Commun. math. Phys. 36, 171—184 (1974)

Block-Spins Interactions in the Ising Model

G. Gallavotti and H. J. F. Knops

Instituut voor Theoretische Fysica, Nijmegen, The Netherlands

Received January 18, 1974

Abstract. We investigate the interactions between block-spins in the large magnetic field region for the nearest neighbour, d-dimensional, Ising model.

1. Introduction

We try to analyse a number of properties of an Ising model in the attempt to make some rigorous statements about the theories connected with the renormalization group approach to critical phenomena [1].

We cannot make any statements about the critical point; however we can say something away from it.

One could, with reasons, argue that this region is not interesting since there is "nothing" to predict. Nevertheless it seems to us of interest to try to put "known things" in a form which could suggest some precise mathematical conjectures about the critical region.

We show that, if the system is away from the critical point, some consequences of renormalization group arguments can be reduced to very general theorems of probability.

Although we point out, in the final comments, some features which might be of interest in the critical region we do not go into an analysis of the relation between our work and the renormalization group approach [1].

In Section 2 we describe some notations and the main result. Since, at first sight, our theorem looks rather obscure, we clarify it with a series of remarks.

In Section 3 we describe the main tools used to prove our theorem. The technique is basically due to Ruelle [2] who worked it out for the continoum case; it was extended to the lattice case in [3] (but the combinatorics was wrong; the right combinatorial algebra can be found, for instance, in [4]).

The main theorem is proven in Section 4 using a technique already employed in [5] for proving some central limit theorems and extended in [4]: we use the notations and results proven in [4].

In Section 5 we make some final remarks, most of which non rigorous; in particular we discuss a linear transformation which falls into the class of the "generalized renormalization transformations" [6] and which, in the gaussian model, reduces to the renormalization transformation considered in [7].

2. Reduced Lattices. Results

Let Z^d be a d-dimensional square lattice. On each site sits a spin $\sigma = \pm 1$. The formal hamiltonian is:

$$H(\boldsymbol{\sigma}) = -\beta J \sum_{\langle x,y \rangle} \sigma_x \sigma_y - \beta h \sum_x \sigma_x , \tag{2.1}$$

or, in the lattice gas language, if $n_x = \dfrac{\sigma_x + 1}{2}$ and $\mu = h - 2Jd$:

$$H(\boldsymbol{n}) = -\Delta J \beta \sum_{\langle x,y \rangle} n_x n_y - \beta \mu \sum_x n_x . \tag{2.2}$$

The sums in the above formulae run over the nearest neighbour couples. We shall fix $4J = 1$.

Let L be an integer ($L = 1, 2, 3, \ldots$); we break up Z^d into disjoint boxes containing L^d points and in such a way that there is one box with the lower corner at the origin of Z^d. We label in the natural way the centers of the boxes with an index $x \in Z^d$: the lattice of the labels will be called a "reduced lattice" (it is fixed and L-independent).

If Λ is a subset of the reduced lattice we denote $L \cdot \Lambda$ the subset of the original lattice whose points fall into Λ in the "contraction" to the reduced lattice (in particular, $L \cdot \{x\}$ will be the set of the sites in the box with label x).

Conversely, if R is a subset of the original lattice we call $\frac{1}{L} \tilde{\cdot} R$ the set of the points of the reduced lattice representing those boxes that have a non-empty intersection with R. Under the contraction many points of R may go into the same point of $\frac{1}{L} \tilde{\cdot} R$; if we assign to each point of $\frac{1}{L} \tilde{\cdot} R$ a multiplicity equal to the number of points contracted in it, we obtain what we shall denote as $\frac{1}{L} \cdot R$.

On each site of the reduced lattice we shall put a random variable. We proceed as follows: let $M_{x,L}$ be the total magnetization of the box $L \cdot \{x\}$ or, alternatively, let $N_{x,L}$ be the number of occupied sites in $L \cdot \{x\}$ (if we deal with the lattice gas language). Then define the block-spin variables (see Kadanoff in [1]):

$$v_x = \frac{M_{x,L} - \langle M_{x,L} \rangle}{L^{d\delta}} = 2\frac{N_{x,L} - \langle N_{x,L} \rangle}{L^{d\delta}}, \tag{2.3}$$

where $\delta = \frac{1}{2}$ (except in Section 5) and the averages $\langle \cdot \rangle$ are taken with respect to the infinite volume Gibbs measure associated with (2.1). The choice $\delta = \frac{1}{2}$ is obvious if we are dealing with a system which is not at the critical point because we expect $M_{x,L}$ to have normal dispersion.

We now fix a box Λ in the reduced lattice and look at the joint probability (with respect to the infinite volume Gibbs measure associated with (2.1)) of the v-variables in Λ: $G_\Lambda^{(L)}(v_1, \ldots, v_{|\Lambda|})$; we relabel the points of Λ from 1 to $|\Lambda|$ for simplicity of notation.

The question we ask is whether this probability distribution will have a limit as $L \to \infty$.

To describe our results let us introduce some more notation.

Let us consider a set T of points lying on the vertices of a unit cell of Z^d. We allow multiple occupancy of the vertices. So, if $\xi_1, \ldots, \xi_{2^d}$ are the 2^d vertices of the unit cell, the set T will be specified by a sequence $\{n_{\xi_i}\}_1^{2^d}$ where $n_\xi = 0, 1, 2, \ldots < \infty$.

Let $\tilde{T}$ the set of the occupied points of T: $\tilde{T} = \{\xi \mid n_\xi \geqq 1\}$. We define the dimension $\sigma(T)$ of T by considering the set $L \cdot \tilde{T}$ (in the notation considered above): this set consists in a union of boxes with sides L having at least one point in common, but in general a hypersurface; $\sigma(T)$ is the dimensionality of the hypersurface common to all the above boxes. So if $d = 3$ and $\tilde{T}$ contains just one point, $\sigma(T) = 3$; if $\tilde{T}$ contains two points ξ, η and $|\xi - \eta| = 1$ then $\sigma(T) = 2$; if $|\xi - \eta| = \sqrt{3}$ then $\sigma(\mathrm{T}) = 0$.

Define also $|T| = \sum_\xi n_\xi$. Let

$$\hat{G}_\Lambda^{(L)}(\omega_1, \ldots, \omega_{|\Lambda|}), \tag{2.4}$$

be the Fourier transform of $G_\Lambda^{(L)}$.

Let $\chi(v_1, \ldots, v_{|\Lambda|})$ be a smooth test function with Fourier transform vanishing when one of the arguments exceeds a finite $M_\chi > 0$ (i.e. $\hat{\chi}$ has compact support).

Consider the expectation:

$$\begin{aligned} F(\chi; L) &= \int G_\Lambda^{(L)}(v_1, \ldots, v_{|\Lambda|})\, \chi(v_1, \ldots, v_{|\Lambda|})\, dv_1 \ldots dv_{|\Lambda|} \\ &= \int \hat{G}_\Lambda^{(L)}(\omega_1, \ldots, \omega_{|\Lambda|})\, \hat{\chi}(\omega_1, \ldots, \omega_{|\Lambda|}) \frac{d\omega_1}{2\pi} \cdots \frac{d\omega_{|\Lambda|}}{2\pi}, \end{aligned} \tag{2.5}$$

(of course the integral with respect to the v-variables is a sum since $G_\Lambda^{(L)}$ is a superposition of delta functions).

Finally, if T is a subset of the lattice Z^d with multiple occupation of the sites, $T=\{n_\xi\}_{\xi\in Z^d}$, $n_\xi \geqq 0$, $|T|=\sum_\xi n_\xi<\infty$, and if ω_ξ are real numbers, we put:

$$\omega(T)=\prod_\xi \omega_\xi^{n_\xi}. \tag{2.6}$$

The following theorem holds:

Theorem. *Given $\beta>0$, assume that μ (cfr. (2.2)) is negative enough (i.e. $\mu<\mu_0(\beta)$ with $\mu_0(\beta)$ defined after formula (3.6) below).*

Let Λ be a finite box in the reduced lattice; then, if k is a fixed integer:

$$\begin{aligned}\hat{G}_\Lambda^{(L)}(\omega_1,\dots,\omega_{|\Lambda|})=\Big(\exp-\sum_{1<|T|\leqq k}^{*} L^{\varepsilon(T)}J_T^{(L)}\omega(T)\Big)\\ \cdot\left(1+O(M^{k+1}L^{-(k-1)d/2}\right),\end{aligned} \tag{2.7}$$

where the sum runs over the sets T with $\tilde{T}$ lying on a unit cell; $M=\max_i |\omega_i|$; furthermore $\varepsilon(T)=\sigma(T)-|T|d/2$ and $\varepsilon(T)\leqq 0$ and:

$$J_T^{(L)}=J_T^{(\infty)}(1+O(L^{-1}))\,;\qquad J_{\xi\xi}^{(\infty)}>0\,, \tag{2.8}$$

where $J_T^{(L)}$ is translation invariant in T, is Λ-independent and $J_T^{(\infty)}$ is an analytic function of $\mu<\mu_0(\beta)$.

This is our main result and, to clarify it, the following comments are justified:

1) The theorem can only be proved in the low density region or, in the magnetic language, for large magnetic field.

Formula (2.7) is interesting because it shows how the averages of the type (2.5) can be computed, to order $O(M^{k+1}L^{-(k-1)d/2})$, from an effective hamiltonian in the "ω-spins" which couples only ω-spins of the box Λ which are not separated by a distance larger than $d^{\frac{1}{2}}$ (and includes many body interactions up to order k). From the proof it will appear that the interactions between ω-spins at distances larger than $d^{\frac{1}{2}}$ contribute only to order $O(\exp-\varkappa L)$, where $\varkappa>0$ is an estimate of the correlation length ($\varkappa=\beta(\mu-\mu_0(\beta))$, see the theorem of the next section).

The spin interpretation of the ω-spins, which are the Fourier conjugates of the block spins v, is purely formal: for instance the coupling constants $J_T^{(L)}$ with $|T|$ odd are imaginary.

2) The leading term in (2.7) is the "self-interaction" term, $|T|=2$, $\sigma(T)=d$, $\varepsilon(T)=0$. It tells us that in the limit $L\to\infty$ the system of random variables $v_1,\dots,v_{|\Lambda|}$ becomes a set of decoupled gaussian random variables each of which has a distribution:

$$(4\pi J_{\xi\xi}^{(\infty)})^{-\frac{1}{2}}\exp-v^2/4J_{\xi\xi}^{(\infty)}\,. \tag{2.9}$$

This is to be expected as a consequence of the central limit theorems which hold under the assumptions of our theorem (see references of [4] and [5]).

3) When $d=3$, the next correction is the "nearest neighbour" interaction, $|T|=2$, $\sigma(T)=d-1$, $\varepsilon(T)=-1$.

To order $O(M_\chi^2 L^{-\frac{3}{2}})$ the averages in (2.5) can be calculated from an effective hamiltonian which has a gaussian form:

$$H^G(\omega_1, \ldots, \omega_{|\Lambda|}) = \sum_{\xi \in \Lambda} J_{\xi\xi}^{(L)} \omega_\xi^2 - \frac{1}{L} \sum_{\langle \xi, \eta \rangle} J_{\xi\eta}^{(\infty)} \omega_\xi \omega_\eta . \tag{2.10}$$

The interaction in the original ν-variables can be obtained by a Fourier transform which, in the present case, can be easily performed and yields again a quadratic form which is generated from the inverse of the interaction matrix that appears in (2.10). The inversion leads to an interaction which has, in leading order, again a self interaction term followed by a nearest neighbour interaction of order L^{-1}.

However one should note that this inversion generates also couplings between ν-spins that are not nearest neighbours, these couplings fall like higher powers of L as $L \to \infty$. This is also the case for spins with distances larger than $\sqrt{d}$ which, in terms of the ω-spins, are only coupled by exponentially decaying couplings (as $L \to \infty$). In this sense the picture in the ω-spins is simpler than that in the ν-spins.

4) The "many-body" interactions appearing for larger values of k are not easily transformed into interactions between ν-spins.

5) The finite support of $\hat{\chi}$ is necessary to make formula (2.7) applicable. This has a physical interpretation: to use (2.7) for large ω's would mean that we could test the fine details of the distribution $\hat{G}_\Lambda^{(L)}(\omega_1, \ldots, \omega_{|\Lambda|})$ using (2.7) as a good approximation, which cannot be true since it clearly corresponds to a smoothed version of the original distribution which is a sum of delta functions.

3. Tools

The basic tool for our theorem is the so called algebraic method for the virial expansion [2]. We must recall a few simple and boring definitions.

We shall think of a spin configuration in a box Γ as a sequence $\{n_\xi\}_{\xi \in \Gamma}$, $n_\xi = 0, 1$ (the zero corresponds to $\sigma_\xi = -1$ and the one to $\sigma_\xi = +1$).

We shall also consider "non-physical" or "generalized" configurations $X = \{n_\xi\}_{\xi \in \Gamma}$ in which $n_\xi = 0, 1, 2, \ldots$. A configuration is "physical" if and only if $n_\xi = 0,1$ $\forall \xi \in \Gamma$.

Let $X=\{n_\xi\}_{\xi\in\Gamma}$ be a generalized configuration and put:

$$
\begin{aligned}
|X| &= \sum_\xi n_\xi && \text{“number of points of } X\text{”}, \\
\tilde{X} &= \{\xi \mid n_\xi > 1\} && \text{“basis of } X\text{”}, \\
X! &= \prod_\xi n_\xi ! && \text{“multiplicity function”}.
\end{aligned}
\tag{3.1}
$$

Clearly X is a physical configuration if and only if $X=\tilde{X}$ (or, if and only if $X!=1$). We only consider $X's$ such that $|X|<\infty$.

Define, for a physical configuration:

$$\varphi_0(X)=\exp\beta\sum_{\langle\xi,\eta\rangle} n_\xi n_\eta \qquad (\text{if } X!=1), \tag{3.2}$$

and

$$\varphi_0(X)=0 \qquad \text{if } X!>1. \tag{3.3}$$

The following estimates form the basis to our analysis:

Theorem. *Given $\beta>0$ there is a translation invariant function $\varphi_0^T(X)$ defined on the finite generalized configurations, such that:*

i) $\exists\,\bar{z}(\beta)>0$, such that, if z_ξ is an arbitrary function with the property $\tilde{z}=\max|z_\xi|<\bar{z}(\beta)$ and if we put $z(X)=\prod_\xi z_\xi^{n_\xi}$, when $X=\{n_\xi\}$, the following is true:

$$\sum_{\tilde{X}\subset\Gamma}\varphi_0(X)\,z(X)=\exp\sum_{\tilde{X}\subset\Gamma}\frac{\varphi_0^T(X)}{X!}\,z(X), \tag{3.4}$$

for arbitrary finite $\Gamma\subset z^d$; the sum is over all the X such that $\tilde{X}\subset\Gamma$.

ii) *For all $k>0$:*

$$\sum_{\substack{\tilde{X}\ni 0\\ |X|=k}}\frac{|\varphi_0^T(X)|}{X!}\leqq(\bar{z}(\beta))^{-k}. \tag{3.5}$$

iii) *For all $k>0$ there is $\delta_k(\beta)$ such that, if $|\xi-\eta|$ denotes the distance between the two lattice points ξ and η, we have:*

$$\sum_{\tilde{X}\ni\xi,\eta}\frac{|\varphi_0^T(X)|}{X!}\,|z(X)|\,|X|^k\leqq\delta_k(\beta)\,|\xi-\eta|^k\left(\frac{\tilde{z}}{\bar{z}}\right)^{|\xi-\eta|}. \tag{3.6}$$

We define $\mu_0(\beta)=\beta^{-1}\log\bar{z}(\beta)$, $\varkappa=-\log\tilde{z}/\bar{z}$.

With some pain the wilful reader will recognize in the above statements the basic ideas and estimates of the Mayer expansion [2].

Here we just remark that the familiar Ursell functions are:

$$U(X)=\sum_Y\frac{\varphi_0^T(X\cup Y)}{Y!}\,z^{|Y|}, \tag{3.7}$$

where z is the activity $z=\exp\beta\mu$.

The Mayer expansion can be easily obtained from (3.4). One finds, under the assumptions of the theorem:

$$P(\beta, z)=\lim_{\Gamma\to\infty} |\Gamma|^{-1}\log\sum_{X\subset\Gamma}\varphi_0(X)\, z^{|X|}=\sum_{\tilde{X}\ni 0}\frac{\varphi_0^T(x)}{X!}\frac{z^{|X|}}{|\tilde{X}|}. \tag{3.8}$$

The proof of the above theorem can be found in [4] (appendix).

4. Proof of the Main Theorem

Consider the Fourier transform of the distribution $G_\Lambda^{(L)}(v_L, \ldots, v_{|\Lambda|})$ introduced in Section 2 (cfr. (2.4)).

The transform can be conveniently written in terms of the original spins if one introduces the function:

$$\lambda_\xi=\exp 2i\omega_{x(\xi)}L^{-d/2} \tag{4.1}$$

where $\xi\in Z^d$ and $x(\xi)$ is the label of the box in which ξ is contained $\left(\text{in the language of Section 2, } x(\xi)=\frac{1}{L}\cdot\{\xi\}\right)$; the ω's are defined to be zero if $x(\xi)\notin\Lambda$.

Consistently with the notation of Section 2 we put:

$$\lambda(X)=\prod_\xi \lambda_\xi^{n_\xi} \quad \text{if} \quad X=\{n_\xi\}. \tag{4.2}$$

It is then easy to see, using (3.4), that, if we consider the system as enclosed in a finite box Γ, the distribution function $\hat{G}_\Lambda^{(L)(\Gamma)}$ corresponding to the definition (2.4) tends, in the thermodynamic limit $\Gamma\to\infty$ to

$$\hat{G}_\Lambda^{(L)}(\omega_1, \ldots, \omega_{|\Lambda|})=\lim_{\Gamma\to\infty}\hat{G}_\Lambda^{(L)(\Gamma)}(\omega_1, \ldots, \omega_{|\Lambda|}) \tag{4.3}$$

$$=\lim_{\Gamma\to\infty}\left[\exp\left(-2i\sum_{s=1}^{|\Lambda|}\frac{\omega_s\langle N_{s,L}\rangle_\Gamma}{L^{d/2}}\right)\right]\cdot\left[\frac{\exp\sum_{\tilde{X}\subset\Gamma} z^{|X|}\lambda(X)\frac{\varphi_0^T(X)}{X!}}{\exp\sum_{\tilde{X}\subset\Gamma} z^{|X|}\frac{\varphi_0^T(X)}{X!}}\right] \tag{4.4}$$

$$=\exp\left(\frac{-2i\langle N_{s,L}\rangle}{L^{d/2}}\sum_{s=1}^{|\Lambda|}\omega_s+\sum_X\frac{z^{|X|}\varphi_0^T(X)}{X!}(\lambda(X)-1)\right), \tag{4.5}$$

where $z=\exp\beta\mu$.

The last sum in (4.5) is finite by virtue of the relation $\lambda(X)\equiv 1$ if $\tilde{X}\cap L\cdot\Lambda=\emptyset$, furthermore use is made of the fact that $\langle N_{s,L}\rangle$ is s-independent [9].

Notice that, if X_k denotes the part of X with base in the box labeled k:

$$\lambda(X) = \exp 2i \sum_{k=1}^{|A|} \omega_k |X_k| \, L^{-d/2} . \tag{4.6}$$

Notice also that (4.4) implies, in the particular case $|A| = 1$, that:

$$\begin{aligned} 0 \equiv \frac{d}{id\omega} \left\langle e^{2i\omega \frac{N_{1,L} - \langle N_{1,L} \rangle}{L^{d/2}}} \right\rangle \Bigg|_{\omega=0} &\equiv \frac{1}{i} \frac{d}{d\omega} \hat{G}^{(L)}_{\{1\}}(\omega) \Bigg|_{\omega=0} \\ = -2 \frac{\langle N_{1,L} \rangle}{L^{d/2}} + \frac{2}{L^{d/2}} \sum_X z^{|X|} \frac{|X_1|}{X!} \varphi_0^T(X) , \end{aligned} \tag{4.7}$$

so that:

$$\langle N_{1,L} \rangle = \sum_X z^{|X|} \frac{|X_1|}{X!} \varphi_0^T(X) . \tag{4.8}$$

This last observation tells us that, if we expand $(\lambda(X) - 1)$ in (4.5) in powers of the ω's, not only the zero order term is absent but also the first order term cancels out.

Expanding $\lambda(X) - 1$ in powers of the ω's up to order k and writing the remainder in the Schlömilch form, we find that the argument of the exponential in (4.5) becomes:

$$\begin{aligned} &\sum_X z^{|X|} \frac{\varphi_0^T(X)}{X!} \sum_{h=2}^{k} \frac{1}{h!} \left(2i \sum_{s=1}^{|A|} \frac{\omega_s |X_s|}{L^{d/2}} \right)^h \\ &\qquad + \sum_X z^{|X|} \frac{\varphi_0^T(X)}{X!} \left(2i \sum_{s=1}^{|A|} \frac{\omega_s |X_s|}{L^{d/2}} \right)^{k+1} \\ &\qquad \cdot \frac{\left(\cos 2\theta' \sum_{k=1}^{|A|} \frac{\omega_k |X_k|}{L^{d/2}} + i \sin 2\theta'' \sum_{k=1}^{|A|} \frac{\omega_k |X_k|}{L^{d/2}} \right)}{(k+1)!} , \end{aligned} \tag{4.9}$$

where θ', θ'' are suitable real numbers such that $|\theta'|, |\theta''| \leqq 1$.

We first estimate the remainder in (4.9). If we put $M = \max |\omega_i|$, and use the estimate (3.5), we see that it does not exceed (in absolute value):

$$\begin{aligned} &2 \sum_{\tilde{X} \cap L \cdot A \neq \emptyset} z^{|X|} \frac{|\varphi_0^T(X)|}{X!} \frac{(2M)^{k+1} \, |X|^{k+1}}{(k+1)! \, L^{(d/2)(k+1)}} \\ &\leqq \frac{2(2M)^{k+1} \, |A| \, L^d}{(k+1)! \, L^{d(k+1)/2}} \sum_{\tilde{X} \ni 0} \frac{z^{|X|} |\varphi_0^T(X)| \, |X|^{k+1}}{X!} \\ &\leqq 2 \left(\sum_{p=1}^{\infty} p^{k+1} \left(\frac{|z|}{\tilde{z}} \right)^p \right) \frac{(2M)^{k+1} \, |A| \, L^{-(k-1)d/2}}{(k+1)!} = O(M^{k+1} L^{-(k-1)d/2}) . \end{aligned} \tag{4.10}$$

The leading term can be written as:

$$\sum_{h=2}^{k} \frac{(2i)^h}{h!} \sum_{s_1,\ldots,s_h}^{1,|\Lambda|} \omega_{s_1} \ldots \omega_{s_h} \sum_X |X_{s_1}| \ldots |X_{s_h}| \frac{\varphi_0^T(X)}{X!} z^{|X|} . \tag{4.11}$$

The set $s_1, \ldots, s_h$ is a set of points on the reduced lattice box Λ with possible repetitions. We denote, again, by $n_1, n_2, \ldots, n_{|\Lambda|}$ the number of times the points $1, 2, \ldots, |\Lambda|$ appear in $(s_1, \ldots, s_h)$ and define $S = \{n_i\}_1^{|\Lambda|}$, $|S| = \sum_i n_i = h$ and $S! = \prod_i n_i!$.

It is easy to check that:

$$\sum_{h=2}^{k} \frac{1}{h!} \sum_{s_1,\ldots,s_h}^{1,|\Lambda|} \cdots = \sum_{\substack{s \\ 2 \leqq |s| \leqq k}} \frac{1}{s!} \cdots . \tag{4.12}$$

So (4.11) can be transformed into:

$$\sum_{\substack{\tilde{s} \subset \Lambda \\ 2 \leqq |s| \leqq k}} \frac{\omega(s)}{s!} \frac{(2i)^{|s|}}{L^{(d/2) \cdot |s|}} \sum_X z^{|X|} \frac{\varphi_0^T(X)}{X!} |X| (s) , \tag{4.13}$$

where $\omega(s) = \prod_{i=1}^{|\Lambda|} \omega_i^{n_i}$ and $|X| (S) = \prod_{i=1}^{|\Lambda|} |X_i|^{n_i}$.

Suppose that S contains two points which are not in the same unit cell of the reduced lattice, then the corresponding boxes in the original lattice are least at a distance L. The remark that $|X| (S)$ vanishes unless X has at least one point in every box whose label appears in S immediately implies, together with (3.6), that the contribution of the configurations S that are not based on the unit cell is at most of the order

$$O\left(L^k \left(\frac{z}{\bar{z}}\right)^L\right) = O(L^k e^{-\varkappa L}) .$$

So we can restrict our attention to the configurations with base in a unit cell, as stated in the theorem.

Let T be a configuration with base in a unit cell and recall that $\sigma(T)$ is the dimensionality of the hypersurface that is common to all the boxes with label in $\tilde{T}$. Note again that the sum over X in (4.13) has only contributions from configurations X with at least one point in every box belonging to $L \cdot \tilde{T}$ but, in order to give a contribution which is not exponentially small as $L \to \infty$, all the points in X must be close to the common hypersurface. It follows, then, from (3.5), that the coupling constant belonging to the interaction term with $\omega(T)$ has the form:

$$\frac{L^{\sigma(T)}}{L^{d|T|/2}}\left(J_T^{(\infty)}+O(L^{-1})\right). \tag{4.14}$$

The translation invariance, also, follows from the translation invariance of the φ_0^T.

We give a few examples:

$$J_{\xi\xi}^{(\infty)}=2\sum_{\tilde{X}\ni 0}\frac{\varphi_0^T(X)}{X!}\frac{z^{|X|}}{|\tilde{X}|}|X|^2=2\chi_2(\beta,z), \tag{4.15}$$

where $\chi_2(\beta,z)$ is, indeed, the susceptibility, as it follows from (3.8); this implies also that $J_{\xi\xi}^{(\infty)}>0$.

If $S=\xi$ and $S=(\xi,\xi,\ldots,\xi)$, $n_\xi=k\geqq 2$ we find:

$$J_{\xi\ldots}^{(\infty)}=\frac{(2i)^k}{k!}\sum_{\tilde{X}\ni 0}\frac{\varphi_0^T(X)}{X!}\frac{z^{|X|}}{|\tilde{X}|}|X|^k=\frac{(2i)^k}{k!}\left(z\frac{d}{dz}\right)^{k-2}\chi_2(\beta,z). \tag{4.16}$$

If ξ,η are nearest neighbours and $\tilde{S}=(\xi,\eta)$:

$$J_{\xi,\eta}^{(\infty)}=\sum_{p=0}^{\infty}\sum_{X_1,X_2}{}^{*}\frac{\varphi_0^T(X_1\cup X_2)}{X_1!\,X_2!}Z^{|X_1|+|X_2|}\frac{|X_1|}{|\tilde{X}_1|}\cdot|X_2|, \tag{4.17}$$

where the sums have to be interpreted as follows: choose a privileged axis, say the axis ξ_1; consider all the possible couples X_1, X_2 such that X_1 lies in the halfspace $\xi_1\leqq 0$ and X_2 lies in the halfspace $\xi_1>0$; assume that X_1 contains a point with abscissa $-P$ on the axis ξ_1 and sum over all the possible choices of X_1, X_2 and p with the just described restrictions.

It follows from the theorem of Section 3 that $J_{\xi,n}^{(\infty)}$ is analytic in z if $0<z<\bar{z}(\beta)$.

5. Concluding Remarks

There is no reason to believe that the asymptotic formula for $\hat{G}_\Lambda^{(L)}(\omega_1,\ldots,\omega_{|\Lambda|})$ breaks down anywhere in the region $\mu<\mu_c$ where μ_c is the Lee-Yang value of the chemical potential (corresponding to zero magnetic field in the spin language), or, by symmetry, in the region $\mu>\mu_c$.

If $\beta<\beta_c$ one can conceive that the above asymptotic expansions hold even at $\mu=\mu_c$ (or $h=0$). In this last case all the odd terms in the expansion will vanish by symmetry (i.e. $J_T^{(L)}=0$ if T is odd), the expressions will become slightly simpler and all the $J_T^{(L)}$ are real.

Of course the above asymptotic expansions must break down when $\beta \to \beta_c$, $h \to 0$ (in the sense that we cannot expect the error estimates to be uniform in this limit nor the $J_T^{(\infty)}$ to be finite: in fact $J_T^{(L)} \to \infty$ as $L \to \infty$ if $\beta = \beta_c, h = 0$).

We can interpret the divergence as due to a bad choice of δ in (2.3). Actually we expect (by definition of η) that $\langle M_{x,L}^2 \rangle_{h=0,\beta=\beta_c} \simeq L^{d+(2-\eta)}$ where η is the susceptibility critical exponent defined via the pair correlation function:

$$\langle \sigma_0 \sigma_R \rangle_{\beta=\beta_c, h=0} \underset{|R| \to \infty}{\cong} |R|^{2-d-\eta}. \tag{5.1}$$

So it would be more natural to use $\delta = \dfrac{1}{2} + \dfrac{2-\eta}{2d}$ when $h = 0$, $\beta = \beta_c$.

With this choice of δ one can see that the first term in the formal expression for $\hat{G}_\Lambda^{(L)}(\omega_1, \ldots, \omega_{|\Lambda|})$ will be $\exp - \sum_s \omega_s^2 \chi_2(L) L^{\eta-2}$ with $\chi_2(L)$ equal to the susceptibility of a subsystem of Ising spins at the critical point but enclosed in a finite box of size L. Since $\chi_2(L)$ is expected to diverge as $L^{2-\eta}$, we see that this term would give a meaningful result in the limit $L \to \infty$. However we expect that, at $\beta = \beta_c$, $h = 0$, all the terms which do not appear in (2.7) on the grounds that they give exponentially small contributions will be as important as the ω^2 terms; furthermore the relative importance of the terms which appear in (2.7) will no longer be measured by $\varepsilon(T)$.

It seems, however, natural and in some sense necessary that, if δ is properly chosen [1], there exists a family of translation invariant J_T (real and zero for odd T) such that the formal hamiltonian:

$$- \sum_{T \subset \Lambda} J_T \omega(T), \tag{5.2}$$

really defines a random field which describes the limit as $L \to \infty$ of the field which is the "Fourier transform" of the ν-field, at $\beta = \beta_c$, $h = 0$, in Λ.

The random ω-field defined (formally) by (5.2) should generate a ν-field on the reduced lattice which, upon further reduction of the lattice, gives rise to a field ν' identical in distribution to ν.

[1] The right choice of delta would be such that $\langle M_{x,L}^2 \rangle L^{-2d\delta}$ becomes L-independent. This might lead to a L-dependent choice of delta. It seems that two things could go wrong once δ is chosen as above, there might appear a Λ dependence of the J_T's, which is only formally excluded by the Λ independent expressions that we have found in the large field region. It might also happen that some of the J's are infinite so that great care should be used in interpreting (5.2). We are disregarding these possibilities mainly to avoid an excessive complication of an argument which is, anyhow, heuristic.

It is very easy to find, formally, that the reduced field v' will have a Fourier transformed field ω' of the form (5.2) with a new set of "couplings" J'_T related by a linear transformation $\mathscr{L}$ to J:

$$J' = \mathscr{L}(J) . \tag{5.3}$$

The new couplings are easily obtained if one realizes that the Fourier transform of the distribution of the block spins v' for blocks of length $p \cdot L$ can be obtained from the Fourier transform of the original distribution by dividing the reduced lattice in cells with side p and, then, equating the ω's with index belonging to the same cells. This yields:

$$J'_T = \frac{1}{q^{|T|}} \sum_{\frac{1}{p} \cdot R = T} J_R , \tag{5.4}$$

where q is the scale parameter $q = p^{d\delta}$ and where we recall from Section 2 that $\frac{1}{p} \cdot R$ is the set contracted from R (with the multiplicities counted in the appropriate way).

It is easy to see that $|R| \equiv |T|$ and, therefore, the above linear transformation decouples into a denumerable number of independent linear transformations (each of which describes the transformation of the coupling constants for a fixed number of interacting spins).

Furthermore the range of J' does no exceed the range of J. So we can look for fixed points $J = \mathscr{L}(J)$ in which J has only k-body coupling constants (with a fixed k) and finite range. This leads to a finite set of linear equations; the eigenvalues are directly related to δ and η and determine them. It is easy to convince oneself, studying a few simple examples, that the fixed points that are found in this way describe a set of independent gaussian v-spins with the "normal" non critical value of $\delta (\delta = \frac{1}{2})$ or, otherwise, lead to physically unacceptable v-fields.

It seems that the really physically relevant fixed points must have infinite range J's (in this case the equation $J = \mathscr{L}(J)$ although linear seems very complicate).

We note also that it is very well possible to have, at the fixed point, a short range interaction in the v-fields language in spite of the long range character of the ω-fields. This can be explicitely seen in the theory of the v-fields and ω-fields that arise in the context of the Gaussian model[2] [7].

[2] In this model the interaction for the ω-fields is a translation invariant two body interaction with a Fourier transform (in space) which is proportional to the susceptibility $\chi(k)$. In Ref. [7] it is, indeed, found that the renormalisation group equations in terms of χ are linear. Since $\chi(k) \approx k^{-2}$ (for small k) at the critical point, the interaction for the ω-fields is of a long range type even though the original interaction can very well be short ranged.

Coming back to the equation $J = \mathscr{L}(J)$ it appears also that the solution will depend very much on the space in which one decides to solve the equation: there might be several possibilities which are physically acceptable, however the one that corresponds to the Ising model's theory cannot be determined by the fixed point equation alone unless one makes further assumptions.

In other words the linearity of the equation is not a very significant feature: the main problem is to find in which space the equation has to be solved or studied. It can happen that one has some "physical guess" at what what is the space where one should look for the fixed point: however this guess will probably not be in terms of the J's and ω's but in terms of other suitable random fields not linearly related to the J's and ω's: so that the problem will become non linear.

We stress that the above considerations do not seem at all in contrast with the Wilson's ideas on the critical point. The breakthrough work of Bleher and Sinai [9] already gives an example of a situation in which the knowledge of the fixed point of some renormalization transformation is not enough to study the critical point behaviour (the reader familiar with [9] will have noticed that there is a ad hoc modification of the hamiltonian which guarantees that the chosen fixed point is really the one that is relevant for the description of the critical behaviour of the model).

The above remarks certainly lack the rigor of the previous sections but it seems to us that they suggest that a better understanding of the linear renormalization (5.4) would be very helpful. For instance it is not known (or not well known) under which condition on J_T (5.2) really defines a random field, nor it is known when a random field of the type (5.2) has a "Fourier transform" random field (the ν-field). Clearly the "spectrum" of $\mathscr{L}$ in various spaces seems an interesting piece of information.

Acknowledgements. We are indebted to A. Weyland for stimulating discussions.

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Communicated by G. Gallavotti

G. Gallavotti
H. J. F. Knops
Instituut voor Theoretische Fysica
Katholieke Universiteit
Nijmegen, The Netherlands