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ON THE ZEROS OF DIRICHLET SERIES ASSOCIATED WITH CERTAIN CUSP FORMS[1]

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As is well known, in 1859 Riemann [6] conjectured that the function $\zeta(s)$ defined in $\operatorname{Re} s > 1$ by the Dirichlet series $\sum_{n=1}^{\infty} n^{-s}$ has all its zeros, apart from the "trivial" zeros at the negative even integers, on the line $\operatorname{Re} s = \frac{1}{2}$. It is known that these "nontrivial" zeros lie symmetrically about the line $\operatorname{Re} s = \frac{1}{2}$ within the strip $0 < \operatorname{Re} s < 1$. The truth of this Riemann Hypothesis would have a profound impact in the theory of numbers, particularly with regard to the distribution of primes.

One of the major achievements in this theory was due to Selberg [7] in 1943. He proved for $\zeta(s)$ that a positive proportion of the nontrivial zeros lie on the critical line. Later authors have given specific numerical values for this proportion. In this note we announce the proof of a similar theorem for Dirichlet series attached to certain cusp forms on the full modular group. We formulate the specific theorem below.

Let $F(z)$ be a holomorphic cusp form of even integral weight k and constant multiplier system for the full modular group $\Gamma(1) = SL(2, \mathbf{Z})/\{\pm I\}$. That is,

$$F(Mz) = (cz+d)^k F(z), \qquad M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma(1),$$

and $F(z)$ vanishes at $i\infty$. Expand $F(z)$ in a "Fourier series" at the cusp $i\infty$ as

$$F(z) = \sum_{l=1}^{\infty} f(l) e^{2\pi i l z}.$$

The Dirichlet series $L_f(s) = \sum_{l=1}^{\infty} f(l) l^{-s}$ converges absolutely for

$$\operatorname{Re} s > (k+1)/2$$

and can be continued to an entire function in the s-plane. Furthermore, $L_f(s)$ has all its nontrivial zeros in the strip $(k-1)/2 < \operatorname{Re} s < (k+1)/2$. Let

$$N(T) = \#\{\rho = \beta + i\gamma \colon 0 < \gamma < T, (k-1)/2 < \beta < (k+1)/2, L_f(\rho) = 0\}$$

and

$$N_0(T) = \#\{\rho = k/2 + i\gamma \colon 0 < \gamma < T, L_f(\rho) = 0\}.$$

It is known [4] that $N(T) \sim cT \log T$ for some constant $c > 0$. We then have the following theorem.

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THEOREM. *If $F(z)$ is an eigenfunction of all the Hecke operators with $f(1) = 1$, then there exists a constant $A > 0$ depending on F such that*

$$N_0(T) > AT \log T,$$

for all sufficiently large T.

The assumptions on F insure that (a) $f(l)$ is a multiplicative function, (b) $|f(p)| \leq 2p^{(k-1)/2}$, p a prime, and (c) $f(l)$ is real for all l, in fact, a totally real algebraic number. Fact (b) is due to Deligne [**1**] and (c) can be found in Ogg [**5**, p. II-11 and p. III-11]. Of course $f(1) = 1$ can always be achieved by the appropriate normalization.

The method of proof uses three main ingredients. First, we modify slightly Selberg's idea to introduce a mollifier $\phi(s)$ which approximates $L_f^{-1/2}$. Secondly, we require an approximate functional equation for $L_f(s)$ which is provided by A. Good [**2**]. Finally we need to extend A. Good's techniques [**3**] for computing

$$\int_0^T \left| L_f\left(\frac{k}{2} + it\right)\right|^2 dt \sim cT \log T. \tag{2}$$

We are then required to estimate expressions like

$$\int_0^T \left| L_f\left(\frac{k}{2} + it\right)\right|^2 \left|\phi\left(\frac{k}{2} + it\right)\right|^4 dt. \tag{2}$$

There are some extra difficulties which we encounter which make this theorem quite difficult. First, the coefficients $f(l)$ are not completely multiplicative. This makes certain arithmetical sums more difficult to analyse. Secondly, as is the difficulty in (1), analysis of the series

$$\sum_{l=1}^{\infty} \frac{f(l)f(l+n)}{(l+n/2)^s}, \qquad n \geq 0,$$

is required. This is obtained by appealing to the spectral theory of the Laplacian acting on $L^2(\Gamma(1)\backslash\mathcal{H})$. (See Good [**3**].) However the introduction of the mollifier complicates significantly the corresponding calculation in (2). In particular we need an analysis of the series

$$\sum_{l=1}^{\infty} \frac{f(l)f((lb+n)/a)}{(lb+n/2)^s}, \qquad n \geq 0,\ (a,b) = 1.$$

We require growth estimates with respect to the $\operatorname{Im} s$, in the region where the series does not converge absolutely, and which are uniform in a and b. This uniformity is the major difficulty. To handle this problem we appeal to spectral theory of $L^2(\Gamma_0(a,b)\backslash\mathcal{H})$ where $\Gamma_0(a,b)$ is the congruence group defined by

$$\Gamma_0(a,b) = \left\{ \begin{pmatrix} * & \beta \\ \gamma & * \end{pmatrix} \in \Gamma(1) \colon \beta \equiv 0 \,(\operatorname{mod} b), \gamma \equiv 0 \,(\operatorname{mod} a) \right\}.$$

And finally, estimates for the Fourier coefficients of the Maass wave forms (the orthonormal basis of eigenfunctions for the discrete spectrum of the Laplacian) which are uniform in a and b are required.

The mollifier we choose is given by

$$\phi(s) = \phi_\xi(s) = \sum_{\nu \leq \xi} \alpha_\nu \left(1 - \frac{\log \nu}{\log \xi}\right),$$

where $\xi \geq 2$ and $\alpha_l = \mu(l) f(l)/d(l)$. Here μ and d are the usual Möbius and divisor functions. We then prove that there exists a number $a > 0$ such that for $\xi = T^a$, $0 \leq h \leq (\log \xi)^{-1/2}$,

$$\text{(a)} \qquad \int_0^T \left| \int_t^{t+h} L_f\left(\frac{k}{2} + iu\right) \left|\phi\left(\frac{k}{2} + iu\right)\right|^2 du \right|^2 dt \ll \frac{T h^{3/2}}{\sqrt{\log \xi}},$$

and

$$\text{(b)} \qquad \int_0^T \int_t^{t+h} \left| L_f\left(\frac{k}{2} + iu\right) \phi^2\left(\frac{k}{2} + iu\right) \right|^2 du\, dt \ll \frac{h^2 T \log T}{\log \xi}.$$

From these estimates the deduction of the theorem follows just as in Selberg's proof.

A classical example to which our theorem applies is the cusp form of weight 12 defined by

$$\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \qquad q = e^{2\pi i z},$$
$$= \sum_{l=1}^{\infty} \tau(l) e^{2\pi i l z},$$

where $\tau(l)$ is Ramanujan's function.

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