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# SINGULAR CONTINUOUS SPECTRUM FOR A CLASS OF ALMOST PERIODIC JACOBI MATRICES[1]

BY JOSEPH AVRON AND BARRY SIMON

ABSTRACT. We consider operators (parametrized by $\alpha$, $\theta$, $\lambda$) on $l_2$ with matrix $\delta_{i,j+1} + \delta_{i,j-1} + a_i\delta_{ij}$ with

$$a_n = \lambda \cos(2\pi\alpha n + \theta).$$

If $\alpha$ is a Liouville number and $\lambda > 2$, we prove that for a.e. $\theta$, the operator's spectral measures are all singular continuous.

We consider the operator $H$ on $l_2(Z)$ depending upon three parameters, $\lambda, \alpha, \theta$,

$$[H(\lambda, \alpha, \theta)u](n) = u(n+1) + u(n-1) + \lambda\cos(2\pi\alpha n + \theta)u(n). \tag{1}$$

In this note we will sketch the proof of the following result whose detailed proof will appear elsewhere [**3**].

THEOREM 1. *Fix $\alpha$, an irrational number obeying*

$$|\alpha - p_k/q_k| \leqslant k^{-q_k} \tag{2}$$

*for a sequence $q_k \longrightarrow \infty$. Fix $\lambda > 2$. Then for a.e. $\theta$, $H(\lambda, \alpha, \theta)$ has purely singular continuous spectrum.*

REMARKS 1. It is not hard to see that uncountably many $\alpha$ obey (2) but that the set of such $\alpha$ has Lebesgue measure zero.

2. One interest in this is that if $\alpha$ is rational, (1) has purely absolutely continuous spectrum by a Bloch wave analysis [**10**]. We believe that if $\lambda > 2$ and $|\alpha - p/q| \geqslant C/q^k$ for all rational $p/q$ (and some fixed $C$, $k$), then (1) has only point spectrum but we can only prove less (see Theorem 3 below).

3. We emphasize that the spectrum ($\equiv$ *closed* support of the spectral measure class) need not have zero Lebesgue measure nor do we prove it is a Cantor set (although we believe it is!).

4. See Pearson [**9**] for another simple looking class of operators with purely singular spectrum.

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We rely, in the first place, on the following result of Gordon [5]:

THEOREM 2 (GORDON). *Let $V$ be a function on $Z$ so that there exist periodic functions $V_m$ on $Z$ of period $T_m \longrightarrow \infty$ with*

(i) $\sup_{m,n} |V_m(n)| < \infty$,

(ii) $\sup_{-2T_m \leqslant n \leqslant 2T_m} |V_m(n) - V(n)| \leqslant Cm^{-T_m}$

*for some $C$. Then, for all $E$,*

$$u(n+1) + u(n-1) + V(n)u(n) = Eu(n)$$

*has no solution in $l_2$, indeed, it has no solution with* $\lim_{n\to\infty} |u(n)| = 0$.

Gordon states his result for $-d^2/dx^2 + V(x)$ but his proof easily extends to the finite difference case. Gordon's result applies to operators of the form (1) with $\alpha$ obeying (2); let $T_m = q_{2m}$ and

$$V_m(n) = \lambda \cos(2\pi\alpha_m n + \theta); \qquad \alpha_m = p_{2m}/q_{2m}.$$

We thus conclude that if (2) holds, $H(\alpha, \lambda, \theta)$ has no point spectrum for all $\lambda, \theta$. Thus Theorem 1 follows from

THEOREM 3. *Fix $\lambda > 2$ and any irrational $\alpha$. Then $H(\lambda, \alpha, \theta)$ has no absolutely continuous spectrum for* a.e. $\theta$.

Aubry and André [1] argued that for any $\alpha$ irrational, if $\lambda > 2$, the spectrum is dense point. Although this claim is inconsistent with Theorem 2, our proof of Theorem 3 follows their ideas combined with some simple functional analysis; their error involves ignoring various sets of measure zero (e.g. the complement of the set $S$ in Theorem 4 below).

Let $\chi_L$ be the characteristic function of the set $\{0, \ldots, L-1\}$ and let $P(\lambda, \alpha, \theta, E)$ be the spectral projection for $H(\lambda, \alpha, \theta)$. Then, one proves [8, 12, 3] that

$$\lim_{L\to\infty} L^{-1} \operatorname{Tr}(P(\lambda, \alpha, \theta, E)\chi_L) = k(\lambda, \alpha, E)$$

exists and (for irrational $\alpha$) is $\theta$ independent; $k(E)$ is called the *integrated density of states.* The second order difference equation (1) can be written as a first order 2 component vector equation and we let $T(0, n; \lambda, \alpha, \theta, E)$ be the matrix relating $(u(n), u(n+1))$ to $(u(0), u(1))$ for solutions of $Hu = Eu$. We say that $H - E$ has *Lyaponov behavior* (with *Lyaponov index* $\gamma(E)$) if

$$\lim_{|N|\to\infty} |N|^{-1} \ln\|T(0, N; \lambda, \alpha, \theta, E)\| \equiv \gamma(E)$$

exists. Note that since $T$ has determinant 1,

$$\gamma(E) \geqslant 0. \tag{3}$$

We prove the following formula:

THEOREM 4. *Fix* $\alpha, \lambda$. *Put Lebesgue measure on* $[0, 2\pi) \times (-\infty, \infty)$. *Then there exists a set* $S$ *whose complement has measure zero so that if* $(\theta, E) \in S$, *then* $H(\lambda, \alpha, \theta) - E$ *has Lyaponov behavior with*

$$\gamma(E) = \int \ln|E - E'|dk(\lambda, \alpha, E). \tag{4}$$

In (4), $dk$ indicates the Steiltjes measure in $E$. (4) is a formula of Thouless [13] originally proven formally for random potentials. Our proof [3] uses his ideas together with some functional analysis including the $L^2$ continuity of the Hilbert transform and the subadditive ergodic theorem. The proof is not special to the cosine potential but works for any almost periodic potential (the range of $\theta$ in $[0, 2\pi)$ is replaced by the hull of the a.p. potential) and there is a "once subtracted" analog for Schrödinger operators, $-d^2/dx^2 + V(x)$. The next result is special to the cosine potential.

THEOREM 5 (AUBRY DUALITY). *Fix* $\alpha$ *irrational. Then*

$$k(\lambda, \alpha, E) = k(4/\lambda, \alpha, 2E/\lambda). \tag{5}$$

Formally, this comes from the fact that under Fourier transform, the finite difference part of $H$ turns into a cosine and the cosine into a finite difference operator. Aubry [2] initially found (5) with a formal proof. Our rigorous proof [3] exploits various continuity properties of $k$ in $\alpha$ (e.g. it is continuous in $\alpha$ at irrational points). Putting (5) into the Thouless formula, we find, following Aubry and André [1], that if $\alpha$ is irrational,

$$\gamma(\lambda, \alpha, E) = \gamma(4/\lambda, \alpha, 2E/\lambda) + \ln(\lambda/2)$$

so by (3), we conclude that

$$\gamma(\lambda, \alpha, E) > 0 \quad \text{if } \lambda > 2.$$

Next, we need the following consequence of a general theorem of Oseledec [7] (see also Ruelle [11]).

THEOREM 6. *If* $H - E$ *has Lyaponov behavior with* $\gamma > 0$, *then there exist (not necessarily distinct) subspaces,* $V_\pm$, *of* $\mathbf{C}^2$, *so that if* $\varphi \in V_\pm$, *then*

$$\lim_{N \to \pm\infty} |N|^{-1} \ln\|T(0, N; \lambda, \alpha, \theta, E)\varphi\| = -\gamma(E)$$

*and if* $\varphi \notin V_\pm$, *the limit is* $\gamma(E)$. *In particular, any solution of* $(H - E)u = 0$ *which is polynomial bounded at* $\infty$ *and* $-\infty$, *falls off exponentially at* $\pm\infty$ *and so is in* $l_2$.

Now, the Berezanski-Gel'fand-Kac [4] generalized eigenfunction expansion implies that for almost all $E$ with respect to the spectral measure class, there are polynomially bounded eigenfunctions. Thus, if $\gamma > 0$ for all $E$, we see that for each fixed $\theta$, $\{E|(\theta, E) \notin S\} \cup \{E|E$ is an eigenvalue of $H\}$ must support a spectral measure. Therefore, for a.e. $\theta$ no absolutely continuous spectrum. This proves Theorem 3 and so Theorem 1.

We close with three remarks and a question. First, we note that Gordon's theorem says that the generalized eigenfunctions in the context of Theorem 1 do not go to zero at $\pm\infty$. This destroys the common belief that singular continuous spectrum is associated to continuum eigenfunctions going to zero at infinity but in a non-$L^2$ way.

Secondly, in the context of Theorem 1, we claim that the complement of $S$ is nonempty: for a.e. $\theta$ this is obvious, since $\{E|(\theta, E) \notin S\}$ must support the spectrum. Moreover, if $E \in \operatorname{spec}(H)$ (which is $\theta$ independent), $\{\theta|(\theta, E) \notin S\}$ is nonempty, for a theorem of Johnson **[6]** assures us that for some $\theta$, $(H(\theta) - E)u = 0$ has a bounded eigenfunction; but if $(\theta, E) \in S$, such an eigenfunction is $L^2$, contradicting Theorem 2.

Thirdly, we note the intuition to understand why there is singular continuous spectrum. Electrons moving under such Hamiltonians will travel long distances thinking they are in a periodic potential and then get reflected, so the behavior will be close to that in Pearson's example **[9]**.

Finally, we note that a detailed analysis of duality suggests that point spectrum and a.c. spectrum are dual. Is it true, as suggested by this, that Theorem 1 remains true for $0 < \lambda < 2$?

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DEPARTMENT OF PHYSICS 452–48, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08544

*Current address* (Joseph Avron): Department of Physics, California Institute of Technology, Pasadena, California 91125

*Current address* (Barry Simon): Department of Mathematics 253–37, California Institute of Technology, Pasadena, California 91125