Commun. math. Phys. 40, 49—54 (1975)

The Invariance of the *S*-Matrix under Point Transformations in Renormalized Perturbation Theory

R. Flume

II. Institut für Theoretische Physik, Universität Hamburg, Hamburg, Federal Republic of Germany

Received June 18, 1974

Abstract. We give a simple proof of the invariance of the *S*-matrix under point transformations of the fields in renormalized perturbation field theory.

I. Introduction

The equivalence theorem of Lagrangian field theory can be stated in the following form: The quantization of two classical Lagrange densities $\mathscr{L}(\phi, \partial_\mu \phi)$ and $\mathscr{L}'(\phi, \partial_\mu \phi)$ related to each other through a point transformation of the fields

$$\phi \to (\phi + h(\phi)), \qquad \left(h(0) = 0, \ \frac{dh}{d\phi} = 0\right)$$

$$\mathscr{L}'(\phi, \partial_\mu \phi) = \mathscr{L}(\phi + h(\phi), \partial_\mu(\phi + h(\phi)))$$

gives rise to the same *S*-matrix.

There exist several formal proofs for this theorem in the literature [1, 2]. Recently Lam [3] proposed a constructive and rigorous proof in renormalized perturbation theory. His method of proof is based on the technique of anisotropically quantized normal products [4]. The purpose of this note is to present a proof of the equivalence theorem, which subsists on a minimum of technicalities and, in particular, avoids the rather involved normal product formalism. Furthermore, in contrast to Lam's approach, we don't need to refer in our proof to the Haag-Ruelle (L.S.Z.)-theorem.

II.

We confine ourselves to a Lagrangian $\mathscr{L}(\phi, \partial^\mu \phi)$ which is a function of a single scalar massive field (of mass $\bar{m}$). We exclude higher than first space-time derivatives. The proof presented below applies modulo a

greater amount of book-keeping also to Lagrangians involving several fields of possibly different type.

To start with, we construct for a given not necessarily renormalizable Lagrangian $\mathscr{L}(\phi, \partial_\mu \phi)$ [1] the S-matrix

$$S(\mathscr{L}) = \lim_{g \to \text{konst.}} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int T(:\bar{\mathscr{L}}(x_1): \ldots : \bar{\mathscr{L}}(x_n):) \cdot g(x_1) \ldots g(x_n)\, dx_1 \ldots dx_n , \tag{1}$$

$$\mathscr{L} = \mathscr{L}_{\text{free}} + \bar{\mathscr{L}} ,$$

by specifying through some ad hoc prescription, which may or may not correspond to a minimal subtraction scheme, the time ordered operator valued distributions $T(:\bar{\mathscr{L}}(x_1): \ldots : \bar{\mathscr{L}}(x_n):)$. One knows from the work of Epstein and Glaser [5, 6] that the adiabatic limit $g \to$ konst in (1) exists [2] and that the theory fulfils in the adiabatic limit all requirements of locality and unitary in the sense of formal power series.

$\mathscr{L}(\phi, \partial_\mu \phi)$ goes under a point transformation $\phi \to (\phi + h(\phi))$ over into $\mathscr{L}_h = \mathscr{L}(\phi + h(\phi), \partial_\mu(\phi + h(\phi)))$. h has to be considered as a formal power series in $\phi (h(0) = 0,\ dh/d\phi = 0)$. We interpolate $\mathscr{L}$ and $\mathscr{L}_h$ by

$$\mathscr{L}_\lambda(\phi) = \mathscr{L}(\phi + \lambda h, \partial_\mu(\phi + \lambda h)) \qquad 0 \leqq \lambda \leqq 1 .$$

Specifying the time ordered products $T_\lambda(x_1 \ldots x_n) = T(:\bar{\mathscr{L}}_\lambda(x_1): \ldots : \bar{\mathscr{L}}_\lambda(x_n):)$ we obtain a one parameter family S_λ of S-matrices. We will define the T_λ in such a way that $dS_\lambda/d\lambda$ vanishes and S_0 is given by Eq. (1). To make the motivation of our procedure clearer we give first a naive argument, which afterwards will be made rigorous. The argument is taken over with slight modifications from Divakaran [2].

$dS_\lambda/d\lambda$ is entirely determined by the derivatives with respect to λ of the time ordered products T_λ

$$\frac{d}{d\lambda} T_\lambda(x_1 \ldots x_n) = \sum_i T\left(:\bar{\mathscr{L}}_\lambda(x_1): \ldots : \bar{\mathscr{L}}_\lambda(x_{i-1}): \ldots : \frac{d}{d\lambda} \mathscr{L}_\lambda(x_i): \ldots : \bar{\mathscr{L}}_\lambda(x_n):\right).$$

We can manipulate $d\mathscr{L}_\lambda/d\lambda$ as follows:

$$\frac{d\mathscr{L}_\lambda}{d\lambda} = \frac{d}{d\lambda} \mathscr{L}(\phi + \lambda h, \partial_\mu(\phi + \lambda h)) = \frac{\delta \mathscr{L}_\lambda}{\delta \psi} h + \frac{\delta \mathscr{L}_\lambda}{\delta(\partial^\mu \psi)} \partial_\mu \phi h'$$

$$\psi = (\phi + \lambda h), \qquad h' = \frac{dh}{d\phi} .$$

[1] $\mathscr{L}(\phi, \partial_\mu \phi)$ is supposed to have a formal power series expansion in ϕ, $\partial_\mu \phi$ around $\phi = \partial_\mu \phi = 0$.

[2] After correct mass and wave function renormalization.

We set:

$$f = \frac{h}{1+\lambda h'}$$

$$\begin{aligned}\frac{d\mathscr{L}_\lambda}{d\lambda} &= \frac{\delta\mathscr{L}_\lambda}{\delta\phi} f + \frac{\delta\mathscr{L}_\lambda}{\delta(\partial^\mu\phi)}\partial^\mu f \\ &= f\left(\frac{\delta\mathscr{L}_\lambda}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathscr{L}_\lambda}{\delta(\partial^\mu\phi)}\right)\right) + \partial_\mu\left(\frac{\delta\mathscr{L}_\lambda}{\delta(\partial^\mu\phi)} f\right).\end{aligned} \tag{2}$$

One expects that the first term in (2) – f multiplied with the equation of motion – does not contribute on the mass shell. The second term should drop out also off mass shell in the adiabatic limit because of the total derivative in front.

We model the inductive construction à la Epstein and Glaser [5, 6] of the time ordered products closely after the naive argument given above.

Assume that all time ordered products with less than n points have already been defined. We include in our induction hypothesis the following assumptions:

$$\begin{aligned}\frac{d}{d\lambda}\overset{(-)}{T}_\lambda(x_1,\ldots,x_m) &= \frac{d}{d\lambda}\overset{(-)}{T}_{\lambda,m} \\ &= \sum_{i=1}^{m}\Big\{(\partial_\mu^{x_i}\overset{(-)}{T}{}^{1,i,\mu}_{\lambda,m} + \overset{(-)}{T}{}^{2,i}_{\lambda,m} \\ &\quad - \sum_{j\neq i}\delta(x_i-x_j)\overset{(-)}{T}{}^{2,j}_{\lambda,m-1}(x_1\ldots x_{i-1}x_{i+1}\ldots x_m)\Big\} \\ &\qquad (m<n).\end{aligned} \tag{3}$$

$\overset{(-)}{T}$ denotes alternatively chronological or antichronological ordering.

In order to avoid notational complications we interpret each operator term in Eq. (3) and in all following equations as an shorthand notation for the set of all Feynman resp. Dirac graphs (which are not vacuum to vacuum) corresponding to the respective operator expression. We subsume therewith the construction and the induction hypothesis for the submonomials of the interaction Lagrangian $\bar{\mathscr{L}}_\lambda$.

$$T^{1,i,\mu}_{\lambda,m} = T\left(:\bar{\mathscr{L}}_\lambda(x_1):\ldots:f\frac{\delta\mathscr{L}_\lambda}{\delta(\partial^\mu\phi)}(x_i):\ldots:\bar{\mathscr{L}}'_\lambda(x_m):\right), \tag{4}$$

$$T^{2,i}_{\lambda,m} = T(:\bar{\mathscr{L}}_\lambda(x_1):\ldots:f\,\bar{\mathscr{L}}_{\lambda,\phi}(x_i):\ldots:\bar{\mathscr{L}}_\lambda(x_m):),$$

$$\bar{\mathscr{L}}_{\lambda,\phi} = \frac{\delta\mathscr{L}_\lambda}{\delta\phi} - \partial_\mu\frac{\delta\mathscr{L}_\lambda}{\delta(\partial^\mu\phi)} + (\Box+\bar{m}^2)\phi\,. \tag{5}$$

$T_{\lambda,m}$, $T^{1,i,\mu}_{\lambda,m}$, and $T^{2,i}_{\lambda,m}$ can be represented as formal power series in λ.

$$T_{\lambda,m} = \sum_n T^{(n)}_{\lambda,m} \lambda^n$$

$$\frac{d}{d\lambda} T_{\lambda,m} = \sum_n T^{(n)}_{\lambda,m} \lambda^{n-1} n .$$

$T^{(0)}_{\lambda,m}$ is equal T_m of Eq. (1). Equation (3) has to be understood as equality in the sense of formal power series. Furthermore we assume that the c-number kernels, which appear in a Wick product expansion of $T_{\lambda,m}$, $T^{i,j,(\mu)}_{\lambda,m}$ have all properties necessary for establishing the adiabatic limits

$$\int T^{(i,j)(\mu)}_{\lambda,m}(x_1, \ldots x_m)\, g(x_1 \ldots g(x_m)\, dx_1 \ldots dx_m,\ g \to \text{konst.}$$

Equation (3) implies $dS_\lambda/d\lambda = 0$: the contributions to $dS_\lambda/d\lambda$ from $T^{2,i}_{\lambda,m}$ are cancelled by those from

$$-\sum_{j \neq i} \delta(x_i - x_j)\, T^{2,i}_{\lambda,m} .$$

The contribution from $\partial^{x_i}_\mu T^{1,i,\mu}_{m,\lambda}$ drops out in the adiabatic limit.

To see that Eq. (3) is in fact a reasonable assumption, one has to verify that it is consistent for the two point function

$$T\left(:\frac{d\mathscr{L}_\lambda}{d\lambda}(x_1): :\bar{\mathscr{P}}_\lambda(x_i):\right).$$

The inductive procedure explained below will than guarantee the same for time ordered products with more than two points.

The time ordered two point product $T\left(:\frac{d\mathscr{L}}{d\lambda}: :\bar{\mathscr{P}}_\lambda:\right)$ can be represented as the sum of a normal two point product (which is well defined) and a retarded commutator containing the ambiguities connected with renormalisation. That is, the starting point of the construction are the well defined two point objects $\left(:\frac{d\mathscr{L}_\lambda}{d\lambda}(x_1): :\bar{\mathscr{P}}_\lambda(x_2):\right)$.

From $:\frac{d\mathscr{L}_\lambda}{d\lambda}:$ we can split off the null operator $:f(\vec{\Box} + \bar{m}^2)\varphi:$

$$\frac{d\mathscr{L}_\lambda}{d\lambda} = :f\,\bar{\mathscr{P}}_{\lambda;\varphi}: + :\partial_\mu\left(\frac{\delta\mathscr{L}_\lambda}{\delta(\partial_\mu\varphi)} f\right): - :f(\vec{\Box} + \bar{m}^2)\varphi:$$

and define afterwards somehow $T(:f\,\bar{\mathscr{P}}_{\lambda;\varphi}(x_1): :\bar{\mathscr{P}}_\lambda(x_2):)$ and

$$T\left(:\partial_\mu \frac{(\delta\mathscr{L}_\lambda f)}{\delta(\partial_\mu\varphi)}(x_1): :\bar{\mathscr{P}}_\lambda(x_2):\right) \stackrel{!}{=} \partial_\mu T\left(:\frac{\delta\mathscr{L}_\lambda f}{\delta(\partial_\mu\varphi)}: :\bar{\mathscr{P}}_\lambda:\right)$$

with the prescription for the derivative as expressed in the last equation.

We arrive at Eq. (3) by exploiting the ambiguity in the definition of the retarded commutator, which allows us to add a term

$$-\delta(x_1 - x_2) : f \mathscr{P}_{\lambda;\varphi}(x_2) : .$$

This term can also be looked at as coming from the treegraph contribution $: f(-\vec{\Box}_{x_1} - \bar{m}^2) \Delta_f(x_1 - x_2) \mathscr{P}_{;\varphi}(x_2) :$, Δ_f being the free Feynman propagator.

In order to construct the time ordered n-point product $T_\lambda(x_1 \dots x_n)$ (along Epstein and Glaser's lines) we consider first

$$D^{(n)}_\lambda(x_1 \dots x_{n-1}; x_n) = \sum_{\substack{J \cup J' = \{x_1 \dots x_{n-1}\} \\ J \cap J' = \emptyset \\ J \neq \emptyset}} [T_\lambda(J', x_n), \bar{T}_\lambda(J)] \, (-1)^{|J|}$$

Applying the induction hypothesis (3) to the factors T_λ, $\bar{T}_\lambda$ (with less than n points) one obtains

$$\begin{aligned} \frac{d}{d\lambda} D_{\lambda,n} = \sum_{k=1}^{n} \Big\{ & \partial^{x_k}_\mu D^{1,\mu,k}_{\lambda,n} + D^{2,k}_{\lambda,n} - \sum_{j \neq k} \delta(x_j - x_k) \\ & \cdot D^{2,j}_{\lambda,n-1}(x_1 x_{k-1} x_{k+1}; x_n) \Big\} . \end{aligned} \tag{6}$$

$D^{1,\mu,k}_{\lambda,n}$, $D^{2,k}_{\lambda,n}$ are similarly defined as $T^{1,\mu,k}_{\lambda,n}$, $T^{2,k}_{\lambda,n}$, that is, the operator $: \bar{\mathscr{P}}'_\lambda(x_k) :$ of $D_{\lambda,n}$ is substituted by $: f \dfrac{\delta \mathscr{L}_\lambda}{\delta(\partial^\mu \phi)}(x_k) :$ and $: f \mathscr{P}_{\lambda,\phi}(x_k) :$ respectively.

A proper definition of the time ordered n-point product is obtained by "splitting" $D_{\lambda,n}$, that is, one looks for a "retarded" product $R_{\lambda,n}(x_1 \dots ; x_n)$, whose support is contained in $V^- = \{x = \{x_1 \dots, x_n\} R^{4n}, (x_i - x_n)^2 \geqq 0, x^0_i - x^0_n > 0, i < n\}$ such that the support of $(D_{\lambda,n} + R_{\lambda,n})$ is in $V^+ = -V^-$. (The support of D is contained in $V^+ \cup V^-$.)

Every term on the right side of Eq. (6) has according to the induction hypothesis the same structure as D_λ. This fact enables us to split every term individually. Afterwards we integrate formally with respect to λ $\left(\lambda^n \to \dfrac{\lambda^{n+1}}{n+1}\right)$ and add $R_{n,\lambda=0}$. This procedure renders an acceptable definition of $R_\lambda(x_1 \dots x_{n-1}; x_n)$ and thereby also a proper definition of the time ordered product

$$\begin{aligned} T_\lambda(x_1 \dots x_n) &= R_\lambda(x_1 \dots x_{n-1}; x_n) - \sum_{\substack{J \cup J' = \{x_1 \dots x_{n-1}\} \\ J \cap J' = \emptyset \\ J \neq \emptyset}} \bar{T}_\lambda(J) \, T_\lambda(J', x_n) \, (-1)^{(J)} \\ &= R_{\lambda,n} - R'_{\lambda,n} \end{aligned}$$

By a computation analogous to the computation of $\dfrac{d}{d\lambda} D_{\lambda,n}$ [Eq. (6)] one can easily verify that $R'_{\lambda,n}$ satisfies by itself the induction hypothesis.

We have to build in all splittings the correct mass and wave function renormalisations in order to guarantee the existence of the adiabatic limit. The term

$$\sum_{j \neq k} \delta(x_k - x_j) D^{2,k}_{\lambda,n-1}$$

of Eq. (6) is handled in the same manner as $D^{2,k}_{\lambda,n-1}$ in the preceding induction step (the δ-functions are unessential). $D_{n,\lambda=0}$ is treated according to some ad hoc prescription as it was fixed after Eq. (1). Concerning the term $\partial^{x_k}_\mu D^{1,k,\mu}_{n,\lambda}$ we first split $D^{1,k,\mu}_{n,\lambda}$ and apply only afterwards the derivative $\partial^{x_k}_\mu$, that is, we leave the derivative outside the T-product. Following these prescriptions we reproduce the induction hypothesis for $T_{\lambda,n}$. One should note that apart from the restrictions for the selfenergy kernels one can choose the splittings of $D^{1,k,\mu}_{\lambda,n}$, $D^{2,k}_{\lambda,n}$ arbitrarily.

Acknowledgment. I am indebted to Professor P. P. Divakaran for a discussion, which initiated this work.

References

1. Chisholm, J. S.: Nucl. Phys. **26**, 469 (1961)
   Kamefuchi, S., O'Raifeartaigh, L., Salam, A.: Nucl. Phys. **28**, 529 (1961)
2. Divakaran, P. P.: Nucl. Phys. **42**, 235 (1963)
3. Lam, Yuk-Ming P.: Phys. Rev. D**7**, No. 10, 2943 (1973). See also Lam, Yuk-Ming P., Schroer, B.: Phys. Rev. D**8**, 657 (1973)
4. Lam, Yuk-Ming P.: Phys. Rev. D**6**, 2145 (1972)
5. Epstein, H., Glaser, V.: The role of locality in perturbation theory. CERN preprint TH 1400 (1971)
6. Epstein, H., Glaser, V.: Adiabatic limit in perturbation theory. CERN preprint TH 1344 (1971)

Communicated by K. Symanzik

R. Flume
II. Institut für Theoretische Physik
der Universität Hamburg
D-2000 Hamburg 52
Bahrenfelder Chaussee 169
Federal Republic of Germany