BOOK REVIEWS

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Maximum principles and their applications, by René Sperb, Mathematics in Science and Engineering, vol. 157, Academic Press, New York, 1981, ix + 224 pp., $29.50.

Maximum principles are among the most useful and best known tools in the study of second order elliptic equations. They generalize the elementary fact that a function $f(x)$ with $f''(x) > 0$ in $[a, b]$ assumes its maximum either in a or in b. The basic principles which are usually referred to are *Hopf's first* and *second principles*. The first principle states that a function u defined in a bounded domain $D \subset \mathbf{R}^N$ and satisfying the differential inequality $Lu \geqslant 0$, where

$$L = \sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i},$$

$x = (x_1, x_2, \ldots, x_N)$, is a uniformly elliptic operator with bounded coefficients, cannot attain its maximum in D unless it is a constant. If $u \not\equiv$ constant assumes its maximum at some point $P \in \partial D$, then it is clear that the outer normal derivative $\partial u/\partial n$ of u at this point is nonnegative. However by Hopf's second maximum principle we have even the sharper estimate $\partial u/\partial n > 0$.

By means of these results it is easy to derive uniqueness theorems for Dirichlet problems and bounds for their solutions in terms of the data. Hopf's maximum principles have been generalized in numerous ways and applied to a wide range of problems both of mathematical and physical interest. Moreover they provide useful tools in the approximation of solutions and in the determination of error bounds for such approximations. An excellent reference is Protter and Weinberger's book **[1]** which at the same time serves students and researchers in this field.

Why a new text on this topic? First of all great progress has been made in the last years and methods were developed which are rooted in the maximum principles, especially in connexion with nonlinear problems. An important method of this type is the method of upper and lower solutions **[2]**. In addition, maximum principles led to nonexistence results for certain Dirichlet problems **[3]**, and they served also to prove symmetry and convexity in elliptic and parabolic equations **[4–7]**.

A group of mathematicians, mainly stimulated by Payne, asked for expressions involving solutions and their derivatives of elliptic and parabolic equations, which obey a maximum principle. This way, they intended to derive gradient bounds. The departure was Payne's observation [8] that for a convex domain D in the plane, the function

$$P := |\nabla u|^2 + 2u,$$

u being the solution of

$$\Delta u + 1 = 0 \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

assumes its maximum where $\nabla u = 0$. Since $|\nabla u|^2$ achieves its maximum on the boundary, we have immediately

$$|\nabla u|^2 \leqslant 2u_{\max}.$$

This particular choice for P was motivated by the 1-dimensional problem

$$u'' + f(u) = 0 \quad \text{in } (-1, 1), \qquad u(-1) = u(1) = 0$$

where the function $P := u'^2 + 2\int_0^u f(s)\, ds$ is constant.

The simplicity and the beauty of this idea attracted a number of researchers, among them R. Sperb. They extended and applied it to various, mostly physically oriented problems and worked out other types of "P-functions". The whole material has been collected in the present book and occupies the main part of it.

The author proposes to proceed as follows: For solutions of $\Delta u + f(u) = 0$ he defines $P := g(u)|\nabla u|^2 + h(u)$ and determines conditions on g and h such that P satisfies an elliptic differential inequality of the form

$$\Delta P + \sum_{i=1}^{N} L_i P_{x_i} / |\nabla u|^2 \geqslant 0.$$

He then asked under which conditions the maximum occurs on ∂D or at a critical value of u. The first case can sometimes be excluded by imposing certain assumptions on the curvature of ∂D. The discussion is particularly transparent for $N = 2$. From these results bounds are obtained for the torsion problem, several eigenvalue problems and nonlinear Dirichlet problems.

This programme was carried out to elliptic equations of divergence form. This research was initiated by Payne and Philippin [9] in order to study the problem of torsional creep, the surfaces of constant mean curvature and the problem of capillary surfaces.

The extension to the general case $Lu + f(u) = 0$ seem to cause much more difficulties. First results for linear equations go back to Protter and Weinberger [10]. Here the author suggests to split the elliptic operator L into a Beltrami operator with the metric tensor $g_{ij} = (a_{ij})^{-1}$ and first order terms. In the spirit of Riemannian geometry he takes as candidates for P functions of the form

$$|\tilde{\nabla} u|^2 g(u) + h(u)$$

where $|\tilde{\nabla} u|^2$ is the differential parameter of the first order.

The conditions for these candidates to have the "right" properties become very involved and are of purely computational nature. Among the most interesting applications are estimates for the first eigenvalue of the Laplacian on manifolds. In fact, in all applications L is a Beltrami operator.

Further chapters are devoted to weakly coupled systems, to fourth-order problems and to parabolic systems where estimates for the blow up time are found.

The author's intention is to stimulate researchers to apply and develop the techniques presented in this book in the form of an informal series of lectures. Almost half of the exposition is a collection of material on elliptic and parabolic problems where a large number of ideas is mentioned with brief comments and references. A chapter deals with Riemannian geometry where some concepts are reviewed in a rather sketchy way. Although geometrical tools are used very often the exposition is of purely computational character.

The style and level are uneven and the systematic development is a rather weak point of this text. In many proofs just the main ideas are given and the details are left to the reader. This has the advantage of making the reading more lively and stimulating. On the other hand it makes it sometimes difficult to verify the correctness of the statements, for example the improvement of Payne and Weinberger's inequality for symmetric plane regions where the interior sphere condition is violated or the results in §8.3 on the free membrane. There are numerous misprints; most of them are obvious to the experienced reader. Like any research survey which is written during a rapid development of the field, accents will probably be put differently in a few years or by other experts.

The author has done the mathematical community a service by giving the reader access to interesting and nice aspects of the maximum principles.

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