\Psi),g_i\rangle=Z_i(\Psi)$ for every $\Psi\in\Omega_0(U)$. Hence the assertion follows by one part of Fubini's theorem. □

Proof of 5.8. Case 1: Assume first that $X\in\sigma(\bar{U})_b$, $X\geqq 0$. Let $X_1\equiv X$ and $X_2\equiv 1$. Then by 2.8 and the local Markov property of P_0, we have that

$$E_{P_0}(X_i e^{-\lambda a_U}\,|\,\sigma(\partial U))=\pi_U(X_i e^{-\lambda a_U}),\quad P_0\text{-a.s.}.$$

By 5.15 there exist $Z_i\in\mathscr{L}$, $Z_i\geqq 0$, such that $\pi_U(X_i e^{-\lambda a_U})=Z_i$ on $\Omega_0(U)$. Therefore, since $P_0(\Omega_0(U))=1$ (cf. 2.5) and $Z_i\in\mathscr{L}$, (5.1) is equivalent to

$$E_P(X\,|\,\sigma(\partial U)) = \frac{Z_1}{Z_2} \quad P\text{-a.s..} \tag{5.4}$$

Since $P(\Omega_0(U)) = 1$ by 5.14, it follows that (5.4) is equivalent to

$$E_P(X\,|\,\sigma(\partial U)) = \pi_U^\lambda(X), \quad P\text{-a.s..} \tag{5.5}$$

Now the Markov property of P implies the equivalence of (5.5) and (1.8) for every $X \in \sigma(\bar{U})_b$. But then (1.8) also holds for $X \in \mathscr{B}$:

Case 2: Assume $X \in \mathscr{B}_b$, $X \geqq 0$. Let $\mathscr{D}_1$ be as in 5.13 and $g \in \mathscr{D}_1$. Let $Z_g \in \sigma(\bar{U})$ be as in 5.13, then it follows that P-a.s. in $\Psi \in \mathscr{D}'$

$$\begin{aligned}
&E_P(\exp i\Phi(g)\,|\,\sigma(U^c))(\Psi)\\
&\quad = \exp i\langle \bar{H}_U(\Psi), g\rangle E_P(\exp i(\Phi(g) - \langle \bar{H}_U(\cdot), g\rangle)\,|\,\sigma(U^c))(\Psi)\\
&\quad = \exp i\langle \bar{H}_U(\Psi), g\rangle E_P(\exp iZ_g\,|\,\sigma(U^c))(\Psi) \quad \text{(by 5.14)}\\
&\quad = \exp i\langle \bar{H}_U(\Psi), g\rangle \pi_U^\lambda(\exp iZ_g)(\Psi) \quad \text{(by Case 1)}\\
&\quad = \exp i\langle \bar{H}_U(\Psi), g\rangle \pi_U^\lambda(\exp i(\Phi(g) - \langle \bar{H}_U(\cdot), g\rangle))(\Psi) \quad \text{(by 5.1(iv))}\\
&\quad = \pi_U^\lambda(\exp i\Phi(g))(\Psi) \quad \text{(by [R3, (7.7)]).}
\end{aligned}$$

Now the assertion follows by monotone class theorems. □

5.16 Remark. The proof shows that the assumption in 5.8 can be weakened. Instead of the "local absolute continuity," it is enough to assume that: $P_{\restriction \sigma(\bar{U})}$ is absolutely continuous with respect to $P_{0\restriction \sigma(\bar{U})}$ and $P(\Omega_0(U)) = 1$. Furthermore, it is enough to have the Markov property only for U.

6. Existence of Gibbs States

In this chapter we are concerned with the question whether G_λ contains at least one element for any given $\lambda \geqq 0$.

It is a well known fact that there are GRS-Gibbs states for every $\lambda \geqq 0$. This was first established for small coupling using a result of Newman [cf. [G/R/S 1, Remark 1, p. 243], [*N*]) and then for general $\lambda \geqq 0$ by Fröhlich and Simon in [Fr/S, 7.2]. Since we know by 5.4 that G_λ is equal to the set of all GRS-Gibbs states, we have the following theorem:

6.1 Theorem. *$G_\lambda \neq \phi$ for every $\lambda \geqq 0$.*

Since this result is very important and the proof is not at all obvious, we want to describe a new approach to prove this theorem using ideas of Preston [P1, Chap. 3] and [P2, Chap. 4]. The main difference to former proofs is that instead of applying Minlos' theorem we use directly Kolmogorov's existence theorem generalized to inductive limits of standard Borel spaces (cf. 6.9 below). So fix $\lambda \geqq 0$.

First we need the following notions:

6.2 Definition. Given $P, P_n \in \mathbb{P}(\mathscr{D}'; \mathscr{B})$, $n \in \mathbb{N}$, $(P_n)_{n\in\mathbb{N}}$ is said to converge *$\mathscr{L}_b$-weakly to P*, if for every $Z \in \mathscr{L}_b$

$$\lim_{n\to\infty} \int Z dP_n = \int Z dP.$$

Since $(\pi_U^\lambda)_{U\in\mathbb{L}}$ is a "local" specification in the sense of 5.15, this notion of convergence is appropriate for our purposes.

6.3 Definition. A set $\mathbb{P}_1 \subset \mathbb{P}(\mathscr{D}'; \mathscr{B})$ is called *locally uniformly (absolutely) continuous* if $V \in \mathscr{U}_c$ and $\varepsilon > 0$ for there exists a probability measure P_ε on $(\mathscr{D}', \sigma(V))$ and $\delta > 0$ such that if $F \in \sigma(V)$ with $P_\varepsilon(F) < \delta$ then $P(F) < \varepsilon$ for all $P \in \mathbb{P}_1$.

Set for L, $t > 0$,

$$U_{L,t} \equiv \{(l,s) \in \mathbb{R}^2 \colon |l| < L/2; |s| < t/2\} \quad \text{and} \quad a_{L,t} \equiv a_{U_{L,t}}.$$

The proof of 6.1 is an immediate consequence of the following three theorems.

6.4 Theorem. *Let P, $P_n \in \mathbb{P}(\mathscr{D}'; \mathscr{B})$, $n \in \mathbb{N}$, such that:*

(i) *$(P_n)_{n\in\mathbb{N}}$ converges $\mathscr{L}_b$-weakly to P.*

(ii) *For each $U \in \mathbb{L}$ there exists $n(U) \in \mathbb{N}$ such that $P_n \in G_\lambda(U)$ for every $n \geqq n(U)$.*

Then $P \in G_\lambda$.

6.5 Theorem. *For every L, t,*

$$\frac{e^{-\lambda a_{L,t}}}{P_0(e^{-\lambda a_{L,t}})} P_0 \in G_\lambda(U_{L,t}),$$

and there exist L_0, $t_0 \geqq 0$ such that

$$\left\{ \frac{e^{-\lambda a_{L,t}}}{P_0(e^{-\lambda a_{L,t}})} P_0 \colon L \geqq L_0, t \geqq t_0 \right\}$$

is locally uniformly continuous.

6.6 Theorem. *Let $\mathbb{P}_1 \subset \mathbb{P}(\mathscr{D}'; \mathscr{B})$ be locally uniformly continuous. Then every sequence in $\mathbb{P}_1$ has a $\mathscr{L}_b$-weakly convergent subsequence.*

Proof of 6.4. Since by (ii) and 5.6 each P_n is locally absolutely continuous with respect to P_0 on every $U \in \mathbb{L}$ such that $n \geqq n(U)$, we conclude by (i) that P is locally absolutely continuous with respect to P_0. Hence it follows by 5.14 that

$$P(\Omega_0(U)) = 1 \quad \text{for each} \quad U \in \mathbb{L}. \tag{6.1}$$

Now let $U \in \mathbb{L}$ and $X \in \mathscr{L}_b$. By 5.15 we can find $Z \in \mathscr{L}_b$ such that $\pi_U^\lambda X = Z$ on $\Omega_0(U)$. Hence by (6.1) and 5.1 (iv) we conclude that

$$P\pi_U^\lambda(X) = P(Z) = \lim_{n\to\infty} P_n(Z) = \lim_{n\to\infty} P_n \pi_U^\lambda(X) = \lim_{n\to\infty} P_n(X) = P(X).$$

Now the usual monotone class theorems and 5.1 (ii) imply the assertion. □

The proof of 6.5 is done in several steps. By [Gl/J, Theorem 8.6.2] the first part of the assertion is contained in the following proposition as a special case which is called the case of "free boundary condition" (cf. [S, §V.1]). In Sect. 7 we will apply this proposition to the most general "boundary conditions" in the sense of [G/R/S 2] (cf. 7.3 below).

6.7 Proposition. *Let* $U\in\mathbb{L}$.

(i) *Let* $\sigma\geqq-\lambda$, $P\in G_\lambda(U)$ *and* $\rho\in\sigma(U^c)$, $\rho>0$, *such that*

$$\int\frac{e^{-\sigma a_U}}{\rho}dP=1.$$

Then

$$\frac{e^{-\sigma a_U}}{\rho}\cdot P\in G_{\lambda+\sigma}(U).$$

In particular, we may take $\rho\equiv\pi_U(e^{-(\lambda+\sigma)a_U})/\pi_U(e^{-\lambda a_U})$.

(ii) *There exists a one to one correspondence between* $G_0(U)$ *and* $G_\lambda(U)$ *given by*

$$G_0(U)\in P\leftrightarrow\frac{e^{-\lambda a_U}}{\pi_U(e^{-\lambda a_U})}P\in G_\lambda(U).$$

Proof. It suffices to prove (i). Let $X\in\mathscr{B}_b$, then

$$\begin{aligned}\int\pi_U^{\lambda+\sigma}(X)\frac{e^{-\sigma a_U}}{\rho}dP&=\int\pi_U^{\lambda+\sigma}(X)\pi_U^{\lambda}(e^{-\sigma a_U})\rho^{-1}dP=\int\pi_U^{\lambda}(Xe^{-\sigma a_U})\rho^{-1}dP\\&=\int Xe^{-\sigma a_U}\rho^{-1}dP.\end{aligned}$$

Hence $e^{-\sigma a_U}/\rho\cdot P\in G_{\lambda+\sigma}(U)$. Since $\pi_U^{\lambda}(e^{-\sigma a_U})=\pi_U(e^{-(\lambda+\sigma)a_U})/\pi_U(e^{-\lambda a_U})$, also the second part of assertion (i) follows. □

In order to prove the locally absolute continuity we need the following result due to Fröhlich and Simon. For the proof see [Fr/S] (in particular, Lemma 7.1 and the proof of Theorem 2.3).

6.8 Proposition. *Let* $V\in\mathscr{U}_c$ *and* $q>4$. *Then there exist constants* $c, t_0, L_0\geqq 0$ *such that for every* $F\in\sigma(V)_b$ *and every* $L\geqq L_0$, $t\geqq t_0$,

$$\left|\int F\frac{e^{-\lambda a_{L,t}}}{E_{P_0}(e^{-\lambda a_{L,t}})}dP_0\right|\leqq c\,\|F\|_{\mathbb{R}^2,q}. \tag{6.2}$$

Proof of 6.5. Let $V\in\mathscr{U}_c$ and $p\equiv(1-q^{-1})^{-1}$. Define

$$\rho_{L,t}\equiv\frac{E_{P_0}(\exp(-\lambda a_{L,t})|\sigma(V))}{E_{P_0}(\exp(-\lambda a_{L,t}))},\quad L,t\geqq 0.$$

Then we know by 6.8 that $\{\rho_{L,t}: L,t\geqq 0\}$ is uniformly bounded in $(L^p(\mathscr{D}',\sigma(V)),\|\ \ \|_{\mathbb{R}^2,p})$ for $L\geqq L_0$, $t\geqq t_0$. Using the Banach/Alaoglu-Theorem (cf. e.g. [Re/S, Theorem IV, 21]) and the Dunford/Pettis-Theorem (cf. e.g. [Me, Chap. II, T23]) we conclude that $\{\rho_{L,t}: L\geqq L_0, t\geqq t_0\}$ is uniformly integrable. Now the second part of 6.5 follows immediately (cf. e.g. [B, 20.7]). □

Theorem 6.6 can be found in an unpublished manuscript of Preston ([P2, Chap. 4]). For completeness we present the proof here. First we recall the following version of Kolmogorov's existence theorem.

6.9 Theorem. *For every* $V\in\mathscr{U}_c$, *let* $P_V\in\mathbb{P}(\mathscr{D}';\sigma(V))$. *If* $(P_V)_{V\in\mathscr{U}_c}$ *is consistent*, i.e.

$$P_V(F)=P_{V'}(F)\ \textit{for all}\ V\subset V',\ F\in\sigma(V),$$

then there exists a probability measure P on $(\mathscr{D}'; \mathscr{B})$ such that

$$P(F) = P_V(F) \text{ for all } V \in \mathscr{U}_c, \quad F \in \sigma(V).$$

Proof. See [Pa, Theorem 4.2, page 143]. We note that for each $V \in \mathscr{U}_c$ the measure space $(\mathscr{D}'; \sigma(V))$ is standard Borel and that its atoms are just of the form,

$$A(\Psi) \equiv \{\Phi \in \mathscr{D}' : \Phi = \Psi \text{ on } \mathscr{D}(V)\},$$

where $\Psi \in \mathscr{D}'$. Hence the condition in [Pa, Theorem 4.2] concerning the atoms of $(\mathscr{D}'; \sigma(V))$ is satisfied. □

Proof of 6.6. Since $(\mathscr{D}'; \sigma(V))$ has uncountably many atoms, we can use the following fact which is an immediate consequence of the main representation theorem for standard Borel spaces (see e.g. [Pa, p. 133]):

There exists a countable algebra $\mathscr{E}(V)$ generating $\sigma(V)$ such that if $P: \mathscr{E}(V) \to \mathbb{R}^+$ is bounded, then P is a measure (i.e. countably additive) if and only if it is finitely additive.

Now let $(V_n)_{n \in \mathbb{N}}$ be a sequence in $\mathscr{U}_c$ such that for each $n \in \mathbb{N}$, $\bar{V}_n \subset V_{n+1}$ and

$$\bigcup_{n \geqq 1} V_n = \mathbb{R}^2.$$

Let

$$\mathscr{A} \equiv \bigcup_{n \geqq 1} \mathscr{E}(V_n),$$

and $(P_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{P}_1$.

Since $\mathscr{A}$ is countable there exists a subsequence $(n_k)_{k \in \mathbb{N}}$ such that

$$Q(F) \equiv \lim_{k \to \infty} P_{n_k}(F)$$

exists in $\mathbb{R}$ for every $F \in \mathscr{A}$. Clearly $Q_{\upharpoonright \mathscr{E}(V_n)}$ is finitely additive, hence countably additive, and thus (by Caratheodory's theorem) extends uniquely to a measure Q_n on $\sigma(V_n)$.

Now we want to apply 6.9. There is no a priori reason for supposing that $(Q_n)_{n \in \mathbb{N}}$ is consistent, but in fact this is implied by the local uniform continuity:

We claim that for all $n \in \mathbb{N}$ and $F \in \sigma(V_n)$,

$$Q_n = \lim_{k \to \infty} P_{n_k}(F). \tag{6.3}$$

Fix $n \in \mathbb{N}$ and $F \in \sigma(V_n)$. Let $\varepsilon > 0$, then there exists a probability measure P_ε on $(\mathscr{D}'; \sigma(V_n))$ and $\delta > 0$ such that if $F \in \sigma(V_n)$ with $P_\varepsilon(F) < \delta$, then $P_{n_k}(F) < \varepsilon$ for all $k \in \mathbb{N}$. Since $\mathscr{E}(V_n)$ generates $\sigma(V_n)$, we can find $A \in \mathscr{E}(V_n)$ such that $Q_n(A \Delta F) < \varepsilon$ and $P_\varepsilon(A \Delta F) < \delta$, and therefore $P_{n_k}(A \Delta F) < \varepsilon$ for every $k \in \mathbb{N}$, where $A \Delta F = (A \backslash F) \cup (F \backslash A)$. Thus

$$\begin{aligned} |Q_n(F) - P_{n_k}(F)| &\leqq |Q_n(A) - P_{n_k}(A)| + Q_n(A \Delta F) + P_{n_k}(A \Delta F) \\ &\leqq |Q_n(A) - P_{n_k}(A)| + 2\varepsilon \end{aligned}$$

and (6.3) is proven.

Equation (6.3) implies the consistency of $(Q_n)_{n\in\mathbb{N}}$, and extending this sequence in the obvious way we get a consistent family $(Q_V)_{V\in\mathscr{U}_c}$ such that $Q_n = Q_{V_n}$ for all $n\in\mathbb{N}$. Now 6.9 and (6.3) imply the assertion. $\square$

7. Martin Boundaries for $\mathscr{P}(\Phi)_2$-Random Fields

Since our Gibbs states are described by means of a specification, we can apply the Martin boundary theory for random fields which was developed by Dynkin and Föllmer (cf. [Dy 1, 2], [F] and also [P1]) to our situation. That means essentially: we can represent an arbitrary Gibbs state in terms of extreme Gibbs states. In the case $\lambda = 0$ it is even possible to give a characterization of the extreme (global) Gibbs states. In the more interesting case of $\lambda > 0$ this is much more difficult. We have a "partial" result also in this case. "Locally" the characterization of the extreme Gibbs states is an almost immediate consequence of the definition of our specification $(\pi_U^\lambda)_{U\in\mathbb{L}}$. This will be the contents of our next theorem. First we need the following notations: Fix $\lambda \geqq 0$ and $U\in\mathbb{L}$.

Given a measure space $(\mathscr{X};\mathscr{A})$, a set $\mathscr{X}'$ and a map T: $\mathscr{X}\to\mathscr{X}'$, define

$$T(\mathscr{A}) \equiv \{F' \subset \mathscr{X}' : T^{-1}(F')\in\mathscr{A}\}.$$

And if $P\in\mathbb{P}\,(\mathscr{X};\mathscr{A})$, we denote by $T(P)$ the image measure of P under T defined on $(\mathscr{X}'\colon T(\mathscr{A}))$. Furthermore, we set for $U\in\mathbb{L}$,

$$\mathbb{P}_U \equiv \mathbb{P}(\tilde{\mathscr{H}}_0(U); \bar{H}_U(\sigma(U^c))).$$

7.1 Theorem. (i) *If* $\Psi\in\mathscr{D}'$, *then*

$$\pi_U^\lambda(\Psi,\cdot) = \pi_U^\lambda(\bar{H}_U(\Psi),\cdot) \tag{7.1}$$

and

$$\pi_U^\lambda(\Psi,\{\Phi\in\mathscr{D}' : \bar{H}_U(\Phi) = \bar{H}_U(\Psi)\}) = 1. \tag{7.2}$$

Furthermore,

$$\partial G_\lambda(U) = \{\pi_U^\lambda(\Psi,\cdot)\colon \Psi\in\mathscr{D}'\}.$$

In particular, the map $h\mapsto\pi_U^\lambda(h,\cdot)$ *is a bijection from* $\tilde{\mathscr{H}}_0(U)$ *to* $\partial G_\lambda(U)$.

(ii) $\bar{H}_U$ *induces a bijection from* $G_\lambda(U)$ *to* $\mathbb{P}_U$. *Under this map the point masses on* $(\tilde{\mathscr{H}}_0(U), \bar{H}_U(\sigma(U^c)))$ *correspond to the extreme elements in* $G_\lambda(U)$.

(iii) $P\in G_\lambda(U)$ *if and only if there exists a (unique)* $\bar{P}\in\mathbb{P}_U$ *(in fact:* $\bar{P} = \bar{H}_U(P)$*) such that*

$$P = \bar{P}\bar{\pi}_U^\lambda, \tag{7.3}$$

where $\bar{\pi}_U^\lambda\colon \tilde{\mathscr{H}}_0(U)\times\mathscr{B}\to[0,1]$ *is a probability kernel which is just the restriction of* π_U^λ *to* $\tilde{\mathscr{H}}_0(U)$ *(in the first variable).*

Proof. (i): (7.1) and (7.2) are consequences of 2.1 (vi) and [R3, 8.7 (ii)], since $\pi_U^\lambda(\Psi,\cdot)$ is by definition absolutely continuous with respect to $\pi_U(\Psi,\cdot)$ for every $\Psi\in\mathscr{D}'$. Now let

$\Psi\in\mathscr{D}'$. Let $P_1, P_2\in G_\lambda(U)$ such that

$$\pi_U^\lambda(\Psi,\cdot)=\tfrac{1}{2}P_1+\tfrac{1}{2}P_2.$$

By (7.2) it follows that $P_2(\{\bar{H}_U=\bar{H}_U(\Psi)\})=P_2(\{\bar{H}_U=\bar{H}_U(\Psi)\})=1$, hence by 5.1 (ii) and (7.1),

$$P_1=P_1\pi_U^\lambda=\pi_U^\lambda(\Psi,\cdot)=P_2\pi_U^\lambda=P_2.$$

Thus, $\pi_U^\lambda(\Psi,\cdot)\in\partial G_\lambda(U)$. Let $P\in\partial G_\lambda(U)$. Since $Z\cdot P\in G_\lambda(U)$ for every $Z\in\sigma(U^c)$ with $P(Z)=1$, it is easy to see that P is trivial on $\sigma(U^c)$. Hence for a dense countable subset $\mathscr{D}_1$ of $\mathscr{D}$ we can find $\Psi\in\mathscr{D}'$ such that for every $g\in\mathscr{D}_1$,

$$\pi_U^\lambda(\exp i\Phi(g))=\pi_U^\lambda(\exp i\Phi(g))(\Psi)\quad P\text{-a.s.}.$$

Since $P\pi_U^\lambda=P$, it follows that $P=\pi_U^\lambda(\Psi,\cdot)$ and (i) is proven.
(ii) + (iii): We have to show that the mappings $\bar{P}\mapsto\bar{P}\bar{\pi}_U^\lambda$, $\bar{P}\in\mathbb{P}_U$ and $P\mapsto\bar{H}_U(P)$, $P\in G_\lambda(U)$ are inverse to each other.

So let $\bar{P}\in\mathbb{P}_U$. Since $(\pi_U^\lambda)_{U\in\mathbb{L}}$ satisfies (1.4), we obtain that $\bar{P}\bar{\pi}_U^\lambda\in G_\lambda(U)$. Using [R3, (7.7)] it follows for $X\in\bar{H}_U(\sigma(U^c))_b$ that

$$\begin{aligned}\int X\,d\bar{H}_U(\bar{P}\bar{\pi}_U^\lambda)&=\int\bar{\pi}_U^\lambda(X\circ\bar{H}_U)\,d\bar{P}=\int X\circ\bar{H}_U\,d\bar{P}\quad&&\text{(by [R3, (7.7)])}\\&=\int X\,d\bar{P}.&&\text{(by 2.1 (vi))}.\end{aligned}$$

And for $P\in G_\lambda(U)$ and $X\in\mathscr{B}_b$, we have that

$$\int X\,d(\bar{H}_U(P)\bar{\pi}_U^\lambda)=\int(\pi_U^\lambda X)\circ\bar{H}_U\,dP=\int\pi_U^\lambda X\,dP=\int X\,dP. \qquad\square$$

7.2 Remark. At this point we want to consider some properties of the measures $\pi_U^\lambda(\Psi,\cdot)$ when restricted to $\sigma(U)$. It is easy to see that if $Z\in\sigma(U)_b$, then (7.1) can be rewritten as

$$\pi_U^\lambda(Z)(\Psi)=\pi_U^\lambda(Z)(\mu_\cdot^U(\Psi)),\quad\Psi\in\mathscr{D}'. \tag{7.4}$$

According to the construction in [R3, Sect. 4] we can change the definition of $\mu_\cdot^U(\Phi)$ for $\Phi\in\mathscr{D}'\backslash\Omega_0(U)$ in such a way such that the new version $\tilde{\mu}_x^U(\Phi)$ is still harmonic as a function of x on U, but also $\sigma(U)\cap\sigma(\partial U)$-measurable in $\Phi\in\mathscr{D}'$. If we set $M_h\equiv\{\Phi\in\mathscr{D}':\tilde{\mu}_\cdot^U(\Phi)=h\}$ for $h\in\mathscr{H}_0(U)$, then by [R3, 4.8(v)]

$$M_h\in\sigma(U)\cap\sigma(\partial U). \tag{7.5}$$

Clearly, $M_h\cap M_{h'}=\phi$ if $h'\in\mathscr{H}_0(U)\backslash\{h\}$. Let $\Psi\in\mathscr{D}'$. By (7.3) and the fact that $\pi_U^\lambda(\Psi,\Omega_0(U))=1$, (cf. 5.1 (iv)) it follows that

$$\pi_U^\lambda(\Psi,\{\tilde{\mu}_\cdot^U(\cdot)=\mu_\cdot^U(\Psi)\})=1. \tag{7.6}$$

Hence on $\sigma(U)$ the measure $\pi_U^\lambda(\Psi,\cdot)$ is either equal to $\pi_U^\lambda(\Phi,\cdot)$, $\Phi\in\mathscr{D}'$, namely if $\mu_x^U(\Psi)=\mu_x^U(\Phi)$ for all $x\in U$ or singular to it otherwise.

Now we want to show that for $\Psi\in\mathscr{D}'$ the measure $\pi_U^\lambda(\Psi,\cdot)$ is in general not absolutely continuous with respect to P_0. By (7.4) we may assume that $\Psi\equiv h\in\mathscr{H}_0(U)$. Of course, it is enough to consider the case $\lambda=0$.

Assume that $\pi_U(h,\cdot)$ is absolutely continuous with respect to P_0 on $\sigma(U)$. Then by (7.6) we know that $P_0(M_h)>0$. Therefore, there are at most countably many measures $\pi_U(h,\cdot)$ which are absolutely continuous with respect to P_0.

It is also easy to see that P_0 is not absolutely continuous with respect to $\pi_U(h,\cdot)$ on $\sigma(U)$. Because, if this were the case, then by (7.6), $P_0(\{\tilde{\mu}^U_\cdot(\cdot)=h\})=1$. Therefore, by 2.8, (7.4) and 5.1 (iv) for every $Z\in\sigma(U)_b$,

$$P_0(Z)=\int 1_{\{\tilde{\mu}^U_\cdot(\cdot)=h\}}\pi_U(Z)dP_0=\pi_U(Z)(h).$$

Therefore, $h=0$, hence $P_0=P_U$ on $\sigma(U)$, which is impossible (consider the corresponding Fourier transforms!).

One of the main consequences of 7.1 is that all "cut-off $\mathscr{P}(\Phi)_2$-measures" (with Wick-ordering with respect to P_0) can be represented in terms of basic ones. More precisely:

7.3 Theorem. *Let $P\in\mathbb{P}(\mathscr{D}';\mathscr{B})$ which corresponds to a $\{Q,l\}$-boundary condition Gaussian field on U in the sense of* [G/R/S 2], i.e. *P is the Gaussian measure on $(\mathscr{D}';\mathscr{B})$ with mean*

$$\int \Phi(f)dP=l(f^{\partial U}),\quad f\in\mathscr{D},$$

and covariance

$$\int \Phi(f)\Phi(g)dP=\int \Phi(f)\Phi(g)dP_U+Q(f^{\partial U},g^{\partial U}),$$

$f,g\in\mathscr{D}$, where l is a bounded linear functional on $(H_{-1},\|\quad\|_E)$ and Q is a bounded, positive definite quadratic form on $(H_{-1},\|\quad\|_E)$. Let $\rho\in\sigma(U^c)$, $\rho>0$, such that

$$\int\frac{e^{-\lambda a_U}}{\rho}dP=1,\quad \textit{then}\quad \frac{e^{-\lambda a_U}}{\rho}\cdot P\in G_\lambda(U).$$

In particular, $e^{-\lambda a_U}/\rho\, P$ can be represented in terms of $\pi_U^\lambda(h,\cdot)$, $h\in\mathscr{H}_0(U)$ in the sense of 7.1.

Proof. By 6.7 it suffices to prove that

$$P\in G_0(U). \tag{7.7}$$

Let $V\in\mathbb{L}$, $V\subset U$, then $P\in\mathbb{M}(V)$ by 2.4 (i). Furthermore, it is clear that we can find a constant $c>0$ such that for every $f\in\mathscr{D}$,

$$\left(\int \Phi(f)^2dP\right)^{1/2}\leqq c\|f\|_E=c\left(\int \Phi(f)^2dP_0\right)^{1/2}.$$

Hence by 2.6

$$P(\Omega_0(V))=1. \tag{7.8}$$

Therefore, by [G/R/S 2, Theorem II. 2], [G/R/S 1, Theorem VII. 2] and 5.16

$$P\in G_0(V)$$

for all $V\in\mathbb{L}$ with $\bar{V}\subset U$. Hence (7.7) follows by (7.8) and [R3, 8.7 (i)]. □

Remark. (i) For the connection between $\{Q,l\}$-boundary condition Gaussian measures on U and Gaussian measures associated with $H-1$ for certain self-adjoint extensions H of $\Delta\upharpoonright\mathscr{D}(U)$ see [G/R/S 2 Theorem II 6].

(ii) The main reason that we were able to establish Theorem 7.3 is, of course, that contrary to the GRS-definition of "U-Gibbsian" (cf. 5.3(i)) an element in $G_\lambda(U)$ is not necessarily absolutely continuous with respect to P_0 on $\sigma(U)$ (cf. 7.2).

(iii) From the point of view of statistical mechanics, Theorem 7.3 is, of course, not surprising, since a "local" Gibbs state in statistical mechanics is understood to represent the most general "boundary conditions."

(iv) We note that what is usually called "Dirichlet boundary condition" in the literature now corresponds to "zero boundary condition," since for $X \in \mathscr{B}_b$,

$$\pi_U^\lambda(X)(0) = \frac{P_U(X e^{-\lambda a_U})}{P_U(e^{-\lambda a_U})}. \tag{7.9}$$

Now we want to consider the global situation. We set

$$\mathscr{B}_\infty \equiv \bigcap_{U \in \mathbb{L}} \sigma(U^c). \tag{7.10}$$

$\mathscr{B}_\infty$ is called the *tail σ-field.*

As mentioned above the general Martin boundary theory for random fields can now be applied. Therefore, we have the following theorem.

7.4 Theorem (Dynkin/Föllmer). *Let $\lambda \geqq 0$. Then there exists a probability kernel π_∞^λ on $(\mathscr{D}'; \mathscr{B})$ such that*

(i) *$\pi_\infty^\lambda(X)$ is $\mathscr{B}_\infty$-measurable for each $X \in \mathscr{B}_b$.*

(ii) *If $P \in G_\lambda$ and $X \in \mathscr{B}_b$, then*

$$E_P(X \mid \mathscr{B}_\infty) = \pi_\infty^\lambda(X), \quad P\text{-a.s.}.$$

(iii) *If $P \in \mathbb{P}(\mathscr{D}'; \mathscr{B})$, then:*

$$P \in G_\lambda \textit{ if and only if } P\pi_\infty^\lambda = P.$$

(iv) $\partial G_\lambda = \{\pi_\infty^\lambda(\Psi, \cdot) : \Psi \in \mathscr{D}'\}$,
and if

$$\Delta(\Psi) \equiv \{\Phi \in \mathscr{D}' : \pi_\infty^\lambda(\Psi, \cdot) = \pi_\infty^\lambda(\Phi, \cdot)\}, \quad \Psi \in \mathscr{D}',$$

then $\Delta(\Psi) \in \mathscr{B}_\infty$ and $\pi_\infty^\lambda(\Psi, \Delta(\Psi)) = 1$.

Proof. For a detailed proof see [P1, Chap. 2] □

7.5 Corollary. Let $\mathscr{M} \equiv \{\Delta(\Psi) : \Psi \in \mathscr{D}'\}$.

(i) The map $\Delta : \mathscr{D}' \mapsto \mathscr{M}$ induces an affine bijection between G_λ and $\mathbb{P}(\mathscr{M}; \Delta(\mathscr{B}_\infty))$. Under this map the point masses on $(\mathscr{M}; \Delta(\mathscr{B}_\infty))$ correspond to the extreme elements of G_λ.

(ii) Let $\bar{\pi}_\infty^\lambda : \mathscr{M} \times \mathscr{B} \to [0,1]$ be the probability kernel defined by

$$\bar{\pi}_\infty^\lambda(\Delta(\Psi), F) \equiv \pi_\infty^\lambda(\Psi, F), \quad \Psi \in \mathscr{D}', \quad F \in \mathscr{B}.$$

Then $P \in G_\lambda$ if and only if there exists a unique $\bar{P} \in \mathbb{P}(\mathscr{M}; \Delta(\mathscr{B}_\infty))$ (in fact: $\bar{P} = \Delta(P)$) such that

$$P = \bar{P}\bar{\pi}_\infty^\lambda. \tag{7.11}$$

Proof. [P1, Proposition 2.4] □

7.6 *Definition.* The space $(\mathscr{M}; \Delta(\mathscr{B}_\infty))$ is called the *Martin boundary for* $(\pi_U^\lambda)_{U \in \mathbb{L}}$ (or G_λ).

7.7 *Remark.* (i) (7.11) is just the representation of a $P \in G_\lambda$ as an integral over the Martin boundary $\mathscr{M}$.

(ii) If we restrict all involved probability measures to $\sigma(U)$, we can now interpret Theorem 7.1 analogously by saying that:

If $U \in \mathbb{L}$, then $\mathscr{H}_0(U)$ is the "*local*" *Martin* boundary for $(\pi_V^\lambda)_{V \in \mathbb{L}}$ on U.

(Here we identify $\Delta(\Psi)$ with $\mu^U_\cdot(\Psi)$, $\Psi \in \mathscr{D}'$).

This characterization of the local Martin boundary is also true globally, if $\lambda = 0$. This is just a special case of Theorem 8.6 in [R3] (cf. also [H/St] for a proof using the GRS-definition of Gibbs states and without constructing the associated specification).

7.8 Theorem. *The Martin boundary for $(\pi_U)_{U \in \mathbb{L}}$ is the set of all harmonic functions on $\mathbb{R}^2$.*

An immediate consequence is the following "uniqueness result":

7.9 Corollary. *Let h be a harmonic function on $\mathbb{R}^2$. Then*

$$G_0 \cap \{P \in \mathbb{P}(\mathscr{D}'; \mathscr{B}) : P(\{\bar{H}_{\mathbb{R}^2} = h\}) = 1\} = \{\pi_{\mathbb{R}^2}(h, \cdot)\} = \{T_h(P_0)\}.$$

In particular, P_0 is the unique measure in G_0 supported by $\{\bar{H}_{\mathbb{R}^2} = 0\}$.

7.8 shows that G_λ, $\lambda \geqq 0$, contains in general more than one element and that (even for $\lambda = 0$) an additional "condition at infinity" is needed to obtain uniqueness (see also [G/R/S 1]). The following considerations show that the condition $\bar{H}_{\mathbb{R}^2} = \text{const.}$ P-a.s., which implies uniqueness in the case $\lambda = 0$, is also at least necessary for uniqueness in the case $\lambda \geqq 0$. In order to state the corresponding theorem which will be the last of this paper we introduce the following notation:

Let $\lambda \geqq 0$. For a harmonic function h on $\mathbb{R}^2$ we define

$$G_{\lambda,h} \equiv G_\lambda \cap \{P \in \mathbb{P}(\mathscr{D}'; \mathscr{B}) : P(\{\bar{H}_{\mathbb{R}^2} = h\}) = 1\}.$$

7.10 Theorem. *∂G_λ is the (pairwise disjoint) union of $\partial G_{\lambda,h}$, h harmonic on $\mathbb{R}^2$, and each $P \in G_{\lambda,h}$ can be represented in terms of elements in $\partial G_{\lambda,h}$.*

Proof. Let $\Psi \in \mathscr{D}'$. Then $\bar{H}_{\mathbb{R}^2}(\Psi)$ is harmonic on $\mathbb{R}^2$ and we may consider the map $x \mapsto \bar{H}_{\mathbb{R}^2}(\Psi)(x)$, $x \in \mathbb{R}^2$. Let $P \in \partial G_\lambda$, then we know by [P1, Theorem 2.1 (1)] that for every $B \in \mathscr{B}_\infty$, $P(B) \in \{0, 1\}$. Since by [R3, 6.12] the map $\Psi \mapsto \bar{H}_{\mathbb{R}^2}(\Psi)(x)$ is $\mathscr{B}_\infty$-measurable for every $x \in \mathbb{R}^2$, we conclude that for P-a.e. $\Psi \in \mathscr{D}'$,

$$\bar{H}_{\mathbb{R}^2}(\Psi)(x) = E_P(\bar{H}_{\mathbb{R}^2}(\cdot)(x) \mid \mathscr{B}_\infty) = E_P(\bar{H}_{\mathbb{R}^2}(\cdot)(x)).$$

Since $\mathbb{R}^2$ is separable, we can therefore find a harmonic function h on $\mathbb{R}^2$ such that $\bar{H}_{\mathbb{R}^2} = h$, P-a.s.. Furthermore, we have

$$\partial G_{\lambda,h} = \partial G_\lambda \cap \{P \in \mathbb{P}(\mathscr{D}'; \mathscr{B}) : P(\{\bar{H}_{\mathbb{R}^2} = h\}) = 1\}.$$

The second part of the assertion is an immediate consequence of 7.4(iii) and 7.5. □

If $\lambda > 0$, but small, then one might hope that the reverse of Theorem 7.10 is also true (as in the case $\lambda = 0$). This would again lead to a complete characterization of the Martin boundary of $(\pi_U^\lambda)_{U \in \mathbb{L}}$. But this seems to be extremely hard to prove. The reason is the fact that the interaction (i.e. $\lambda > 0$) produces in a sense a non-linear situation whereas the map $\bar{H}_{\mathbb{R}^2}$ is linear. This problem will be the subject of future study.

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