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Application of the Riemann Method to the Bethe-Salpeter Equation

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Abstract. The Bethe-Salpeter equation describing the interaction of two scalar particles via the exchange of a third scalar particle with mass $\mu \neq 0$ is in configuration space a hyperbolic partial differential equation of fourth order which will be studied with the help of the Riemann method. This method yields two Volterra equations the solutions of which are special solutions of the Bethe-Salpeter equation. The wave function is a superposition of the special solutions. For the coefficients one gets a system of two integral equations. The Fredholm determinant of the system is the generalization of the nonrelativistic Jost function.

1. Introduction

An exhaustive treatment of the Schrödinger equation has been given by Newton. Crucial for the success of this method is the introduction of several modified Green's functions leading to Volterra integral equations. The Volterra equations can be solved by iteration for all values of the potential-strength. Despite the fact that the Schrödinger equation is an ordinary differential equation while the Bethe-Salpeter equation is a partial differential equation the generalization of this method to the Bethe-Salpeter case is possible. The Volterra equations in two variables can be established with help of the Riemann method [2, 3] or formally by splitting the Green's function into a Riemann function and two residual terms. The solutions of the integral equations which can be obtained by iteration are special solutions of the Bethe-Salpeter equation. The solution with causal boundary conditions is a superposition of the special solutions. For the coefficients in this expansion we get a system of two integral equations in one variable. The Fredholm determinant of the system is the generalization of the nonrelativistic Jost function [4].

In Sec. II we treat the radial Schrödinger equation. For convenience we confine us as in the Bethe-Salpeter case to zero angular momentum. Only those aspects are written down which can be already generalized. In Chapt. 3 we write down the differential form of the Bethe-Salpeter equation for two scalar particles with masses m_1 and m_2 which interact via a potential. Here we have in mind the Yukawa potential which describes the interaction via the exchange of a third scalar particle of mass $\mu \neq 0$ but also other potentials belonging to a certain class are admissable. Then we make use of the Riemann method by which we get the Volterra integral equations. The difficulties for getting the Riemann function are by-passed by the introduction of the Green's function and its decomposition. The proof that the Volterra equations can be solved in fact by iteration for all values of the strength of the potential is given in this paper only for the bound state case. A generalization of this proof to scattering seems not trivial to us.

2. The Schrödinger Equation

First let us study the radial Schrödinger equation for zero angular momentum:

$$l\psi(k, r) = \lambda V(r)\, \psi(k, r) \tag{1a}$$

with

$$l := \frac{d^2}{dr^2} + k^2 .$$

The potential $V(r)$ may satisfy

$$\int_0^\infty dr \frac{r}{1+r} |V(r)| < \infty .$$

The free equation

$$l\psi(k, r) = 0 \tag{1b}$$

has the solutions e^{-ikr} and e^{+ikr} or $\sin kr$ and $\cos kr$.

Now define to l an adjoint operator m so that $v(r)\, l u(r) - u(r)\, m v(r)$ will be a divergence:

$$v(r)\, l u(r) - u(r)\, m v(r) = \frac{d}{dr} W_r(v, u) .$$

In this case $m = l$ and $W_r(v, u) = vu' - uv'$; $u' = \frac{d}{dr} u(r)$. If u and v satisfy (1a) or (1b), then $W_r(v, u)$ does not depend on r. Let us consider

$$\int_0^r dr' [A(r, r')\, l' u(r') - u(r')\, m' A(r, r')] = W_{r'}(A(r, r'), u(r')) \Big|_{r'=0}^{r'=r} \tag{2}$$

where l' and m' act on the primed coordinates. Let $u(r')$ be a solution of (1a) and let $u(r')$ tend to $u^0(r')$ for $r' \to 0$ where $u^0(r)$ may be a solution of (1b). Further may be

$$m' A(r, r') = 0$$

with

$$A(r, r')|_{r'=r} = 0, \quad \frac{\partial}{\partial r'} A(r, r')|^{r'=r} = -1 .$$

It is easy to see that in this case

$$A(r, r') = \frac{1}{k} \sin k(r - r')$$

and

$$\begin{aligned} W_{r'}(A(r, r'), u(r'))|_{r'=0} &= W_{r'}(A(r, r'), u^0(r'))|_{r'=0} \\ &= W_{r'}(A(r, r'), u^0(r'))|_{r'=r} = -u^0(r) \end{aligned}$$

because of

$$m A = l u^0 = 0 .$$

Further it is

$$W_{r'}(A(r, r'), u(r'))|^{r'=r} = u(r) .$$

Thus taking in mind $l' u(r') = \lambda V(r') u(r')$ and $m' A(r, r') = 0$ (2) can be written as

$$u(r) = u^0(r) + \lambda \int_0^r dr' A(r, r') V(r') u(r') . \tag{3a}$$

For $u^0(r) = \sin kr$ denote the solution of (3a) by $\varphi(k, r)$ in agreement with Newton [1]. (3a) can be solved by iteration for every λ. The proof for this statement can be found in Newton's paper.

There is another way for getting this Volterra equation. In the scattering theory one usually converts (1a) in an integral equation which incorporates the boundary conditions that the solution of the Schrödinger equation without separation of the angular momentum consists of a plane wave plus an outgoing spherical wave; thus for zero angular momentum the integral equation reads

$$\psi(k, r) = \sin kr + \lambda \int_0^\infty dr' \, G(k, r, r') \, V(r') \, \psi(k, r') \tag{4}$$

where $G(k, r, r')$, the Green's function, is

$$\frac{2}{\pi} \int_0^\infty dp \frac{\sin pr \sin pr'}{-p^2 + k^2 + i\varepsilon} = -\frac{1}{k} \left(\theta(r - r') e^{ikr} \sin kr' + \theta(r' - r) e^{ikr'} \sin kr\right) .$$

(4) can be solved by iteration only for small enough λ [5]. The aim is now to transform (4) into another equation like equation (3a), which can be solved always by iteration.

This is done easily by writing

$$\begin{aligned} G(k, r, r') &= -\frac{1}{k} e^{ikr'} \sin kr + \theta(r - r') \left[-\frac{1}{k} (e^{ikr} \sin kr' - e^{ikr'} \sin kr)\right] \\ &= -\frac{1}{k} e^{ikr'} \sin kr + \theta(r - r') A(r, r') . \end{aligned} \tag{5}$$

So the Green's function is split into a Volterra kernel and a separable kernel. It is worthwhile for our further investigations to exhibit how the splitting can be performed in momentum space:

We have

$$\frac{1}{-p^2+k^2+i\varepsilon}=\frac{1}{-(p+i\varepsilon)^2-k^2}-2\pi i\theta(p)\,\delta(p^2-k^2)$$

and

$$-\frac{1}{i\pi}\int_{-\infty}^{+\infty}dp\,\frac{e^{ipr'}\sin pr}{(p+i\varepsilon)^2-k^2}=\frac{1}{k}\,\theta(r-r')\,\frac{1}{2i}\left(e^{-ikr'}\,e^{ikr}-e^{ikr'}\,e^{-ikr}\right)$$
$$=\theta(r-r')\,A(r,r')\,.$$

Thus

$$G(k,r,r')=-\frac{1}{i\pi}\int_{-\infty}^{+\infty}dp\,\frac{\sin pr\,e^{ipr'}}{p^2-k^2-i\varepsilon}=-\frac{1}{i\pi}\int_{-\infty}^{+\infty}dp\,\frac{\sin pr\,e^{ipr'}}{(p+i\varepsilon)^2-k^2}$$
$$-2\int_{-\infty}^{+\infty}dp\,\theta(p)\,\delta(p^2-k^2)\,e^{ipr'}\sin pr$$
$$=\theta(r-r')\,A(r,r')-\frac{1}{k}\,e^{ikr'}\sin kr\,.$$

For the Bethe-Salpeter case such a splitting in the integral representation will be the simplest way to get a Volterra kernel. Now, inserting (5) into (4) we get

$$\psi(k,r)=\sin kr-\frac{1}{k}\sin kr\int_0^\infty dr'\,e^{ikr'}\,\lambda\,V(r')\,\psi(k,r')$$
$$+\lambda\int_0^r dr'\,A(r,r')\,V(r')\,\psi(k,r')\,.$$

Recalling (3a) we see that

$$\psi(k,r)=a(k)\,\varphi(k,r)$$

with

$$a(k)=k-\lambda\int_0^\infty dr'\,e^{ikr'}\,V(r')\,\psi(k,r')$$

and so

$$a(k)=k-a(k)\,\lambda\int_0^\infty dr'\,e^{ikr'}\,V(r')\,\varphi(k,r')$$

or

$$a(k)=\frac{k}{f(-k)}$$

where $1+\lambda\int_0^\infty dr'\,e^{ikr'}\,V(r')\,\varphi(k,r')$ is denoted by $f(-k)$ again in agreement with Newton.

In a completely analogous manner we can find the other Volterra equations of Newton

$$f(\mp k,r)=e^{\pm ikr}+\lambda\int_r^\infty dr'\,\tilde{A}(r,r')\,V(r')\,f(\mp k,r') \qquad (3\text{b})$$

and the corresponding expansion of the solution $\psi(k, r)$ in terms of these special solutions is

$$\psi(k, r) = -\frac{1}{2i} f(k, r) + \frac{1}{2i} S(k) f(-k, r)$$

with

$$S(k) = \frac{1 + k^{-1} \cdot \lambda \int_0^\infty dr' \sin kr' \, V(r') \, f(k, r')}{1 + k^{-1} \cdot \lambda \int_0^\infty dr' \sin kr' \, V(r') \, f(-k, r')} .$$

The crucial points are the steps from (2) to (3a) and the splitting in (5) which can be generalized to the Bethe-Salpeter case.

3. The Bethe-Salpeter Equation

3.1. The Equation and its Free Solutions

Let us now consider a system of two relativistic particles with masses m_1, m_2, coordinates x_1, x_2 and momenta p_1, p_2. We write the equation in its differential form

$$(\square_1 + m_1^2)(\square_2 + m_2^2)\, \psi(x_1, x_2) = \lambda V(x_1 - x_2)\, \psi(x_1, x_2)$$

where

$$\square_i = \frac{\partial^2}{\partial x_{i0}^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_{ij}^2} .$$

Following SCHWARTZ and ZEMACH [6] we make a complete canonical transformation

$$P = p_1 + p_2 \qquad X = \mu_1 x_1 + \mu_2 x_2$$
$$k = \mu_2 p_1 - \mu_1 p_2 = (k_0, \mathbf{k}) \qquad x = x_1 - x_2 = (x_0, \mathbf{r})$$

where μ_1, μ_2 are constants restricted by $\mu_1 + \mu_2 = 1$ so that

$$dp_1 \, dp_2 = dP \, dk, \quad dx_1 \, dx_2 = dX \, dx .$$

In the center-of-mass frame we can write

$$p_1 = (\omega_1, \mathbf{k}), \quad p_2 = (\omega_2, -\mathbf{k}) ,$$

where

$$\omega_i = \sqrt{m_i^2 + \mathbf{k}^2}$$

thus

$$P = (E, \mathbf{o}); \; E = \omega_1 + \omega_2 .$$

In the bound state case we have $\mathbf{k}^2 < 0$; $k = \sqrt{\mathbf{k}^2} = i\varkappa$. Further let us split off with SCHWARTZ and ZEMACH the c.m. momentum writing $\psi(x_1, x_2) = e^{-iPX} \psi'(x)$ and make the phase transformation $\psi' \to e^{i\nu x_0} \psi'$ where $\nu = \mu_2 \omega_1 - \mu_1 \omega_2$. The incident wave is written as

$$\psi_0(x_0, \mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \text{independent of } x_0 .$$

Assuming that $V(x)$ only depends on x_0 and $r = |\mathbf{r}|$ we make a partial wave analysis and consider the Bethe-Salpeter equation for zero angular momentum:

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial r^2} - 2i\omega_1 \frac{\partial}{\partial x_0} - k^2\right)\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial r^2} + 2i\omega_2 \frac{\partial}{\partial x_0} - k^2\right)\psi(x_0, r) = \lambda V(x_0, r)\,\psi(x_0, r)\,. \tag{6}$$

This is a hyperbolic partial differential equation of fourth order in two variables. It is convenient to introduce the characteristic coordinates [2, 3] $z_1 = x_0 + r$ and $z_2 = x_0 - r$ so that (6) can be written in the form

$$L_1 L_2 \psi(z_1, z_2) = \lambda V(z_1, z_2)\,\psi(z_1, z_2) \tag{7}$$

where

$$L_1 = 4\frac{\partial^2}{\partial z_1 \partial z_2} - 2i\omega_1\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right) - k^2$$

$$L_2 = 4\frac{\partial^2}{\partial z_1 \partial z_2} + 2i\omega_2\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right) - k^2.$$

Evidently $L_1 L_2 = L_2 L_1$.

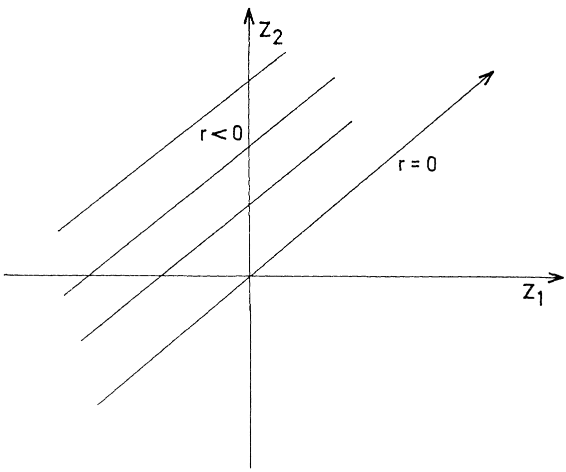

Fig. 1

Because of $r > 0$ we have $z_1 > z_2$. $z_1 \equiv 0$ and $z_2 \equiv 0$ are the boundaries of the light-cone $x^2 = x_0^2 - r^2 = z_1 z_2 = 0$.

The fourth quadrant corresponds to the spacelike domain, the lower half of the first quadrant to the forward cone and the lower half of the third quadrant to the backward cone.

The potential $V(z_1, z_2)$ will have to fulfill come conditions which will show up in the further calculation.

The free equation

$$L_1 L_2 \varphi(z_1, z_2) = 0 \tag{7b}$$

has the following special solutions:

a) two manifolds of solutions to $L_1 \varphi = 0$

$$\omega_{11}(q, z_1, z_2) = e^{i\frac{\omega_1}{2}(z_1+z_2)} e^{-i\frac{m_1^2}{4q}z_1} e^{-iqz_2} \tag{8a}$$

and

$$\omega_{12}(q, z_1, z_2) = e^{i\frac{\omega_1}{2}(z_1+z_2)} e^{-iqz_1} e^{-i\frac{m_1^2}{4q}z_2} \tag{8b}$$

with $q > 0$ or $q < 0$.

b) two manifolds of solutions to $L_2 \varphi = 0$

$$\omega_{21}(q, z_1, z_2) = e^{-i\frac{\omega_2}{2}(z_1+z_2)} e^{-i\frac{m_2^2}{4q}z_1} e^{-iqz_2} \tag{8c}$$

and

$$\omega_{22}(q, z_1, z_2) = e^{-i\frac{\omega_2}{2}(z_1+z_2)} e^{-iqz_1} e^{-i\frac{m_2^2}{4q}z_2} \tag{8d}$$

with $q > 0$ or $q < 0$.

c) two solutions $e^{\pm ikr} = e^{\pm i\frac{k}{2}(z_1-z_2)}$ of $L_1 \varphi = 0$ *and* $L_2 \varphi = 0$ which can be obtained by setting in a): $q = \frac{\omega_1}{2} \pm \frac{k}{2}$ or in b): $q = -\frac{\omega_2}{2} \pm \frac{k}{2}$.

For convenience let us define

$$u_1(q, z_1, z_2) = \frac{1}{2i}\left(\omega_{11}(q, z_1, z_2) - \omega_{12}(q, z_1, z_2)\right) \tag{8e}$$

$$u_2(q, z_1, z_2) = \frac{1}{2i}\left(\omega_{21}(q, z_1, z_2) - \omega_{22}(q, z_1, z_2)\right) \tag{8f}$$

and

$$u_0(k, z_1, z_2) = \sin kr = u\left(\frac{\omega_1}{2} + \frac{k}{2}, z_1, z_2\right) = u_2\left(-\frac{\omega_2}{2} + \frac{k}{2}, z_1, z_2\right). \tag{8g}$$

3.2. The Riemann Method

Again let us define an adjoint operator M_i to L_i so that $u L_i v - v M_i u$ will be a divergence:

$$u L_i v - v M_i u = \frac{\partial}{\partial z_1} U_i + \frac{\partial}{\partial z_2} V_i\,.$$

It is easy to see that

$$M_1 = 4\frac{\partial^2}{\partial z_1\, d z_2} + 2i\omega_1\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right) - k^2$$

$$M_2 = 4\frac{\partial}{\partial z_1\, \partial z_2} - 2i\omega_2\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2}\right) - k^2$$

and

$$\begin{aligned} U_1 &= v \cdot 4 \frac{\partial u}{\partial z_2} - 2 i \omega_1 v u \\ V_1 &= -u \cdot 4 \frac{\partial v}{\partial z_1} - 2 i \omega_1 v u \\ U_2 &= v \cdot 4 \frac{\partial u}{\partial z_2} + 2 i \omega_2 v u \\ V_2 &= -u \cdot 4 \frac{\partial v}{\partial z_2} + 2 i \omega_2 v u \, . \end{aligned}$$

Thus

$$\begin{aligned} v L_2 L_1 u - u M_1 M_2 v &= v L_1 (L_2 u) - (L_2 u) M_1 v + (M_1 v) L_2 u - u M_2 (M_1 v) \\ &= v L_2 (L_1 u) - (L_1 u) M_2 v + (M_2 v) L_1 u - u M_1 (M_2 v) \end{aligned}$$

so that in the first case we have

$$v L_2 L_1 u - u M_1 M_2 v = \frac{\partial}{\partial z_1} W_{11}(v, u) + \frac{\partial}{\partial z_2} W_{12}(v, u) \tag{9a}$$

and in the second case

$$v L_2 L_1 u - u M_1 M_2 v = \frac{\partial}{\partial z_1} W_{21}(v, u) + \frac{\partial}{\partial z_2} W_{22}(v, u) \tag{9b}$$

with

$$W_{11}(v, u) = \left[\left(4 \frac{\partial}{\partial z_2} - 2 i \omega_1\right) L_2 u\right] v + M_1 v \left[\left(4 \frac{\partial}{\partial z_2} + 2 i \omega_2\right) u\right] \tag{10a}$$

$$W_{12}(v, u) = (L_2 u) \left[\left(-4 \frac{\partial}{\partial z_1} - 2 i \omega_1\right) v\right] + u \left[\left(-4 \frac{\partial}{\partial z_1} + 2 i \omega_2\right) M_1 v\right] \tag{10b}$$

$$W_{21}(v, u) = \left[\left(4 \frac{\partial}{\partial z_2} + 2 i \omega_2\right) L_1 u\right] v + M_2 v \left[\left(4 \frac{\partial}{\partial z_2} - 2 i \omega_1\right) u\right] \tag{10c}$$

$$W_{22}(v, u) = (L_1 u) \left[\left(-4 \frac{\partial}{\partial z_1} + 2 i \omega_2\right) v\right] + u \left[-4 \frac{\partial}{\partial z_1} - 2 i \omega_1\right) M_2 v\right] . \tag{10d}$$

Now consider

$$\begin{aligned} \iint_D dz_1' \, dz_2' \, [R(k, z_1, z_2, z_1', z_2') \, L_1' L_2' u(z_1', z_2') & \\ - u(z_1', z_2') \, M_1' M_2' R(k, z_1, z_2, z_1', z_2')] & \end{aligned} \tag{11}$$

where D will be the domain noted in Fig. 2 and $R(k, z_1, z_2, z_1', z_2')$ will be an integralkernel denoted in the following by Riemann function.

Because of (9a, b) we can write for (11):

$$\begin{aligned} & \int_{z_2}^{\tilde{z}_1} dz_2' \, W_{i1}'(R, u)|^{z_1' = \tilde{z}_1} + \int_{z_1}^{\tilde{z}_1} dz_2' \, W_{i1}'(R, u)|_{z_1' = z_2'} + \int_{z_2}^{z_1} dz_2' \, W_{i1}'(R, u)|_{z_1' = z_1} \\ & + \int_{z_1}^{\tilde{z}_1} dz_1' \, W_{i2}'(R, u)|_{z_2' = z_2} + \int_{z_1}^{\tilde{z}_1} dz_1' \, W_{i2}'(R, u)|^{z_2' = z_1'}; \quad i = 1 \text{ or } 2 \end{aligned} \tag{12}$$

where in $W_{ij}'(R, u)$ the differential operators will act on the primed coordinates.

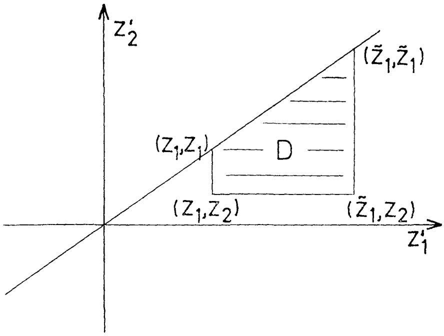

Fig. 2

Now let $\tilde{z}_1$ tend to ∞ and choose R and u so that they meet the following conditions:

First of all:

$$M_1' M_2' R(k, z_1, z_2, z_1', z_2') = 0 \tag{13a}$$

$$L_1' L_2' u(z_1', z_2') = \lambda V(z_1', z_2') u(z_1', z_2') \tag{14a}$$

so that (11) on one hand can be written as

$$\lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' dz_2' \, \theta(z_1' - z_2') R(k, z_1, z_2, z_1', z_2') V(z_1', z_2') u(z_1', z_2')$$

then, for $R(k, z_1, z_2, z_1', z_2')$

on $z_1' = z_1$:

$$R = 0; \frac{\partial}{\partial z_2'} R = 0; \left(4 \frac{\partial}{\partial z_2'} + 2 i \omega_1\right) M_2' R = 0; \left(4 \frac{\partial}{\partial z_2'} - 2 i \omega_2\right) M_1' R = 0 \tag{12b}$$

on $z_2' = z_2$:

$$R = 0; \frac{\partial}{\partial z_1'} R = 0; \left(4 \frac{\partial}{\partial z_1'} + 2 i \omega_1\right) M_2' R = 0; \left(4 \frac{\partial}{\partial z_1'} - 2 i \omega_2\right) M_1' R = 0 \tag{12c}$$

on $z_1' = z_2' \geqq z_1$:

$$R = 0; \; M_1' R = 0; \; M_2' R = 0 \tag{13d}$$

in $z_1' = z_1$; $z_2' = z_2$:

$$M_1' R = M_2' R = \frac{1}{4} \tag{13e}$$

and finally for $u(z_1, z_2)$:

a) on $z_1' = z_2'$: $u(z_1', z_2') = 0$; $L_1' u(z_1', z_2') = 0$ (14b)

for $z_1' \to \infty$ $u(z_1', z_2')$ may tend to $u_1^\infty(z_1', z_2')$ where $L_1' u_1^\infty(z_1', z_2') = 0$ (14c)

or b) on $z_1' = z_2'$: $u(z_1', z_2') = 0$; $L_2' u(z_1', z_2') = 0$ (14d)

for $z_1' \to \infty$ $u(z_1', z_2')$ may tend to $u_2^\infty(z_1', z_2')$ where $L_2' u_2^\infty(z_1', z_2') = 0$. (14e)

We denote the solution of (14a, b, c) and (14a, d, e) by $u_1(z_1, z_2)$ and $u_2(z_1, z_2)$ respectively.

Some of the conditions on R are redundant. If they are eliminated we can state the remaining conditions as a complete mixed boundary value problem.

With these conditions on $R(k, z_1, z_2, z_1', z_2')$ and with (14a, d, e) on $u(z_1', z_2')$ (12) can be greatly simplified.

Consider the case $i = 1$. The second, fourth and fifth term vanish, in the third term a partial integration yields

$$\int_{z_2}^{z_1} dz_2' \, W_{11}'(R, u_2)|_{z_1' = z_1}$$

$$= \int_{z_2}^{z_1} dz_2' \left[\left(\left(4 \frac{\partial}{\partial z_2'} - 2 i \omega_1\right) L_2' u_2(z_1', z_2')\right) R(k, z_1, z_2, z_1', z_2') \right.$$
$$\left. + u_2(z_1', z_2') \left(-4 \frac{\partial}{\partial z_2'} + 2 i \omega_2\right) M_1' R(k, z_1, z_2, z_1', z_2')\right]\Bigg|_{z_1' = z_1}$$
$$+ 4 M_1' R(k, z_1, z_2, z_1', z_2') \, u_2(z_1', z_2')\Big|_{z_2' = z_2,\, z_1' = z_1}^{z_2' = z_1} = u_2(z_1, z_2) \, .$$

The first term yields

$$\lim_{\tilde{z}_1 \to \infty} \int_{z_2}^{\tilde{z}_1} dz_2' \, W_{11}'(R, u_2)|^{z_1' = \tilde{z}_1} = \lim_{\tilde{z}_1 \to \infty} \int_{z_2}^{\tilde{z}_1} dz_2' \, W_{11}'(R, u_2^{\infty})|^{z_1' = \tilde{z}_1}$$

$$= \int_{z_2}^{z_1} dz_2' \, W_{11}(R, u_2^{\infty})|^{z_1' = z_1} = 4 M_1' R(k, z_1, z_2, z_1', z_2') \, u_2^{\infty}(z_1', z_2')\Big|_{z_2' = z_2}^{z_2' = z_1,\, z_1' = z_1}$$

$$= - u_2^{\infty}(z_1, z_2)$$

where we have used the fact that $\int_{z_2}^{z_1'} dz_2' \, W_{11}(R, u_2^{\infty})$ is independent of z_1'. Thus, from (12) we obtain $u_2(z_1, z_2) - u_2^{\infty}(z_1, z_2)$ and hence from (11) the equation

$$u_2(z_1, z_2) = u_2^{\infty}(z_1, z_2)$$
$$+ \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2') \, R(k, z_1, z_2, z_1', z_2') \, V(z_1', z_2') \, u_2(z_1', z_2')$$

and analogously with (14a, b, c) instead of (14a, d, e) for u considering (12) for $i = 2$:

$$u_1(z_1, z_2) = u_1^{\infty}(z_1, z_2)$$
$$+ \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2') \, R(k, z_1, z_2, z_1', z_2') \, V(z_1', z_2') \, u_1(z_1', z_2') \, .$$

Inserting for $u_i^{\infty}(z_1, z_2)$ the free solutions (8e, f) we obtain two manifolds of special solutions of the Bethe-Salpeter equation

$$\varphi_i(q, z_1, z_2) = u_i(q, z_1, z_2) + \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2) \cdot R(k, z_1, z_2, z_1', z_2') \, V(z_1', z_2') \, \varphi_i(q, z_1', z_2') \tag{15}$$

which will behave as $u_i(q, z_1, z_2)$ for $z_1 \to \infty$ and will vanish as $u_i(z_1, z_2)$ on $z_1 = z_2$. In a subsequent section it will be shown that for a certain class of potentials and also for the Yukawa-potential these equations can be solved by iteration.

All this is quite analogous to the nonrelativistic Schrödinger case. The $\varphi_i(q, z_1, z_2)$ correspond to the single special solution $\varphi(k, r)$ and as in the Schrödinger case where $\psi(k, r) = \frac{k}{f(-k)} \varphi(k, r)$ we will see that the wave function $\psi(k, z_1, z_2)$ can be expressed as a superposition of the $\varphi_i(q, z_1, z_2)$:

$$\psi(k, z_1, z_2) = \int dq \, f_1(q, k) \, \varphi_1(q, z_1, z_2) + \int dq \, f_2(q, k) \, \varphi_2(q, z_1, z_2) \, .$$

In the Schrödinger case it was easy to construct the Volterra kernel $A(r, r')$ and the coefficient $\frac{k}{f(-k)}$. Both problems are much more difficult in the Bethe-Salpeter case but splitting the Green's function will help here, too.

3.3. Splitting of the Green's Function

The Green's function with causal boundary conditions is

$$G(k, x, x') = (2\pi)^{-4} \int d^4 p \frac{e^{-ip(x-x')}}{[(p_0 + \omega_1)^2 - \mathbf{p}^2 - m_1^2 + i\varepsilon][(p_0 - \omega_2)^2 - \mathbf{p}^2 - m_2^2 + i\varepsilon]} \, .$$

SCHWARTZ und ZEMACH have shown that the wave function defined by the integral equation

$$\psi(\mathbf{k}, x) = e^{i\mathbf{k}\cdot\mathbf{r}} + \lambda \int d^4 x' \, G(k, x, x') \, V(x') \, \psi(\mathbf{k}, x') \tag{16}$$

has the desired behaviour for $r \to \infty$, namely

$$\psi(\mathbf{k}, x) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}' \leftarrow \mathbf{k}) \frac{e^{ikr}}{r} \, ; \quad \mathbf{k}' = \mathbf{r} \cdot \frac{|\mathbf{k}|}{|\mathbf{r}|}$$

with

$$f(\mathbf{k}' \leftarrow \mathbf{k}) = \frac{i}{8\pi E} \lambda \int d^4 x' \, e^{-i\mathbf{k}'\mathbf{r}'} \, V(x') \, \psi(\mathbf{k}, x') \, .$$

For zero angular momentum (16) reads

$$\psi(k, x_0, r) = \sin kr + \lambda \int_{-\infty}^{+\infty} \int_{0}^{+\infty} dx_0' \, dr' \, G(k, x_0, r, x_0', r') \, V(x_0', r') \, \psi(k, x_0', r')$$

where

$$G(k, x_0, r, x_0', r') = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_0^{\infty} dp_0\, dp \frac{e^{-ip_0(x_0 - x_0')} \sin pr \sin pr'}{[(p_0 + \omega_1)^2 - p^2 - m_1^2 + i\varepsilon]\,[(p_0 - \omega_2)^2 - p^2 - m_2^2 + i\varepsilon]}$$

and with $z_1' = x_0' + r'$, $z_2' = x_0' - r'$ where $dx_0\, dr' = \frac{1}{2} dz_1'\, dz_2'$ we obtain

$$\begin{aligned} \psi(k, z_1, z_2) = u_0(k, z_1, z_2) + \lambda \iint_{-\infty}^{+\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2') \\ \times G(k, z_1, z_2, z_1', z_2')\, V(z_1', z_2')\, \psi(k, z_1', z_2') \end{aligned} \tag{17}$$

where we have set $G(k, z_1, z_2, z_1', z_2') = \frac{1}{2} G(k, x_0, r, x_0', r')$.

The scattering amplitude for zero angular momentum is

$$f(k, k') = \frac{2i}{\pi} \lambda \iint_{-\infty}^{+\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2')\, u_0(k', z_1', z_2')\, V(z_1', z_2')\, \psi(k, z_1', z_2')$$

and on the mass shell we evidently have $k' = k$.

Keeping in mind the method of splitting the Green's function in the Schrödinger case we presume that we have to split from

$$\frac{1}{[(p_0 + \omega_1)^2 - p^2 - m_1^2 + i\varepsilon]\,[(p_0 - \omega_2)^2 - p^2 - m_2^2 + i\varepsilon]}$$

the term

$$\frac{1}{[(p_0 + \omega_1 - i\varepsilon)^2 - p^2 - m_1^2]\,[(p_0 - \omega_2 - i\varepsilon)^2 - p^2 - m_2^2]}$$

and that

$$\hat{R}(k, z_1, z_2, z_1', z_2') = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_0^{+\infty} dp_0\, dp \frac{e^{-ip_0(x_0 - x_0')} \sin pr \sin pr'}{[(p_0 + \omega_1 - i\varepsilon)^2 - p^2 - m_1^2]\,[(p_0 - \omega_2 - i\varepsilon)^2 - p^2 - m_2^2]}$$

will be a Volterra kernel and will lead us to the Riemann function. So let us make the following decomposition:

$$\begin{aligned} &\frac{1}{[(p_0 + \omega_1)^2 - p^2 - m_1^2 + i\varepsilon]\,[(p_0 - \omega_2)^2 - p^2 - m_2^2 + i\varepsilon]} \\ &= -\frac{2\pi i \theta(p_0 + \omega_1)\, \delta((p_0 + \omega_1)^2 - p^2 - m_1^2)}{(p - \omega_2)^2 - p^2 - m_2^2} \\ &\quad - \frac{2\pi i\, \theta(p_0 - \omega_2)\, \delta((p_0 - \omega_2)^2 - p^2 - m_2^2)}{(p_0 + \omega_1)^2 - p^2 - m_1^2} \\ &\quad + \frac{1}{[(p_0 + \omega_1 - i\varepsilon)^2 - p^2 - m_1^2]\,[(p_0 - \omega_2 - i\varepsilon)^2 - p^2 - m_2^2]} \end{aligned} \tag{18}$$

thus

$$\begin{aligned} G(k, x_0, r, x_0', r') = A_1(k, x_0, r, x_0', r') \\ + A_2(k, x_0, r, x_0', r') + \hat{R}(k, x_0, r, x_0', r') \end{aligned}$$

with

$$A_1(k, x_0, r, x_0', r') = -2\pi i \frac{1}{\pi^2} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} dp_0\, dp \cdot \theta(p_0+\omega_1)\, \delta((p_0+\omega_1)^2 - p^2 - m_1^2) \frac{e^{-ip_0(x_0-x_0')} \sin pr \sin pr'}{(p_0-\omega_2)^2 - p^2 - m_2^2}$$

and an analogous definition for A_2.

Firstly we will show that $\hat{R}(k, x_0, r, x_0', r')$ leads us to the Riemann function.

Introducing the new integration variables $q_1 = \frac{p_0+p}{2}$, $q_2 = \frac{p_0-p}{2}$ where $dp_0\, dp_1 = 2dq_1\, dq_2$ writing again $x_0 + r = z_1$, $x_0 - r = z_2$ and taking in mind $G(k, z_1, z_2, z_1', z_2') = \frac{1}{2} G(k, x_0, r, x_0', r')$ we have

$$\hat{R}(k, z_1, z_2, z_1', z_2') = \frac{1}{4\pi^2} \int\int_{-\infty}^{+\infty} dq_1 dq_2 \cdot \frac{e^{-iq_1(z_1-z_1')}\, e^{-iq_2(z_2-z_2')} - e^{-iq_1(z_1-z_2')}\, e^{-iq_2(z_2-z_1')}}{[(q_1+q_2+\omega_1-i\varepsilon)^2 - (q_1-q_2)^2 - m_1^2]\,[(q_1+q_2-\omega_2-i\varepsilon)^2 - (q_1-q_2)^2 - m_2^2]}. \tag{19}$$

Yet another representation will be convenient in the following. Introducing a Feynman parameter like SCHWARTZ and ZEMACH we obtain

$$\hat{R}(k, z_1, z_1', z_2') = \frac{1}{4\pi^2} \frac{1}{E} \int_{-\omega_2}^{\omega_1} dy\, e^{\frac{iy}{2}(z_1+z_2-z_1'-z_2')} \cdot \frac{\partial}{\partial Q^2(y)} \int\int_{-\infty}^{+\infty} dq_1\, dq_2 \frac{e^{-iq_1(z_1-z_1')}\, e^{-iq_2(z_2-z_2')} - (z_1' \leftrightarrow z_2')}{4(q_1-i\varepsilon)(q_2-i\varepsilon) - Q^2(y)} \tag{20}$$

where $Q^2(y) = y^2 - k^2$.

Evidently $\hat{R}$ can be written in the form

$$\begin{aligned} \hat{R}(k, z_1, z_2, z_1', z_2') &= A(k, z_1, z_2, z_1', z_2') - A(k, z_1, z_2, z_2', z_1') \\ \text{or} \quad &= A(k, z_1, z_2, z_1', z_2') - A(k, z_2, z_1, z_1', z_2') \end{aligned} \tag{21}$$

and the integral appearing in (20) can be calculated [7]:

$$\int\int_{-\infty}^{+\infty} dq_1\, dq_2 \frac{e^{-iq_1(z_1-z_1')}\, e^{-iq_2(z_2-z_2')}}{4(q_1-i\varepsilon)(q_2-i\varepsilon) - Q^2} = -\pi^2 \theta(z_1'-z_1)\, \theta(z_2'-z_2)\, \mathscr{I}_0(Q\sqrt{(z_1'-z_1)(z_1'-z_2)}) \tag{20a}$$

where $\mathscr{I}_0(z)$ is the Bessel function of zero order.
So we have

$$A(k, z_1, z_2, z_1', z_2') = \theta(z_1'-z_1)\, \theta(z_2'-z_2) \cdot \frac{1}{8E} \int_{-\omega_2}^{\omega_1} dy\, e^{\frac{iy}{2}(z_1+z_2-z_1'-z_2')} \frac{\sqrt{(z_1'-z_1)(z_2'-z_2)}}{Q(y)} \mathscr{I}_1(Q(y)\sqrt{(z_1'-z_1)(z_2'-z_2)})$$

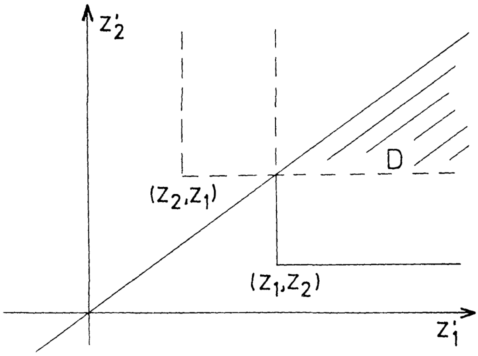

Fig. 3

and taking advantage of the series for $\mathcal{I}_1(z)$ we can write:

$$A(k, z_1, z_2, z_1', z_2') \tag{22}$$
$$= \theta(z_1' - z_1)\,\theta(z_2' - z_2)\frac{1}{8E}\sum_{l=0}^{\infty}\frac{(-)^l\, 2^{-2l-1}}{l!(l+1)!}F_l[(z_1' - z_1)(z_2' - z_2)]^{l+1}$$

with

$$F_l = \int_{-\omega_2}^{\omega_1} dy\, e^{\frac{iy}{2}(z_1 + z_2 - z_1' - z_2')}(y^2 - k^2)^l .$$

The first term of $\hat{R}$, $A(k, z_1, z_2, z_1', z_2')$ does not vanish only in the first quadrant starting from (z_1, z_2) while the second term $A(k, z_2, z_1, z_1', z_2')$ does not vanish only in the first quadrant starting from (z_2, z_1). On $z_1' = z_2'$ both terms are equal, hence $\hat{R} = 0$. Because we consider the domain $z_1' > z_2'$ in D, the second term contributes only in the hatched domain. Applying the differential operator M_1' or M_2' to $\hat{R}$ we get

$$M_1'\hat{R}(k, z_1, z_2, z_1', z_2')$$
$$= e^{\frac{-i\omega_2}{2}(z_1 + z_2 - z_1' - z)}\frac{(-1)}{4\pi^2}\iint_{-\infty}^{+\infty} dq_1\, dq_2\, \frac{e^{-iq_1(z_1 - z_1')}\, e^{-iq_2(z_2 - z_2')} - (z_1' \leftrightarrow z_2')}{4(q_1 - i\varepsilon)(q_2 - i\varepsilon) - m_2^2}$$

hence with (20a):

$$M_1'\hat{R}(k, z_1, z_2, z_1', z_2') = \frac{1}{4}e^{-i\frac{\omega_2}{2}(z_1 + z_2 - z_1' - z_2')}[\theta(z_1' - z_1)\,\theta(z_2' - z_2) \tag{23}$$
$$\cdot \mathcal{I}_0(m_2\sqrt{(z_1' - z_1)(z_2' - z_2)}) - \theta(z_1' - z_2)\,\theta(z_2' - z_1)\,\mathcal{I}_0(m_2\sqrt{(z_1' - z_2)(z_2' - z_1)})]$$

and analogously we find:

$$M_2'\hat{R}(k, z_1, z_2, z_1', z_2') = \frac{1}{4}e^{\frac{i\omega_1}{2}(z_1 + z_2 - z_1' - z_2')}[\theta(z_1' - z_1)\,\theta(z_2' - z_2) \tag{24}$$
$$\cdot \mathcal{I}_0(m_1\sqrt{(z_1' - z_1)(z_2' - z_2)}) - \theta(z_1' - z_2)\,\theta(z_2' - z_1)\,\mathcal{I}_0(m_1\sqrt{(z_1' - z_2)(z_2' - z_1)})]$$

Now it is easily seen that $\hat{R}$ will be identical with the Riemann function if we omit the $\theta(z_1' - z_1)\,\theta(z_2' - z_2)$ in front of $A(k, z_1, z_2, z_1', z_2')$. (13b, c) follows from (22), (23), (24) taking in mind the series of $\mathscr{I}_0(z)$ [7]

$$\mathscr{I}_0(m_i\sqrt{(z_1' - z_1)\,(z_2' - z_2)}) = 1 - \frac{m_i^2}{4}\,(z_1' - z_1)\,(z_2' + z_2) + - \cdots$$

(13d) follows from (21), (23), (24); (13e) follows from (23), (24).

It is $M_1' M_2' \hat{R}(k, z_1, z_2, z_1', z_2') = \delta(z_1' - z_1)\,\delta(z_2' - z_2) - \delta(z_1' - z_2)\,\delta(z_2' - z_1)$ but the second term on the right hand side always vanishes because the support of the δ-functions is in the inadmissable domain $r' < 0$. Hence after cancellation of the $\theta(z_1' - z_1)\,\theta(z_2' - z_2)$ in front of $A(k, z_1, z_2, z_1', z_2')$ which produce the $\delta(z_1' - z_1)\,\delta(z_2' - z_2)$ also (13a) will be fulfilled. So it is shown that

$$\begin{aligned}
&R(k, z_1, z_2, z_1', z_2') \\
&= \frac{1}{4E}\int_{-\omega_2}^{\omega_1} dy\, e^{\frac{iy}{2}(z_1 + z_2 - z_1' - z_2')}\,\frac{\sqrt{(z_1' - z_1)\,(z_2' - z_2)}}{2Q(y)}\,\mathscr{I}_1(Q(y)\sqrt{(z_1' - z_1)\,(z_2' - z_2)}) \\
&\quad - \theta(z_1' - z_2)\,\theta(z_2' - z_1) \\
&\times \frac{1}{4E}\int_{-\omega_2}^{\omega_1} dy\, e^{\frac{iy}{2}(z_1 + z_2 - z_1' - z_2')}\,\frac{\sqrt{(z_1' - z_2)\,(z_2' - z_1)}}{2Q(y)}\,\mathscr{I}_1(Q(y)\sqrt{(z_1' - z_2)\,(z_2' - z_1)})
\end{aligned}$$

is a solution of the boundary value problem. But this solution is unique, hence R is the Riemann function.

It may be noted that

$$\begin{aligned}
&\int_{z_1}^{\infty}\int_{z_2}^{\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2')\,\hat{R}(k, z_1, z_2, z_1', z_2')\, V(z_1', z_2')\, u(z_1', z_2') \\
&= \int_{z_1}^{\infty}\int_{z_2}^{\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2')\, R(k, z_1, z_2, z_1', z_2')\, V(z_1', z_2')\, u(z_1', z_2')\,.
\end{aligned}$$

Riemann function R and retarded Green's function $\hat{R}$ are different only through the $\theta(z_1' - z_1)\,\theta(z_1' - z_2)$ in front of $A(k, z_1, z_2, z_1', z_2')$. So it is $M_1' M_2' \hat{R} = \delta(z_1' - z_1)\,\delta(z_2' - z_2)$ while $M_1' M_2' R = 0$.

3.4. *Proof of Convergence*

Now we can use the knowledge of the Riemann function to prove that the integral equation (15) can be solved by iteration for all λ. As potential we take the Yukawa potential

$$V(x) = \frac{i}{\pi^2}\int d^4 k\,\frac{e^{-ikx}}{k^2 - \mu^2 + i\varepsilon}$$

or

$$V(z_1, z_2) = 2\pi\,\mu^2\,\frac{H_1^{(2)}(\mu\sqrt{z_1 z_2 - i\varepsilon})}{\mu\sqrt{z_1 z_2 - i\varepsilon}}\,.$$

This proof will at the same time exhibit a whole class of potentials for which the iteration of the integral equation does converge.

Using the integral representation for $A(k, z_1, z_2, z_1', z_2')$

$$A(k, z_1, z_2, z_1', z_2') = \frac{1}{4\pi^2} \int\int_{-\infty}^{+\infty} dq_1\, dq_2 \frac{e^{-iq_1(z_1 - z_1')}\, e^{-iq_2(z_2 - z_2')}}{[4q_1q_2 + 2\omega_1(q_1+q_2) + k^2 - 2i\varepsilon(q_1+q_2+\omega_1)] \cdot [4q_1q_2 - 2\omega_2(q_1+q_2) + k^2 - 2i\varepsilon(q_1+q_2+\omega_2)]}$$

and performing the q_1-integration we get

$$A(k, z_1, z_2, z_1', z_2') = \theta(z_1' - z_1)\,\theta(z_2' - z_2)\frac{1}{4\pi^2}\sum_{i=1}^{4} I_i(k, z_1, z_2, z_1', z_2') \tag{25}$$

where

$$I_1(k, z_1, z_2, z_1', z_2') = \frac{-i\pi}{4E} e^{\frac{i\omega_1}{2}(z_1+z_2-z_1'-z_2')} \int_0^\infty dq \frac{e^{+i\frac{m_1^2}{4q}(z_1'-z_1)}\, e^{+iq(z_2'-z_2)}}{\left(q - \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}}$$

$$I_2(k, z_1, z_2, z_1', z_2') = \frac{-i\pi}{4E} e^{\frac{i\omega_1}{2}(z_1+z_2-z_1'-z_2')} \int_0^\infty dq \frac{e^{-i\frac{m_1^2}{4q}(z_1'-z_1)}\, e^{-iq(z_2'-z_2)}}{\left(q + \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}}$$

$$I_3(k, z_1, z_2, z_1', z_2') = \frac{i\pi}{4E} e^{-\frac{i\omega_2}{2}(z_1+z_2-z_1'-z_2')} \int_0^\infty dq \frac{e^{i\frac{m_2^2}{4q}(z_1'-z_1)}\, e^{+iq(z_2'-z_2)}}{\left(q + \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}}$$

$$I_4(k, z_1, z_2, z_1', z_2') = \frac{i\pi}{4E} e^{-\frac{i\omega_2}{2}(z_1+z_2-z_1'-z_2')} \int_0^\infty dq \frac{e^{-i\frac{m_2^2}{4q}(z_1'-z_1)}\, e^{-iq(z_2'-z_2)}}{\left(q - \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}}.$$

By the method of stationary phase [8] we can show that for the bound state case i.e. for $\mathbf{k}^2 < 0$ each $I_i(k, z_1, z_2, z_1', z_2')$ can be estimated by

$$|I_i(k, z_1, z_2, z_1', z_2')| \leqq c \frac{1}{1 + [(z_1' - z_1)(z_2' - z_2)]^{1/4}}$$

hence we can write:

$$|A(k, z_1, z_2, z_1', z_2')| \leqq c \frac{|z_1' - z_1|}{1 + |z_1' - z_1|^{5/4}} \frac{|z_2' - z_2|}{1 + |z_2' - z_2|^{5/4}}.$$

Now it is readily seen that for $z_1 < 0$ or $z_2 < 0$ the possible upper bound for the first iteration

$$\int_{z_1}^{\infty}\int_{z_2}^{\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2')\, |R(k, z_1, z_2, z_1', z_2')|\, |V(z_1', z_2')|\, |u_i^\infty(z_1', z_2')| \leqq$$

$$\leqq \int_{z_1}^{\infty}\int_{z_2}^{\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2')\, |R(k, z_1, z_2, z_1', z_2')|\, |V(z_1', z_2')|$$

does not exist because of the bad decrease of $V(z_1, z_2)$ as well as the $\left(\frac{1}{z_1 z_2 - i\varepsilon}\right)$-singularities on $z_1 z_2 = 0$. Thus the proof can not be established in the usual manner. A way out is the following:

First we take into account the asymptotic expansion of $H_1^{(2)}(z)$ and the expansion about $z_1 z_2 = 0$ [7], thus decompose $V(z_1, z_2)$ as follows:

$$V(z_1, z_2) = V_1(z_1, z_2) + V_2(z_1, z_2) + V_3(z_1, z_2)$$

with

$$V_1(z_1, z_2) = 4i \frac{1}{z_1 z_2 - i\varepsilon}$$

$$V_2(z_1, z_2) = (8\pi\mu)^{1/2} e^{i\,3/4\pi} (z_1 \cdot z_2)^{-3/4} e^{-i\mu\sqrt{z_1 z_2}}$$

then we can write for $V_3(z_1, z_2)$ the estimation:

$$|V_3(z_1, z_2)| \leqq c\,|z_1 z_2|^{-3/4} \frac{1}{1 + |z_1 z_2|^{1/4}}\,.$$

Then we consider

$$\begin{aligned} R^{(2)}(k, z_1, z_2, z_1'', z_2'') &= \lambda R V R \\ &= \theta(z_1'' - z_1)\,\theta(z_2'' - z_2)\,\lambda \int_{z_1}^{z_1''} \int_{z_2}^{z_2''} dz_1'\,dz_2'\,A(k, z_1, z_2, z_1', z_2') \\ &\quad \cdot V(z_1', z_2')\,A(k, z_1', z_2', z_1'', z_2'') \\ &\quad - \theta(z_1'' - z_2)\,\theta(z_2'' - z_1)\,\lambda \int_{z_2}^{z_1''} \int_{z_1}^{z_2''} dz_1'\,dz_2'\,A(k, z_2, z_1, z_1', z_2') \\ &\quad \cdot V(z_1', z_2')\,A(k, z_1', z_2', z_1'', z_2'')\,. \end{aligned}$$

It can be shown [9] that for $R^{(2)}$ we have the following estimation:

$$\begin{aligned} &|R^{(2)}(k, z_1, z_2, z_1'', z_2'')| \\ &\leqq c \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4-\varepsilon}} \frac{1}{1 + |z_2|^{1/4-\varepsilon}} \frac{1}{1 + |z_1''|^{1/4-\varepsilon}} \frac{1}{1 + |z_2''|^{1/4-\varepsilon}} \end{aligned} \tag{26}$$

$\varepsilon > 0$.

Now write

$$\begin{aligned} \varphi^0(z_1, z_2) &= \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1'\,dz_2'\,\theta(z_1' - z_2')\,R(k, z_1, z_2, z_1', z_2') \\ &\quad \cdot V(z_1', z_2')\,u_j(q, z_1', z_2') \\ &\quad + \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1'\,dz_2'\,\theta(z_1' - z_2')\,R^{(2)}(k, z_1, z_2, z_1', z_2') \\ &\quad \cdot V(z_1', z_2')\,u_j(q, z_1', z_2'); \quad j = 1 \text{ or } 2 \end{aligned}$$

$$\begin{aligned} \varphi_1^{n+1}(z_1, z_2) &= \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1'\,dz_2'\,\theta(z_1' - z_2')\,R^{(2)}(k, z_1, z_2, z_1', z_2') \\ &\quad \cdot V_1(z_1', z_2')\,\varphi^n(z_1', z_2') \end{aligned}$$

$$\begin{aligned} \varphi_2^{n+1}(z_1, z_2) &= \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1'\,dz_2'\,\theta(z_1' - z_2')\,R^{(2)}(k, z_1, z_2, z_1', z_2') \\ &\quad \cdot \left(V_2(z_1', z_2') + V_3(z_1', z_2')\right)\,\varphi^n(z_1', z_2') \end{aligned}$$

and let us take $z_1 < 0$, $z_2 < 0$; for the other cases the proof is not so difficult so that we may omit it here.

The estimations

$$|\varphi^n(z_1, z_2)| = |\varphi_1^n(z_1, z_2) + \varphi_2^n(z_1, z_2)|$$
$$\leqq \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4-\varepsilon}} \frac{1}{1 + |z_2|^{1/4-\varepsilon}} \frac{1}{n!} \left[\int_{z_1+z_2}^{\infty} dz'\, F(z') \right]^n \tag{27a}$$

$$|\varphi^n(z_1, z_2) - \varphi^n(z_1, 0) - \varphi^n(0, z_2)|$$
$$\leqq \frac{|z_1|}{1 + |z_1|} \frac{|z_2|}{1 + |z_2|} \frac{1}{n!} \left[\int_{z_1+z_2}^{\infty} dz'\, F(z') \right]^n \tag{27b}$$

where[1]

$$F(z) = c \frac{1}{1 + |z|^{1/4-\varepsilon}} \frac{1}{1 + |z|^{1/4-\varepsilon}} |z|^{-3/4}$$

will now be shown by induction. For $n = 0$ the estimations are shown in the appendix.

Firstly because of $|V_2(z_1, z_2) + V_3(z_1, z_2)| \leqq c|z_1 z_2|^{-3/4}$ we have

$$|\varphi_2^{n+1}(z_1, z_2)| \leqq c \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4-\varepsilon}} \frac{1}{1 + |z_2|^{1/4-\varepsilon}}$$
$$\cdot \int_{z_1}^{\infty}\int_{z_2}^{\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2') \frac{1}{1 + |z_1'|^{1/4-\varepsilon}} \frac{1}{1 + |z_2'|^{1/4-\varepsilon}} |z_1' z_2'|^{-3/4}$$
$$\cdot \frac{1}{1 + |z_1'|^{1/4-\varepsilon}} \frac{1}{1 + |z_2'|^{1/4-\varepsilon}} \cdot \frac{1}{n!} \left[\int_{z_1'+z_2'}^{\infty} dz''\, F(z'') \right]^n .$$

Taking $z' = z_1' + z_2'$, $r' = z_1' - z_2'$ as new variables we can perform the r'-integration. The integral can be estimated by

$$c \int_0^{\infty} dr' \frac{1}{1 + |z' + r'|^{1/4-\varepsilon}} \frac{1}{1 + |z' - r'|^{1/4-\varepsilon}} |z'^2 - r'^2|^{-3/4}$$
$$\cdot \frac{1}{1 + |z' + r'|^{1/4-\varepsilon}} \frac{1}{1 + |z' - r'|^{1/4-\varepsilon}} \leqq F(z')$$

hence we get

$$|\varphi_2^{n+1}(z_1, z_2)| \leqq \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4-\varepsilon}} \frac{1}{1 + |z_2|^{1/4-\varepsilon}} \frac{1}{n!}$$
$$\cdot \int_{z_1+z_2}^{\infty} dz'\, F(z') \left[\int_{z'}^{\infty} dz''\, F(z'') \right]^n$$
$$= \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4-\varepsilon}} \frac{1}{1 + |z_2|^{1/4-\varepsilon}} \frac{1}{(n+1)!} \left[\int_{z_1+z_2}^{\infty} dz'\, F(z') \right]^{n+1} .$$

[1] All inessential factors may be gathered in the following in c.

To estimate $\varphi_1^{n+1}(z_1, z_2)$ we write

$$|\varphi_1^{n+1}(z_1, z_2)| \leqq 4\lambda \int_{z_1}^{\infty}\int_{z_2}^{\infty} dz_1'\, dz_2'\, \theta(z_1' - z_2')\, |R^{(2)}(k, z_1, z_2, z_1', z_2')| \left|\frac{1}{z_1' z_2'}\right|$$
$$\cdot\, |\varphi^n(z_1', z_2') - \varphi^n(z_1', 0) - \varphi^n(0, z_2')|$$
$$+ 4\lambda \int_{z_1}^{\infty} dz_1' \left| \int_{z_2}^{\infty} dz_2'\, \theta(z_1' - z_2')\, R^{(2)}(k, z_1, z_2, z_1', z_2') \frac{1}{z_1' z_2' - i\varepsilon} \right| |\varphi^n(z_1', 0)|$$
$$+ 4\lambda \int_{z_2}^{\infty} dz_2' \left| \int_{z_1}^{\infty} dz_1'\, \theta(z_1' - z_2')\, R^{(2)}(k, z_1, z_2, z_1', z_2') \frac{1}{z_1' z_2' - i\varepsilon} \right| |\varphi^n(0, z_2')|\,.$$

Take $z_1' > 0$ and let a be a number with $z_2 < -a < 0 < a < z_1'$ then

$$\left| \int_{z_2}^{\infty} dz_2'\, \theta(z_1' - z_2')\, R^{(2)}(k, z_1, z_2, z_1', z_2') \frac{1}{z_1' z_2' - i\varepsilon} \right|$$
$$= \frac{1}{|z_1'|} \left| \int_{z_2}^{-a} dz_2'\, \theta(z_1' - z_2')\, R^{(2)}(k, z_1, z_2, z_1', z_2') \frac{1}{z_2' - \frac{i\varepsilon}{z_1'}} \right.$$
$$+ \int_{a}^{\infty} dz_2'\, \theta(z_1' - z_2')\, R^{(2)}(k, z_1, z_2, z_1', z_2') \frac{1}{z_2' - \frac{i\varepsilon}{z_1'}}$$
$$+ \int_{-a}^{+a} dz_2'\, \theta(z_1' - z_2')\, (R^{(2)}(k, z_1, z_2, z_1', z_2') - R^{(2)}(k, z_1, z_2, z_1', 0)) \frac{1}{z_2' - \frac{i\varepsilon}{z_1'}}$$
$$\left. + R^{(2)}(k, z_1, z_2, z_1', 0) \int_{-a}^{+a} dz_2' \frac{1}{z_2' - \frac{i\varepsilon}{z_1'}} \right|$$
$$\leqq c \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4-\varepsilon}} \frac{1}{1 + |z_2|^{1/4-\varepsilon}} \frac{1}{1 + |z_1'|^{1/4-\varepsilon}} \frac{1}{|z_1'|}\,.$$

Here we have estimated the four terms separately and we have used (25) and

$$|R^{(2)}(k, z_1, z_2, z_1', z_2') - R^{(2)}(k, z_1, z_2, z_1', 0)|$$
$$\leqq c \frac{|z_2'|}{1 + |z_2'|} \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4-\varepsilon}} \frac{1}{1 + |z_2|^{1/4-\varepsilon}} \frac{1}{1 + |z_1'|^{1/4-\varepsilon}}\,.$$

Evidently the above estimation is correct also for $z_1' \leqq 0$. In the same manner we get

$$\left|\int\limits_{z_1}^{\infty} dz_1'\, \theta(z_1'-z_2')\, R^{(2)}(k, z_1, z_2, z_1', z_2') \frac{1}{z_1' z_2' - i\varepsilon}\right|$$
$$\leqq c \frac{|z_1-z_2|}{1+|z_1-z_2|} \frac{1}{1+|z_1|^{1/4-\varepsilon}} \frac{1}{1+|z_2|^{1/4-\varepsilon}} \frac{1}{1+|z_2'|^{1/4-\varepsilon}} \frac{1}{|z_2'|}$$

hence

$$|\varphi_1^{n+1}(z_1, z_2)| \leqq c \frac{|z_1-z_2|}{1+|z_1-z_2|} \frac{1}{1+|z_1|^{1/4-\varepsilon}} \frac{1}{1+|z_2|^{1/4-\varepsilon}}$$
$$\cdot \left\{\int\limits_{z_1}^{\infty} dz_1' \frac{1}{1+|z_1'|^{1/4-\varepsilon}} \frac{1}{1+|z_1'|} \frac{1}{1+|z_1'|^{1/4-\varepsilon}} \frac{1}{n!}\left[\int\limits_{z_1}^{\infty} dz''\, F(z'')\right]^n\right.$$
$$+\int\limits_{z_1}^{\infty}\int\limits_{z_2}^{\infty} dz_1'\, dz_2'\, \theta(z_1'-z_2') \frac{1}{1+|z_1'|^{1/4-\varepsilon}} \frac{1}{1+|z_2'|^{1/4-\varepsilon}} \frac{1}{|z_1' z_2'|} \frac{|z_1'|}{1+|z_1'|} \frac{|z_2'|}{1+|z_2'|}$$
$$\cdot \frac{1}{n!}\left[\int\limits_{z_1'+z_2'}^{\infty} dz''\, F(z'')\right]^n + \int\limits_{z_2}^{\infty} dz_2' \frac{1}{1+|z_2'|^{1/4-\varepsilon}} \frac{1}{1+|z_2'|}$$
$$\left.\cdot \frac{1}{1+|z_2'|^{1/4-\varepsilon}} \frac{1}{n!}\left[\int\limits_{z_2'}^{\infty} dz''\, F(z'')\right]^n\right\}.$$

Again we introduce in the double integral $z' = z_1' + z_2'$, $r' = z_1' - z_2'$ and estimate the r'-integral by

$$\int\limits_0^{\infty} dr' \frac{1}{1+|z'+r'|^{1/4-\varepsilon}} \frac{1}{1+|z'-r'|^{1/4-\varepsilon}} \frac{1}{1+|z'+r'|} \frac{1}{1+|z'-r'|}$$
$$\leqq c \frac{1}{1+|z'|^{1/4-\varepsilon}} \frac{1}{1+|z'|} \leqq F(z').$$

Likewise we set in the other integrals

$$\frac{1}{1+|z_i|^{1/4-\varepsilon}} \frac{1}{1+|z_i|} \frac{1}{1+|z_i|^{1/4-\varepsilon}} \leqq F(z_i) \quad i = 1, 2$$

so that finally we obtain

$$|\varphi_1^{n+1}(z_1, z_2)| \leqq \frac{|z_1-z_2|}{1+|z_1-z_2|} \frac{1}{1+|z_1|^{1/4-\varepsilon}} \frac{1}{1+|z_2|^{1/4-\varepsilon}}$$
$$\cdot \left\{\frac{1}{(n+1)!}\left[\int\limits_{z_1}^{\infty} dz''\, F(z'')\right]^{n+1}\right.$$
$$\left.+\frac{1}{(n+1)!}\left[\int\limits_{z_1+z_2}^{\infty} dz''\, F(z'')\right]^{n+1} + \frac{1}{(n+1)!}\left[\int\limits_{z_2}^{\infty} dz''\, F(z'')\right]^{n+1}\right\}.$$

Because of $z_1 < 0$, $z_2 < 0$ it is $\int\limits_{z_i}^{\infty} dz'' F(z'') < \int\limits_{z_1+z_2}^{\infty} dz'' F(z'')$ so we get the desired bound for $\varphi_1^{n+1}(z_1, z_2)$ and hence for $\varphi^{n+1}(z_1, z_2)$.

The estimation for $|\varphi^{n+1}(z_1, z_2) - \varphi^{n+1}(z_1, 0) - \varphi^{n+1}(0, z_2)|$ is done in the same manner, we have to use here

$$|R^{(2)}(k, z_1, z_2, z_1'', z_2'') - R^{(2)}(k, z_1, 0, z_1'', z_2'') - R^{(2)}(k, 0, z_2, z_1'', z_2'')|$$
$$\leq c \frac{1}{1+|z_1''|^{1/4-\varepsilon}} \frac{1}{1+|z_2''|^{1/4-\varepsilon}} \frac{|z_1|}{1+|z_1|} \frac{|z_2|}{1+|z_2|}.$$

The solution of (15) is now

$$\varphi(z_1, z_2) = u_3(q, z_1, z_2) + \sum_{n=0}^{\infty} \varphi^n(z_1, z_2)$$

which thus can be estimated by

$$|\varphi(z_1, z_2)| \leq c' \frac{|z_1 - z_2|}{1+|z_1 - z_2|}$$
$$+ \frac{|z_1 - z_2|}{1+|z_1 - z_2|} \frac{1}{1+|z_1|^{1/4-\varepsilon}} \frac{1}{1+|z_2|^{1/4-\varepsilon}} e^{\int\limits_{z_1+z_2}^{\infty} dz'' F(z'')}.$$

The bad decrease of $V(z_1, z_2)$ for $|z_1 z_2| \to \infty$ forced us to go to the iterated kernel $R^{(2)}(k, z_1, z_2, z_1', z_2')$. But having done the estimation for $R^{(2)}$ and the above mentioned estimation we see immediately that all potentials which decrease faster than $|z_1 z_2|^{-3/4-\varepsilon}$, $\varepsilon > 0$ for $|z_1 z_2| \to \infty$ and which are less singular as $(z_1 z_2)^{-2+\varepsilon}$, $\varepsilon > 0$ on $z_1 z_2 = 0$ admit cruder methods of estimation. For all other potentials like the Yukawa potential we have to take into account their special properties like the oscillating function $e^{-i\mu\sqrt{z_1 z_2}}$ if generally a proof can be established.

3.5. Representation of the Wave Function

We turn now to the remaining terms in the decomposition of $G(k, z_1, z_2, z_1', z_2')$. Proceeding analogously as with $\hat{R}(k, z_1, z_2, z_1', z_2')$ we get

$$A_1(k, z_1, z_2, z_1', z_2')$$
$$= \frac{-1}{2i\pi} e^{\frac{i\omega_1}{2}(z_1+z_2-z_1'-z_2')} \iint\limits_{-\infty}^{+\infty} dq_1\, dq_2\, \theta(q_1+q_2)\, \delta(4q_1 q_2 - m_1^2)$$
$$\cdot \frac{e^{iq_1(z_1'-z_1)}\, e^{iq_2(z_2'-z_2)} - (z_1' \leftrightarrow z_2')}{(q_1+q_2-i\varepsilon)^2 - (q_1-q_2)^2 - m_2^2} = \frac{1}{\pi} \int\limits_0^{\infty} dq \frac{u_1(q, z_1, z_2)\, \omega_{11}^*(q, z_1', z_2')}{8E\left(\left(q - \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}\right)}$$

where we have used the definitions (8a—g).

Similarly

$$A_2(k, z_1, z_2, z_1', z_2') = \frac{1}{\pi} \int\limits_0^{\infty} dq \frac{u_2(q, z_1, z_2)\, \omega_{21}^*(q, z_1', z_2')}{-8E\left(\left(q + \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}\right)}.$$

From the integral equation (17) we obtain

$$\psi(k_1, z_1, z_2) = u_0(k, z_1, z_2) + \lambda \int\int_{-\infty}^{+\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2') \, [A_1(k, z_1, z_2, z_1', z_2')$$

$$+ A_2(k, z_1, z_2, z_1', z_2') + \hat{R}(k, z_1, z_2, z_1', z_2')] \, V(z_1', z_2') \, \psi(k, z_1', z_2')$$

$$= u_0(k, z_1, z_2) + \int_0^\infty dq \frac{u_1(q, z_1, z_2) \, \Gamma_1(q, k)}{8E\left(\left(q - \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}\right)}$$

$$+ \int_0^\infty dq \frac{u_2(q, z_1, z_2) \, \Gamma_2(q, k)}{-8E\left(\left(q + \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}\right)}$$

$$+ \lambda \int_{z_1}^\infty \int_{z_2}^\infty dz_1' \, dz_2' \, \theta(z_1' - z_2') \, R(k, z_1, z_2, z_1', z_2') \, V(z_1', z_2') \, \psi(k, z_1', z_2')$$

where

$$\Gamma_i(q, k) = \frac{1}{\pi} \lambda \int\int_{-\infty}^{+\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2) \, \omega_{i1}^*(q, z_1', z_2') \, V(z_1', z_2') \, \psi(k, z_1', z_2') \, .$$

Remembering the special solutions $\varphi_i(q, z_1, z_2)$ defined by the integral equations (15) and putting

$$\varphi_1\left(\frac{\omega_1}{2} + \frac{k}{2}, z_1, z_2\right) = \varphi_2\left(-\frac{\omega_2}{2} + \frac{k}{2}, z_1, z_2\right) = \varphi_0(k, z_1, z_2)$$

we see that $\psi(k, z_1, z_2)$ can be written as a superposition of these special solutions:

$$\begin{aligned} \psi(k, z_1, z_2) = \varphi_0(k, z_1, z_2) &+ \int_0^\infty dq \frac{\varphi_1(q, z_1, z_2) \, \Gamma_1(q, k)}{8E\left(\left(q - \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}\right)} \\ &+ \int_0^\infty dq \frac{\varphi_2(q, z_1, z_2) \, \Gamma_2(q, k)}{-8E\left(\left(q + \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}\right)} \, . \end{aligned} \tag{28}$$

Thus we naturally obtain the expansion of the wave function in terms of the special solutions from the decomposition of the Green's function. The coefficients $\Gamma_i(q, k)$ in this expansion also can be expressed easily, inserting (28) into the definition of $\Gamma_i(q, k)$. We obtain

$$\begin{aligned} \Gamma_i(q, k) = K_{i0}(q, k) &+ \int_0^\infty dq' \frac{K_{i1}(q, q') \, \Gamma_1(q', k)}{8E\left(\left(q' - \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}\right)} \\ &+ \int_0^\infty dq' \frac{K_{i2}(q, q') \, \Gamma_2(q', k)}{-8E\left(\left(q' + \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}\right)} \end{aligned} \tag{29}$$

with

$$K_{ij}(q, q') = \frac{1}{\pi} \int\int_{-\infty}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2') \, \omega_{i1}^*(q, z_1', z_2') \, V(z_1', z_2') \, \varphi_j(q', z_1', z_2')$$

$$j = 0, 1, 2 .$$

In the Schrödinger case we have instead of (29):

$$a(k) = k - g(k) \, a(k) \tag{30}$$

with

$$g(k) = \int_0^\infty dr' \, e^{ikr'} \, V(r') \, \varphi(k, r') .$$

This is an equation which can be solved readily. In the Bethe-Salpeter case because of the partial differential equation and the two differential operators this equation is blown up to a system of integral equations for the two unknown functions $\Gamma_i(q, k)$. Remarkable is also the similar structure of the kernels $K_{ij}(q, q')$ and $g(k)$.

We have bound states in (30) if the homogeneous equation can be solved i.e. if $a(k) = -g(k) \, a(k)$ or $f(-k) = 0$. In the Bethe-Salpeter case we have bound states for those k for which the homogeneous system of (29) can be solved, i.e. the Fredholm determinant of the system is zero. Thus the Fredholm determinant is the natural generalization of the Jost function [4].

Instead of (18) we could have made the following decomposition

$$\begin{aligned}
&\frac{1}{[(p_0 + \omega_1)^2 - p^2 - m_1^2 + i\varepsilon] \, [(p_0 - \omega_2)^2 - p^2 - m_2^2 + i\varepsilon]} \\
&\qquad = \frac{-2\pi i \theta(-p_0 - \omega_1) \, \delta((p_0 + \omega_1)^2 - p^2 - m_1^2)}{(p_0 - \omega_2)^2 - p^2 - m_2^2} \\
&\qquad - \frac{2\pi i \theta(-p_0 + \omega_2) \, \delta((p_0 - \omega_2)^2 - p^2 - m_2^2)}{(p_0 + \omega_1)^2 - p^2 - m_1^2} \\
&\qquad + \frac{1}{[(p_0 + \omega_1 + i\varepsilon)^2 - p^2 - m_1^2] \, [p_0 - \omega_2 + i\varepsilon)^2 - p^2 - m_2^2]}
\end{aligned}$$

and thus we would have obtained

$$G(k, z_1, z_2, z_1', z_2') = B_1(k, z_1, z_2, z_1', z_2') + B_2(k, z_1, z_2, z_1', z_2') + \tilde{R}(k, z_1, z_2, z_1', z_2')$$

with

$$B_1(k, z_1, z_2, z_1', z_2') = \frac{1}{\pi} \int_{-\infty}^{0} dq \, \frac{u_1(q, z_1, z_2) \, \omega_{11}^*(q, z_1', z_2')}{-8E\left(\left(q - \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}\right)}$$

and

$$B_2(k, z_1, z_2, z_1', z_2') = \frac{1}{\pi} \int_{-\infty}^{0} dq \, \frac{u_2(q, z_1, z_2) \, \omega_{21}^*(q, z_1', z_2')}{8E\left(\left(q + \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}\right)}$$

and

$$\tilde{R}(k, z_1, z_2, z_1', z_2') = \frac{1}{4\pi^2}\frac{1}{E}\int_{-\omega_2}^{\omega_1} dy\, e^{\frac{iy}{2}(z_1+z_2-z_1'-z_2')}\frac{\partial}{\partial Q^2(y)}\int\int_{-\infty}^{+\infty} dq_1\, dq_2 \cdot \frac{e^{-iq_1(z_1-z_1')}\, e^{-iq_2(z_2-z_2')} - (z_1' \leftrightarrow z_2')}{4(q_1+i\varepsilon)(q_2+i\varepsilon) - Q^2(y)}.$$

Similar to $\hat{R}\tilde{R}$ leads to a Riemann function. This Riemann function is zero outside the domain D' (Fig. 4) and satisfies similar conditions on the boundary of D' as R on the boundary of D.

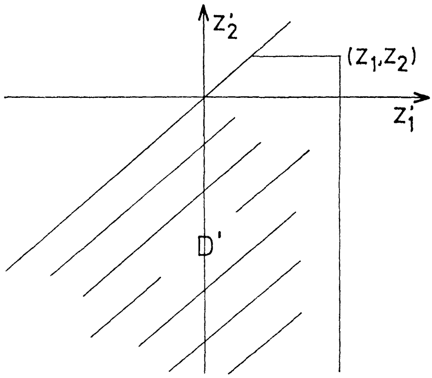

Fig. 4

In the same manner we can define special solutions:

$$\tilde{\varphi}_i(q, z_1, z_2) = u_i(q, z_1, z_2) + \lambda \int_{-\infty}^{z_1}\int_{-\infty}^{z_2} dz_1'\, dz_2'\, \theta(z_1' - z_2') \cdot \tilde{R}(k, z_1, z_2, z_1', z_2')\, V(z_1', z_2')\, \tilde{\varphi}_i(q, z_1', z_2'); \quad i = 0, 1, 2$$

and these integral equations can also be solved by iteration.

Thus we get the decomposition

$$\psi(k, z_1, z_2) = \tilde{\varphi}_0(k, z_1, z_2) + \int_{-\infty}^{0} dq \frac{\tilde{\varphi}_1(q, z_1, z_2)\, \Gamma_1(q, k)}{-8E\left(\left(q - \frac{\omega_1}{2}\right)^2 - \frac{k^2}{4}\right)} + \int_{-\infty}^{0} dq \frac{\varphi_2(q, z_1, z_2)\, \Gamma_2(q, k)}{8E\left(\left(q + \frac{\omega_2}{2}\right)^2 - \frac{k^2}{4}\right)}$$

in terms of the special solutions $\tilde{\varphi}_i(q, z_1, z_2)$. The asymptotic behaviour for $z_2 \to -\infty$ is determined by $u_i(q, z_1, z_2)$. Furthermore $\tilde{\varphi}_i(q, z_1, z_2)$ have to vanish on $z_1 = z_2$ as $u_i(q, z_1, z_2)$. The coefficients $\Gamma_i(q, k)$ are the same as in (28) but the integration now runs from $-\infty$ to 0.

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Appendix

In this appendix we will establish the estimations (27) for $n = 0$ [9]: Consider:

$$\psi_2^1(z_1, z_2) = \frac{1}{2i} \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, A(k, z_1, z_2, z_1', z_2') \, V_2(z_1', z_2') \, \omega_{j1}(q, z_1', z_2')$$

$$j = 1 \text{ or } 2 .$$

Using the representation (25) for A and writing

$$I_i(k, z_1, z_2, z_1', z_2') = f_i(z_1, z_2, z_1', z_2') \, g_i(z_1, z_2, z_1', z_2')$$

where e.g.

$$f_1(z_1, z_2, z_1', z_2') = e^{\frac{i\omega_1}{2}(z_1 + z_2 - z_1' - z_2')} \, e^{i m_1 \sqrt{(z_1' - z_1)(z_2' - z_2)}}$$

we have

$$\psi_2^1(z_1, z_2) = \frac{1}{2i} \frac{1}{4\pi^2} \sum_{i=1}^{4} \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, g_i(z_1, z_2, z_1', z_2') \, h_i'(z_1, z_2, z_1', z_2')$$

with

$$h_i'(z_1, z_2, z_1', z_2') = f_i(z_1, z_2, z_1', z_2') \, (8\pi\mu)^{1/2} e^{i\,3/4\pi} (z_1' \cdot z_2')^{-3/4} e^{-i\mu\sqrt{z_1' z_2'}} \omega_{j1}(q, z_1', z_2').$$

Let us study the decrease in z_1: then clearly $z_1 > 0$ but it may be $z_2 < 0$. This is the most difficult case:

The contributions

$$(8\pi\mu)^{1/2} e^{i\,3/4\pi} \int_{z_1}^{\infty} dz_1' \int_{z_2}^{0} dz_2' \, g_i(z_1, z_2, z_1', z_2') \, f_i(z_1, z_2, z_1', z_2')$$

$$\cdot |z_1'|^{-3/4} |z_2'|^{-3/4} e^{-\mu\sqrt{|z_1' z_2'|}} \omega_{j1}(q, z_1', z_2')$$

can easily be estimated by

$$c \int_{z_1}^{\infty} dz_1' |z_1' - z_1|^{-1/4} |z_1'|^{-3/4} \int_{z_2}^{0} dz_2' |z_2'|^{-3/4} e^{-\mu\sqrt{|z_1' z_2'|}}$$

$$\leqq c \int_{z_1}^{\infty} dz_1' |z_1' - z_1|^{-1/4} |z_1'|^{-1} \leqq c |z_1|^{-1/4}$$

where we have used for $g_i(z_1, z_2, z_1', z_2')$ the estimation:

$$|g_i(z_1, z_2, z_1', z_2)| \leqq c |z_1' - z_1|^{-1/4}$$

which can be read off from the estimation for $I_i(k, z_1, z_2, z_1', z_2')$. To estimate

$$\int_{z_1}^{\infty} \int_{0}^{\infty} dz_1' \, dz_2' \, g_i(z_1, z_2, z_1' z_2') \, h_i'(z_1, z_2, z_1', z_2')$$

define

$$h_i(z_1, z_2, z_1', z_2') = \int_{0}^{z_2'} d\tilde{z}_2 \, h_i'(z_1, z_2, z_1', \tilde{z}_2')$$

so that by a partial integration in $\tilde{z}_2$ we get

$$\int_{z_1}^{\infty} dz_1' \, g_i(z_1, z_2, z_1', z_2') \, h_i(z_1, z_2, z_1', z_2') \Big|_{z_2'=0}^{z_2'=\infty}$$

$$- \int_{z_1}^{\infty} \int_{0}^{\infty} dz_1' \, dz_2' \left(\frac{\partial}{\partial z_1'} g_i(z_1, z_2, z_1', z_2') \right) h_i(z_1, z_2, z_1', z_2') \, .$$

The integrated term vanishes because $h_i(z_1, z_2, z_1', z_2' = 0) = 0$ and $g_i(z_1, z_2, z_1', z_2')$ tends to zero for $z_2' \to +\infty$ while h_i remains bounded. By the method of stationary phase [8] we obtain the estimations

$$\left| \frac{\partial}{\partial z_2'} g_i(z_1, z_2, z_1', z_2') \right| \leqq c \frac{|z_1' - z_1|}{1 + |z_1' - z_1|^{5/4}} \frac{1}{1 + |z_2' - z_2|^{5/4}}$$

$$|h_i(z_1, z_2, z_1', z_2')| \leqq c \frac{1}{|z_1'|}$$

so that the remaining double integral can be estimated by

$$c \int_{z_1}^{\infty} dz_1' |z_1' - z_1|^{-1/4} |z_1'|^{-1/4} \int_{0}^{\infty} dz_2' \frac{1}{1 + |z_2' - z_2|^{5/4}} \leqq c |z_1|^{-1/4} \, .$$

Analogously the same decrease in z_2 is shown, hence

$$|\psi_2^1(z_1, z_2)| \leqq c \frac{1}{1 + |z_1|^{1/4}} \frac{1}{1 + |z_2|^{1/4}}$$

and thus

$$\left| \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2') \, R(k, z_1, z_2, z_1', z_2') \, V_2(z_1', z_2') \, u_j(q, z_1', z_2') \right|$$

$$\leqq c \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4}} \frac{1}{1 + |z_2|^{1/4}} \, .$$

With cruder methods and with methods indicated by the estimation for $\varphi_1^{n+1}(z_1, z_2)$ we obtain $(i = 1, 3)$

$$\left| \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2') \, R(k, z_1, z_2, z_1', z_2') \, V_i(z_1', z_2') \, u_j(q, z_1', z_2') \right|$$

$$\leqq c \frac{|z_1 - z_2|}{1 + |z_1 - z_2|} \frac{1}{1 + |z_1|^{1/4 - \varepsilon}} \frac{1}{1 + |z_2|^{1/4 - \varepsilon}} \, .$$

Hence the same estimation can be derived for

$$\chi^1(z_1, z_2) = \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2')$$

$$\cdot R(k, z_1, z_2, z_1', z_2') \, V(z_1', z_2') \, u_j(q, z_1', z_2') ; \quad j = 1 \text{ or } 2 \, .$$

Now it is easily seen that we can get the same bounds also for

$$\chi^2(z_1, z_2) = \lambda \int_{z_1}^{\infty} \int_{z_2}^{\infty} dz_1' \, dz_2' \, \theta(z_1' - z_2') \, R(k, z_1, z_2, z_1', z_2') \, V(z_1', z_2') \, \chi^1(z_1', z_2') \, .$$

Thus the estimation (27a) for $n = 0$ is proved. The prove for (27b) runs similarly.

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