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On Some Representations of the Anticommutations Relations

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Abstract. We study representations of the canonical anticommutation relations having the form:

$$A(f) = a(Hf) + b^*(Kf)$$

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where $a(f)$, $b^*(f)$ and their adjoints are two basic anticommuting fields in a Fock Space.

A complete determination of the type in terms of $|K| = (K^*K)^{1/2}$ and a sufficient condition for quasi-equivalence are given.

I. Introduction

Let $\mathfrak{E}$ be a complex Hilbert space of test functions, denoted by $f, g, h, \ldots$ To each element f of $\mathfrak{E}$ correspond two bounded operators on a Hilbert space $\mathfrak{F}$, $a(f)$ and $b^*(f)$, depending linearly and continuously on f in the uniform topology of operators. We denote briefly their adjoints by $a^*(f)$ and $b(f)$; therefore, these are semi-linear in f. We impose the relations:

$$\begin{aligned} &[a(f), a(g)]_+ = [b^*(f), b^*(g)]_+ = [a(f), b^*(g)]_+ = [a(f), b(g)]_+ = 0 \\ &[a(f), a^*(g)]_+ = [b(g), b^*(f)]_+ = (f, g) \qquad (1) \\ &f, g \in \mathfrak{E}, \quad [A, B]_+ = AB + BA\,, \end{aligned}$$

and we take for $\mathfrak{F}$ the customary Fock-space associated with these two anticommutating fields. Id est, we have in $\mathfrak{F}$ a vector Ω_0 such that:

$$a(f)\Omega_0 = b(g)\Omega_0 = 0\,, \quad f, g \in \mathfrak{E} \qquad (2)$$

and all the linear combinations of vectors having the form:

$$a^*(f_1) \ldots a^*(f_m)\, b^*(g_1) \ldots b^*(g_n)\Omega_0$$

are a dense set in $\mathfrak{F}$.

Now, if H and K are operators in $\mathfrak{L}(\mathfrak{E})$ which satisfy:

$$H^*H + K^*K = I \qquad (3)$$

we set:

$$A(f) = a(Hf) + b^*(Kf)\,. \qquad (4)$$

Clearly, $A(f)$ is linear and norm-continuous in f. Moreover, if $A^*(f)$ is the adjoint of $A(f)$, a simple calculation gives:

$$[A(f), A(g)]_+ = 0\,, \qquad [A(f), A^*(g)]_+ = (f, g)\,. \tag{5}$$

Thus, we have defined by (4) a representation of the canonical anti-commutation relations (CAR in the following). These representations have been introduced in [1] and are useful for describing gauge invariant generalized free fermion field [2], in particular, a free fermion gas with constant density at finite temperature [3]. Their study mainly from a mathematical point of view, is the purpose of this paper.

First, we recall some facts about the CAR. The most out-standing is the existence of a canonical C^*-algebra $\mathfrak{A}$ which can be viewed as generated by the $A(f)$'s and their adjoints. Detailed constructions of it can be found in [4]. $\mathfrak{A}$ is a uniformly hyperfinite C^*-algebra [5].

With the concept of C^*-algebra is associated the concept of state: a state ω is a positive linear functional on the C^*-algebra with norm one [6]. In our case, a state ω is uniquely determined by the quantities:

$$\omega(A^*(f_1)\ldots A^*(f_n)\,A(g_1)\ldots A(g_m))$$

that is, if we have, for two states ω_1 and ω_2:

$$\omega_1(A^*(f_1)\ldots A^*(f_n)\,A(g_1)\ldots A(g_m))$$
$$= \omega_2(A^*(f_1)\ldots A^*(f_n)\,A(g_1)\ldots A(g_m))$$

for all f_i and g_j in $\mathfrak{E}$, these states are identical.

A representation π of the C^*-algebra $\mathfrak{A}$ defined in the Hilbert space H_π is cyclic if there exits in H_π a vector Ω such that the set of vectors $\{\pi(x)\Omega,\, x \in \mathfrak{A}\}$ is a total one in H_π. If Ω is normed to one, the quantity:

$$\omega(x) = (\pi(x)\Omega, \Omega)\,, \qquad x \in \mathfrak{A}$$

defines a state on $\mathfrak{A}$. Conversely, to each state on $\mathfrak{A}$ can be associated canonically a cyclic representation. We have the evident result of which we shall make use in the following:

If the same state is ascribed to distinct cyclic representations these representations are equivalent.

An important notion, which is basic for our work, is the quasi-equivalence of two representations [6]. Among many definitions, we take the following:

Two representations π_1 and π_2 of a C^*-algebra are quasi-equivalent if there exist a multiple of π_1 and a multiple of π_2 which are equivalent.

It should be noticed that the quasi-equivalence is a true equivalence relation.

II. Some Auxilary Results on Quasi-Equivalence

Let V_H and V_K closed subspaces of $\mathfrak{E}$ spanned by the values of H and K:

$$V_H = \overline{\{Hf, f \in \mathfrak{E}\}} \qquad V_K = \overline{\{Kf, f \in \mathfrak{E}\}}$$

and $V_H^{\perp}$, $V_K^{\perp}$ be their complementary subspaces. Generally, $V_H^{\perp}$ and $V_K^{\perp}$ are distincts from the null space. We denote by h_i, $i = 1, 2, \ldots$ and k_i, $i = 1, 2, \ldots$ some orthonormal basis in each of them. Now let $\mathfrak{F}^{HK}_{i_1 \ldots i_p; j_1 \ldots j_q}$ be the closed subspace of $\mathfrak{F}$ spanned by the vectors:

$$A^*(f_1) \ldots A^*(f_n) \, A(g_1) \ldots A(g_m) \, a^*(h_{i_1}) \ldots a^*(h_{i_p}) \, b^*(k_{j_1}) \ldots b^*(k_{j_q}) \Omega_0 \tag{6}$$

$$n = 0, 1, 2, \ldots; \quad m = 0, 1, 2, \ldots; \quad f_i \in \mathfrak{E}; \quad g_j \in \mathfrak{E}.$$

One the one hand, $\mathfrak{F}^{HK}_{i_1 \ldots i_p; j_1 \ldots j_q}$ is an invariant subspace for the representation (4) in which this representation is restricted to a cyclic representation with $a^*(h_{i_1}) \ldots a^*(h_{i_p}) \, b^*(k_{j_1}) \ldots b^*(k_{j_p}) \Omega_0$ as cyclic vector. These subrepresentations are all equivalent because the states generated on the C^*-algebra of the CAR by the various cyclic vector are identical. This results almost immediately from the anticommutation of the $A(f)$'s and $A^*(f)$'s with the $a^*(h_i)$'s and $b^*(k_j)$'s.

On the other hand, we have:

$$\mathfrak{F} = \bigoplus_{p, q} \bigoplus_{\substack{i_1 \ldots i_p \\ j_1 \ldots j_q}} \mathfrak{F}^{HH}_{i_1 \ldots i_p; j_1 \ldots j_q}.$$

Indeed, it can be proved easily by induction that each vector in $\mathfrak{F}$ having the form:

$$a^*(Hf_1) \ldots a^*(Hf_n) \, b^*(Kg_1) \ldots b^*(Kg_n) \, a^*(h_{i_1}) \ldots a^*(h_{i_p}) \, b^*(k_{j_1}) \ldots b^*(k_{j_q}) \Omega_0 \tag{7}$$

can be written as a linear combination of vectors having the form (6). Let now, in (7), $f_1, \ldots, f_n, g_1, \ldots, g_m$ run over $\mathfrak{E}$, p and q run over all integers and $i_1 \ldots i_p$ and $j_1 \ldots j_q$ over all choice of the indices, we obtain a set of vectors which is a total one in $\mathfrak{F}$; then we get:

Theorem 1. *The representation of the CAR defined by* (4) *is a multiple of a cyclic representation. The multiplicity is equal to* 2^{r+s}, *where* r *and* s *are the dimensions of* $V_H^{\perp}$ *and* $V_K^{\perp}$.

In our case, the state which defines the cyclic representation satisfy:

$$\omega(A(f_1) \ldots A(f_n) \, A(g_1) \ldots A(g_m)) = (A^*(f_1) \ldots A^*(f_n) \, A(g_1) \ldots$$

$$\ldots A(g_m) \Omega_0, \Omega_0) = (-1)^{\frac{n(n-1)}{2}} \delta_{nm} \det(K^* K g_i, f_j).$$

Since the knowledge of these quantities characterizes completely the state, it is clear that two representations (4) with the same value of $K^* K$ can differ only by their multiplicity, and then are quasi-equivalent.

Corollary. *All the representations* (4) *having the same value of* $|K| = (K^* K)^{1/2}$ *are quasi-equivalent.*

In particular, the representation (4) is quasi-equivalent to the representation defined by:

$$A(f) = a(|H| f) + b^*(|K| f) .$$

Lemma 1. *If H and K in $\mathfrak{L}(\mathfrak{E})$ satisfy the relations:*

$$\begin{aligned} H^*H + K^*K &= I \qquad HH^* + KK^* = I \\ H^*K &= K^*H \qquad HK^* = KH^* \end{aligned} \tag{8}$$

and if K is an Hilbert-Schmidt operator, there exits in $\mathfrak{L}(\mathfrak{F})$ a unitary operator U such that:

$$\left.\begin{aligned} A(f) &= a(Hf) + b^*(Kf) = U a(f) U^* \\ B^*(f) &= -a(Kf) + b^*(Hf) = U b(f) U^* . \end{aligned}\right\} \tag{9}$$

Proof. The representation of two anticommuting fields defined by the right member in (9) is irreducible. Indeed, from (8), we get:

$$\begin{aligned} a(f) &= A(H^*f) - B^*(K^*f) \\ b^*(f) &= A(K^*f) + B(H^*f) . \end{aligned}$$

Therefore, an operator commuting with $A(f)$, $B^*(f)$ commutes with $a(f)$, $b^*(f)$ and thus is scalar.

Now, if K is an Hilbert-Schmidt operator, it can be written [7]:

$$Kf = \sum_i \lambda_i g_i (f, f_i) , \qquad \lambda_i \geqq 0 \qquad \Sigma \lambda_i^2 < \infty ,$$

where $\{f_i, i = 1, 2, \ldots\}$ is an orthonormal basis and $\{g_i, i = 1, 2, \ldots\}$ an orthonormal set of vectors in $\mathfrak{E}$. The λ_i's are eigenvalues of $|K|$. By (8), we have for H:

$$Hf = \sum_i \sqrt{1 - \lambda_i^2}\, h_i (f, f_i) ,$$

where $\{h_i, i = 1, 2, \ldots\}$ is an orthonormal set of vectors. Moreover, by the first Eq. (8), we have:

$$\lambda_i^2 (h_i, g_j) = \lambda_j^2 (h_i, g_j) .$$

So, $(h_i, g_j) = 0$ if $\lambda_i \neq \lambda_j$. But for $\lambda_i \neq 0$, $\lambda_i = \lambda_j$ is possible only for a finite number of j, since K is Hilbert-Schmidt. Let $i_1, i_2, \ldots i_n$ be these values of j. From the second Eq. (8), we get:

$$(g_{i_k}, h_{i_l}) = (h_{i_k}, g_{i_l}) .$$

Hence the matrix with elements (g_{i_k}, h_{i_l}) is unitary and hermitian. So, its eigenvalues are ± 1 and there exists a unitary finite dimensional matrix u, such that:

$$\sum_k h_{i_k} u_{kl} = \pm \sum_k g_{i_k} u_{kl} .$$

Setting:

$$g'_{il} = \sum_k g_{i_k} u_{kl}$$

we have:

$$g_{i_k} = \sum_l \bar{u}_{kl} g'_{il}, \; h_{i_k} = \sum_l \bar{u}_{kl} \varepsilon_l g'_{il}, \qquad \varepsilon_l = \pm 1$$

and the terms $\lambda_i \sum_k g_{i_k}(f, f_{i_k}), \sqrt{1-\lambda_i^2} \sum_k h_{i_k}(f, f_{i_k})$ in the expansions of Hf and Kf become respectively:

$$\lambda_i \sum_k g'_{ik} \left(f, \sum_l f_{il} u_{lk}\right), \qquad \sqrt{1-\lambda_i^2} \sum_k \varepsilon_k g'_{ik} \left(f, \sum_l f_{il} u_{lk}\right) \qquad \varepsilon_k = \pm 1 \, .$$

If λ_i is zero, the corresponding f_i and g_i in the expansion of K are immaterial and we can always assume $h_i = g_i$ for all i with $\lambda_i = 0$, so that, setting now:

$$f'_{ik} = \sum_l f_{il} u_{lk}$$

and dropping the dashes, we can write for Hf and Kf:

$$Kf = \sum_i \lambda_i g_i (f, f_i), \quad Hf = \sum_i \sqrt{1-\lambda_i^2} \, \varepsilon_i g_i (f, f_i) \qquad \varepsilon_i = \pm 1$$

where, this time, g_i, $i = 1, 2, \ldots$ is an orthonormal basis in $\mathfrak{E}$, as implied by $HH^* + KK^* = I$.

Then, we have:

$$A(f) = \sum_i (f, f_i) \left(\sqrt{1-\lambda_i^2} \, \varepsilon_i a(g_i) + \lambda_i b^*(g_i)\right)$$

$$B(f) = \sum_i (f, f_i) \left(-\lambda_i a^*(g_i) + \sqrt{1-\lambda_i^2} \, \varepsilon_i b(g_i)\right) . \qquad \varepsilon_i = \pm 1$$

Now the vector:

$$\Omega = \Pi_i \left(\sqrt{1-\lambda_i^2} - \varepsilon_i \lambda_i a^*(g_i) \, b^*(g_i)\right) \Omega_0$$

is in $\mathfrak{F}$, as follows from the condition $\sum_i \lambda_i^2 < \infty$, and a straightforward calculation gives:

$$A(f) \Omega = B(g) \Omega = 0 \, , \qquad f, g \in \mathfrak{E} \; ;$$

Thus, the $A(f)$, $B^*(g)$ defined in (9) determine an irreducible Fock representation in the same way as the $a(f)$'s and $b^*(g)$'s, therefore these two representations are unitarily equivalent. This completes the proof of the lemma.

As a first application, we can derive the:

Lemma 2. *If, in* (4), *H and K are positive operators, we can always assume that K does not admit zero or one as eigenvalues.*

Proof. Let $\{f_i, i = 1, 2, \ldots\}$ and $\{g_j, j = 1, 2, \ldots\}$ be orthonormal basis in the eigenspaces of $\mathfrak{E}$ corresponding to the eigenvalues one and zero respectively. For f in $\mathfrak{E}$, we may write:

$$f = \sum_i \alpha_i f_i + \sum_j \beta_j g_j + h\,, \quad (h, f_i) = (h, g_j) = 0, i, j = 1, 2, \ldots \tag{10}$$

Moreover, by (3):

$$H f_i = 0\,, \quad i = 1, 2, \ldots, H g_j = g_j \quad i = 1, 2, \ldots$$

Now, let $\{\theta_i, i = 1, 2, \ldots\}$ and $\{\omega_j, j = 1, 2, \ldots\}$ be two sets of arguments such that:

$$0 < \theta_i < \frac{\pi}{2}, \quad 0 < \omega_j < \frac{\pi}{2}$$

$$\sum \theta_i^2 < \infty\,, \quad \sum \omega_j^2 < \infty\,.$$

From the preceding lemma, we deduce the existence of an unitary operator V in $\mathfrak{L}(\mathfrak{F})$ such that:

$$\begin{aligned} V a(f) U^* &= \sum_i \alpha_i (\cos\theta_i a(f_i) + \sin\theta_i b^*(f_i)) \\ &\quad + \sum_j \beta_j (\cos\omega_j a(g_j) + \sin\omega_j b^*(g_j)) + a(h)\,, \\ U b^*(f) U^* &= \sum_i \alpha_i (-\sin\theta_i a(f_i) + \cos\theta_i b^*(f_i)) \\ &\quad + \sum_j \beta_j (-\sin\omega_j a(g_j) + \cos\omega_j b^*(g_j)) + b^*(h)\,. \end{aligned}$$

Since the space spanned by the vectors h of (10) is invariant by H and K, we have:

$$\begin{aligned} A'(f) = U A(f) U^* &= a(Hh) + b(Kh) + \sum_i \alpha_i (\cos\theta_i b^*(f_i) - \sin\theta_i a(f_i)) \\ &+ \sum_j \beta_j (\cos\omega_j a(g_j) + \sin\omega_j b^*(g_j)) = a(H'f) + b^*(K'f) \end{aligned} \tag{11}$$

with:

$$K'f = Kf - \sum_i (1 - \cos\theta_i)\,(f, f_i) f_i + \sum_j \sin\omega_j (f, g_j) g_j$$

$$H'f = Hf - \sum_i \sin\theta_i (f, f_i) f_i - \sum_j (1 - \cos\omega_j)\,(f, g_j) g_j\,.$$

If f given by (10) must verify $K'f = 0$, we have:

$$Kh + \sum_i \alpha_i \cos\theta_i f_i \sum_j \beta_j \sin\omega_j g_j = 0\,.$$

The inequalities imposed on θ_i and ω_j imply $\alpha_i = \beta_j = 0$; but, from $Kh = 0$, follows $h = 0$ by the definition of h. In the same way, we shall show that $K'f = f$ implies $f = 0$. Thus we have constructed a representation with the required property, unitarily equivalent to the initial one.

Remark. One can show easily that in the conditions of the lemma, the representation is cyclic with Ω_0 as cyclic vector. This remark shall be of some use in the following (cf. section III).

Let us now consider two representations of the type (4):

$$A_1(f) = a(H_1 f) + b(K_1 f)\,, \tag{12}$$

$$A_2(f) = a(H_2 f) + b(K_2 f)\,, \tag{13}$$

where H_1, K_1 and H_2, K_2 are assumed positive. We state:

Theorem 2. *If* $(H_1K_2 - K_1H_2)$ *is an Hilbert-Schmidt operator, the representations* (12) *and* (13) *are unitarily equivalent.*

Proof. Let us consider:

$$a'(f) = a((H_1H_2 + K_1K_2)f) - b^*((H_1K_2 - K_1H_2)f)\,,$$
$$b'(f) = a((H_1K_2 - K_1H_2)f) + b^*((H_1H_2 + K_1K_2)f)\,.$$

Since $[H_1, K_1] = [H_2, K_2] = 0$, the operators $H = H_1H_2 + K_1K_2$ and $K = K_1H_2 - H_1K_2$ have the properties stated in the lemma 1. Therefore there exists a unitary operator U in $\mathfrak{L}(\mathfrak{F})$ with:

$$a'(f) = U a(f) U^*\,, \qquad b'(f) = U b(f) U^*\,.$$

But:

$$a'(H_2 f) + b'^*(K_2 f) = a(H_1 f) + b^*(K_1 f) = A_1(f)\,.$$

Hence:

$$A_1(f) = U A_2(f) U^*\,.$$

III. The Main Results

Since, by (3), K is bounded by one, we can define an operator Θ by:

$$\Theta = \operatorname{Arc\,sin}|K|\,.$$

Theorem 3. *Each representation* (4) *is quasi-equivalent to a similar representation for which the corresponding* $|K|$ *has a discrete spectrum.*

Proof. Let us assume Θ have a spectrum with a continuous part. von Neumann has shown [8] that we can add to Θ a Hilbert-Schmidt operator, A such that $\Theta + A$ has a discrete spectrum contained in $[0, \pi/2]$ as Θ spectrum is[1]. Moreover, A can be chosen so that $\|A\| < \varepsilon$, where ε is some positive number. We can form $\cos(\Theta + A)$ and $\sin(\Theta + A)$ which are positive operators and consider the representation defined by:

$$A'(f) = a(\cos(\Theta + A)f) + b(\sin(\Theta + A)f)\,. \tag{14}$$

Using a technical device customary in quantum field theory, we have:

$$e^{i(\Theta + A)} = e^{i\Theta}\Biggl(1 + i\int_0^1 A(t)\,dt + i^2\int_0^1 dt_1\int_0^{t_1} dt_2 A(t_1)A(t_2)$$
$$+ \cdots + i^n\int_0^1 dt_1\int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n A(t_1)\,A(t_2)\ldots A(t_n) + .\Biggr)$$

[1] We owe this remark to Dr. O. E. Lanford.

Here $A(t) = e^{-it\Theta} A e^{it\Theta}$. Since $A(t)$ is a Hilbert-Schmidt operator with the same Hilbert-Schmidt norm as A, the series in the right member, minus his first term, converges in the Hilbert-Schmidt norm and defines an Hilbert-Schmidt operator. Thus we have:

$$e^{i(\Theta+A)} = e^{i\Theta} + T ,$$

where T is a Hilbert-Schmidt operator. Taking the adjoints of the two members, we get:

$$e^{-i(\Theta+A)} = e^{-i\Theta} + T$$

and these relations imply:

$$\cos(\Theta + A) = \cos\Theta + \frac{T + T^*}{2} , \qquad \sin(\Theta + A) = \sin\Theta + \frac{T - T^*}{2i} .$$

Now:

$$\sin(\Theta + A)\cos\Theta - \cos(\Theta + A)\sin\Theta = \frac{1}{2}(T e^{i\Theta} + T^* e^{-i\Theta}) .$$

Since the right member is a Hilbert-Schmidt operator, it results from the lemma 2 that the representation (14) is equivalent unitarily to the representation defined by:

$$A''(f) = a(\cos\Theta f) + b^*(\sin\Theta f) = a(|H| f) + b^*(|K| f) .$$

But, as a by-product of the theorem 1, this last representation is quasi-equivalent to the original one.

Therefore, if we do not make any distinction between quasi-equivalent representations, we may assume that K in (4) is a positive operator with a complete discrete spectrum. Moreover, if we take in account the lemma 2, we may even assume that K does not admit zero or one as eigenvalues. Thus we write:

$$K = \sum_i \sin\theta_i P_{f_i} , \qquad 0 < \theta_i < \pi/2$$

where P_{f_i} is the projection operator on the eigenvector f_i. Since the orthonormal set f_i, $i = 1, 2, \ldots$ is complete and Ω_0 is a cyclic vector, we may characterize the representation by the set of numbers:

$$\begin{aligned}&(A^*(f_{i_1}) \ldots A^*(f_{i_n}) A(f_{j_1}) \ldots A(f_{j_m}) \Omega_0, \Omega_0)\\ &= \omega(A^*(f_{i_1}) \ldots A^*(f_{i_n}) A(f_{j_1}) \ldots A(f_{j_m})) ,\end{aligned}$$

which define uniquely a state ω on the C^*-algebra of the CAR. All these numbers are zero, except if the sets $(i_1, \ldots i_n)$ and $(j_1, \ldots j_m)$ are identical up to the order. By the CAR, these last quantities can be written as linear combinations of $\omega(N(f_{i_1}) \ldots N(f_{i_n}))$ where $N(f_i) = A^*(f_i) A(f_i)$, and a direct calculation gives:

$$\omega(N(f_{i_1}) \ldots N(f_{i_n})) = \prod_{k=1}^{n} \sin^2\theta_{i_k} .$$

We must remark that the adjunction of an Hilbert-Schmidt operator to a symetric operator does not change the limit points of the spectrum.

Thus if Θ has a continuous spectrum, the discrete spectrum of $\Theta + A$ admits all the points of a segment as accumulation points. This remark shall be used in the following.

IV. An Equivalent Representation

We recall briefly in this section the construction of a family of representations of the CAR, the detailed study of which was made in [9, 10] in a slightly different form. To each i, $i = 1, 2, \ldots$, let us correspond two two-dimensional Hilbert-spaces $\mathscr{H}_i^{(1)}$ and $\mathscr{H}_i^{(2)}$ and let be $\mathscr{H}_i = \mathscr{H}_i^{(1)} \otimes \mathscr{H}_i^{(2)}$. If θ_i, $i = 1, 2, \ldots$, is a set of arguments verifying the inequalities $0 < \theta_i < \pi/2$, we denote by I_1 the set of indices such that $0 < \theta_i \leqq \pi/4$ and by I_2 the set of indices such that $\pi/4 < \theta_i < \pi/2$. We choose in each $\mathscr{H}_i^{(\tau)}$ the basis vectors $e_{ij}^{(\tau)}$, $j = 1, 2$, and we consider f_i in $\mathscr{H}_i$ given by:

$$f_i = \cos\theta_i e_{i,1}^{(1)} \otimes e_{i,1}^{(2)} + \sin\theta_i e_{i,2}^{(1)} \otimes e_{i,2}^{(2)} \quad \text{for} \quad i \in I_1$$

or:

$$f_i = \sin\theta_i e_{i,1}^{(1)} \otimes e_{i,1}^{(2)} + \cos\theta_i e_{i,2}^{(1)} \otimes e_{i,2}^{(2)} \quad \text{for} \quad i \in I_2 .$$

Let us form the incomplete direct product $\mathfrak{H}$ of the $\mathscr{H}_i$ generated by the f_i's [11]:

$$\mathfrak{H} = \bigotimes^{f_i} \mathscr{H}_i .$$

If a_i, a_i^* are operators in $\mathscr{H}_i^{(1)}$ [2] verifying:

$$\begin{aligned} a_i e_{i,1}^{(1)} &= 0 \qquad a_i^* e_{i,1}^{(1)} = e_{i,2}^{(1)} \\ a_i e_{i,2}^{(1)} &= 0 \qquad a_i^* e_{i,2}^{(1)} = 0 \qquad \text{for} \quad i \in I_1 \end{aligned}$$

or:

$$\begin{aligned} a_i e_{i,1}^{(1)} &= e_{i,2}^{(1)} \qquad a_i^* e_{i,1}^{(1)} = 0 \\ a_i e_{i,2}^{(1)} &= 0 \qquad a_i^* e_{i,1}^{(1)} = e_{i,2}^{(1)} \qquad \text{for} \quad i \in I_2 \end{aligned}$$

we obtain a representation of the CAR in $\mathfrak{L}(\mathfrak{H})$ by setting:

$$A_i = \prod_{j<i} (1 - 2a_j^* a_j) a_i , \qquad A_i^* = \prod_{j<i} (1 - 2a_j^* a_j) a_i^* , \tag{15}$$

where we have adopted the same notations for a_i and a_i^* in $\mathfrak{L}(\mathscr{H}_i^{(1)})$ and their extension to $\mathfrak{L}(\mathfrak{H})$.

The assumption on the θ_i's implies the cyclicity of $\otimes f_i$ and an easy but tedious calculation shows that the corresponding state is identical with the state of the preceding section. Since a cyclic representation of a C^*-algebra is characterized up to an equivalence by its defining state, we get the theorem:

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[2] These a_i, a_i^* must not be confused with the $a(f)$, $a^*(f)$ of the preceding section.

Theorem 4. *For each representation of the CAR defined by* (4), *there is a set of* θ_i, $i = 1, 2, \ldots$ *with* $0 < \theta_i < \pi/2$ *and a complete orthonormal basis in* $\mathfrak{C}$, $\{f_i, i = 1, 2, \ldots\}$ *such that the representation is equivalent to the product representation defined by* (15) *where* $A_i = A(f_i)$.

The representations (15) can be completely classified according to their types [12, 14]. We quote here the statements of [14], adapted to our particular case:

1) The representations are of type I_∞ *if, and only if:*

$$\sum_{i \in I_1} (1 - \cos^2\theta_i) + \sum_{i \in I_2} (1 - \sin^2\theta_i) < \infty\,. \tag{16}$$

2) The representations are of type II_1 *if, and only if:*

$$\sum_{i \in I} \left(1 - \frac{1}{\sqrt{2}}(\cos\theta_i + \sin\theta_i)\right) < \infty\,. \tag{17}$$

3) The representations are of type III if, and only if, for some $c > 0$:

$$\sum_{i \in I_1} \sin^2\theta_i \left|\frac{\cos^2\theta_i}{\sin^2\theta_i} - 1\right|_c^2 + \sum_{i \in I_2} \cos^2\theta_i \left|\frac{\sin^2\theta_i}{\cos^2\theta_i} - 1\right|_c^2 = \infty\,. \tag{18}$$

where $|x|_c = \inf(|x|, c)$.

4) In all other cases, the representations are of type II_∞.

Let us now give the detailed consequences for $|K|$. In case (1), we denote by E the orthogonal projection on the subspace spanned by the vectors f_i, $i \in I_1$. Since inequality (16) is equivalent to:

$$\sum_{i \in I_1} \theta_i^2 + \sum_{i \in I_2} \left(\frac{\pi}{2} - \theta_i\right)^2 < \infty$$

if we take into account $(\Theta + A) f_i = \theta_i f_i$, we deduce:

$$\left(\Theta + A - \frac{\pi}{2}\right)(1 - E) + (\Theta + A) E = \text{Hilbert-Schmidt operator.}$$

But A is also a Hilbert-Schmidt operator. Then:

$$\Theta = \frac{\pi}{2}(1 - E) + \text{Hilbert-Schmidt operator.}$$

Hence:

$$|K| = \sin\Theta = 1 - E + \text{Hilbert-Schmidt operator.}$$

Conversely, if $|K|$ has this form, the representation is of type I_∞ because, from theorem 2, it is quasi-equivalent to the representation corresponding to $|K| = 1 - E$, which is a discrete one in the sense of Wightman-Garding [13].

Let us now consider the case (2). Inequality (17) is equivalent to:

$$\sum_{i \in I} \left(\frac{\pi}{4} - \theta_i\right)^2 < \infty$$

which implies, since A is Hilbert-Schmidt operator:

$$\Theta = \frac{\pi}{4} I + \text{Hilbert-Schmidt operator}$$

or, also:

$$|K| = \frac{1}{\sqrt{2}} I + \text{Hilbert-Schmidt operator.}$$

The discussion of the case (3) is a little more involved. Let us denote by I_1^1, I_1^2 the subsets of I_1 such that:

$$\left(\frac{\cos^2\theta_i}{\sin^2\theta_i} - 1\right) > c \quad \text{if} \quad i \in I_1^1 \,, \left(\frac{\cos^2\theta_i}{\sin^2\theta} - 1\right) < c \quad \text{if} \quad i \in I_1^2 \,.$$

Similarly, let us denote by I_2^1, I_2^2, the subsets of I_2 such that:

$$\left(\frac{\sin^2\theta_i}{\cos^2\theta_i} - 1\right) < c \quad \text{if} \quad i \in I_2^1 \,, \left(\frac{\sin^2\theta_i}{\cos^2\theta_i} - 1\right) > c \quad \text{if} \quad i \in I_2^2 \,.$$

Thus, (18) can be written:

$$c^2 \sum_{i \in I_1^1} \sin^2\theta_i + \sum_{i \in I_1^2} \frac{\cos^2 2\theta_i}{\sin^2\theta_i} + \sum_{i \in I_2^1} \frac{\cos^2 2\theta_i}{\cos^2\theta_i} + c^2 \sum_{i \in I_2^2} \cos^2\theta_i = \infty \,.$$

At least, one of the four series in the left member should be divergent. This is evidently the case if the θ_i's have an accumulation point distinct of 0, $\pi/4$ and $\pi/2$, or, equivalently, if $|K|$ spectrum has a limit point distinct of 0, $1/\sqrt{2}$ and 1. In the opposite situation, we can always find three projection operators in $\mathfrak{C}$, E_1, E_2, E_3, commuting with $|K|$ and so that $|K|E_1$, $|K|E_2$, $|K|E_3$ must have the unique limit point 0, $1/\sqrt{2}$, and 1 respectively[3]. The corresponding representation is of type III if, and only if, at least one of the $|K|E_1$, $|K|E_2 - \frac{1}{\sqrt{2}} E_2$, $|K|E_3 - E_3$ is compact but not Hilbert-Schmidt operator.

It is now very easy to characterize type II_∞ representations: $|K|$ spectrum possesses effectively the three limit points 0, $1/\sqrt{2}$ and 1 and the three operators $|K|E_1$, $|K|E_2 - \frac{1}{\sqrt{2}} E_2$, $|K|E_3 - E_3$ are all Hilbert-Schmidt operators[4].

All this discussion can be summed up in the theorem:

Theorem 5. *Let us consider the representation of the CAR defined by* (4). *We state:*

1°) *The representation is of type I_∞ if, and only if, $|K|$ has the form:*

$$|K| = 1 - E + \textit{Hilbert-Schmidt operator}$$

where E is some projection operator in $\mathfrak{L}(\mathfrak{C})$.

2°) *The representation is to type II_1 if, and only if, $|K|$ has the form:*

$$|K| = \frac{1}{\sqrt{2}} I + \textit{Hilbert-Schmidt operator.}$$

---

[3] It may be that one or two of the E_i's are the null operator.

[4] It may be that E_1 or E_3 is the null operator.

3°) *The representation is of type* II_∞ *if, and only if, the spectrum of* K *has the three limit points* $0, 1/\sqrt{2}$ *and* 1 *and the three operators* $|K|E_1$, $|K|E_2 - \frac{1}{\sqrt{2}}E_2$, $|K|E_3 - E_3$ *are Hilbert-Schmidt operators, where* E_1, E_2, E_3 *are projection operators commuting with* $|K|$ *and chosen so that* $|K|E_1$, $|K|E_2$, $|K|E_3$ *have spectra with the unique limit point* $0, 1/\sqrt{2}$ and 1 *respectively.*

4°) *In all other cases, the representation is of type* III.

In particular, if $|K|$ has a spectrum partly continuous, we have necessarily a limit point distinct of $0, 1/\sqrt{2}$ and 1 and consequently the representation is of type III. This is precisely the situation for the representation given in [3] and describing an infinite free fermion gas with constant density at a temperature T which is finite and not zero.

V. Complementary Results

We can use theorem 5 for etablishing the reciprocal statement of the lemma 1, and, to some extent, of the theorem 2. Indeed, we have:

Lemma 1'. *If* H *and* K *in* $\mathfrak{L}(\mathfrak{E})$ *satisfy the relations* (8) *and if there exists* U *unitary in* $\mathfrak{L}(\mathfrak{F})$ *such that relations* (9) *are satisfied,* K *is a Hilbert-Schmidt operator.*

Proof. Since the representation of the CAR provided by the first line in (9) is of type I_∞, it results from the first part of theorem 5 that:

$$|K| = 1 - E + \text{Hilbert-Schmidt operator.}$$

with E some projection operator in $\mathfrak{L}(\mathfrak{E})$. Now, since this representation is equivalent to Fock-representation, it can be shown easily that E is unity operator. Thus, $|K|$ is actually a Hilbert-Schmidt operator; and the same is evidently true for K.

Theorem 2'. *Let us consider the two sets of operators:*

$$\left.\begin{aligned} A_1(f) &= a(H_1 f) + b^*(K_1 f) \\ B_1^*(f) &= -a(K_1 f) + b^*(H_1 f) \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} A_2(f) &= a(H_2 f) + b^*(K_2 f) \\ B_2^*(f) &= -a(K_2 f) + b^*(H_2 f) \end{aligned}\right\} \tag{20}$$

where H_i, K_i *are positive and satisfy:*

$$H_i^2 + K_i^2 = I\,.$$

If there exists U *unitary in* $\mathfrak{L}(\mathfrak{F})$ *such that:*

$$A_2(f) = U A_1(f) U^*\,, \qquad B_1^*(f) = U B_1^*(f) U^*\,,$$

the operator $H_1K_2 - K_1H_2$ *is necessarily Hilbert-Schmidt.*

Proof. We can write:

$$a(f) = A_1(H_1 f) - B_1^*(K_1 f)$$
$$b^*(f) = A_1(K_1 f) + B_1^*(H_1 f)\,.$$

Hence:

$$U a(f) U^* = A_2(H_1 f) - B_2^*(K_1 f) = a((H_2 H_1 + K_2 K_1) f) + b^*((K_2 H_1 - H_2 K_1) f)\,,$$
$$U b^*(f) U^* = A_2(K_1 f) + B_2^*(K_1 f) = -\,a((K_2 H_1 - H_2 K_1) f) + b^*((H_2 H_1 + K_2 K_1) f)\,.$$

From the preceding lemma, $(K_2 H_1 - H_2 K_1)$, hence its adjoint, is necessarily Hilbert-Schmidt.

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