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BOOK REVIEWS

Three-dimensional problems of the mathematical theory of elasticity and thermoelasticity, by V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Applied Mathematics and Mechanics, vol. 25, North-Holland, 1979, xix + 929 pp., $158.50.

The classical theory of elasticity, with which the book under review is concerned, is one of the more highly developed and satisfactory branches of mathematical physics, in that mathematical rigour and physical intuition are combined with rich results. Over the years, problems which were originally investigated for purely practical reasons have been reformulated in precise terms and have led to mathematical investigations of great interest and elegance. Unfortunately, it is not a subject which is taught nowadays in mathematics departments, even at graduate level, and many mathematicians are unaware of the stimulus that research in it during the last 150 years has given to the study of partial differential equations, singular integral equations, complex function theory, variational inequalities and bifurcation theory.

The mathematical foundations of the classical theory were laid down by Cauchy and as Truesdell, in his preface to [38], has reminded us: "For a long time it was a favourite subject of mathematicians and was regularly taught in mathematics departments. In this century both Hadamard and Hilbert lectured upon it, as had Poincaré and many others in the last. Of the mathematicians of that time who are best known for their work in what is now called 'pure' mathematics, we may collect a long list naming those who made at least one important addition to elasticity–Beltrami, Betti, G. D. Birkhoff, Cesàro, Christoffel, Clebsch, Fredholm, Hadamard, Korn, Lamé, Levi-Civita, Lipschitz, Morera, Volterra, Weingarten, Weyl." Truesdell goes on to highlight the contribution made to elasticity by distinguished Italian mathematicians who specialized in the subject: Almansi, Cerruti, Lauricella, Piola, Signorini, Somigliana and Tedone. One suspects that it is only his reluctance to name living mathematicians that causes Truesdell to omit the name of Gaetano Fichera whose own account [11] of the Italian contribution to the theory of elasticity makes fascinating reading. Nor do I think that Truesdell would object to his own name being added to the list: the rigour of his mathematical proofs, the depth of his physical insight, and the elegance of his writings put him firmly in the Italian tradition. He has certainly been one of the leaders of the renaissance of continuum mechanics.

The only national school which might lay claim to have made a contribution of the same magnitude is that of the U.S.S.R.; its contribution has been mainly to the two-dimensional theory and to the development of the attendant techniques in function theory, and also to the potential theory methods described in the book by Kupradze *et al.* A modest account of its achievements is contained within [29].

The mathematical theory of elasticity is concerned with the calculation of the strain and stress fields within a solid body when it is subject to the action of a self-equilibrating system of forces. The body under considerations is said to be *elastic* if the stress within it is determined by the strain at a given instant and not by the entire past history of the motion.

The first scientist to consider the deformation of a solid body due to the application of an external force was Galileo [13] who considered the problem of determining the resistance of a beam, one end of which is built into a wall, when the tendency to break it arises from its own or an applied weight. Galileo concluded that the beam tends to turn about an axis perpendicular to its length, and in the plane of the wall. He had to treat the beam as inelastic since he was not in possession of any physical law relating the magnitude of the applied forces to the displacements in which they resulted. The experimental basis of such a law was provided by Hooke [17], [15] who gave in 1678, the famous law of proportionality of stress and strain which bears his name.

In the interval of nearly a century and a half between the publication of Hooke's law and that of the field equations of elasticity, the attention of those mathematicians who occupied themselves with the theory of elasticity such as James Bernoulli, Euler, Coulomb and Sophie Germain, was directed mainly to the solution and extension of Galileo's problem, and the related theories of the vibrations of bars and plates.

The general theory of elasticity may be said to have its origin in a paper read before the Paris Academy of Sciences by Navier in May 1821, (but not published until 1827) but it was Cauchy who put the mathematical theory of homogeneous isotropic elastic bodies on a firm foundation. Cauchy's interest in elasticity was first prompted by his appointment to the Commission set up by the Paris Academy to report on a paper by Navier on elastic plates presented to the Academy in August 1820.

By the summer of 1822 Cauchy had set up the elements of the mathematical theory of elasticity; his memoir on the subject was communicated to the Paris Academy in September 1822 but it was not published. An abstract appeared in the *Bulletin des Sciences à la Societé Philomathique*, 1823 and the contents were fully described in three articles in Cauchy's *Exercises de Mathématique* (1827, 1828). The third of these, published in the volume for 1828, entitled "Sur les équations qui experiment les conditions d'equilibre ou les lois de mouvement intérieur d'un corps solide" established the correct equations to model the behaviour of an isotropic elastic body.

To describe Cauchy's work we use modern notation. We consider a solid body identified with the region $B \subset \mathbf{R}^3$ it occupies in a fixed reference configuration. A *deformation* of B is a smooth homeomorphism $\mathbf{f}$: $B \rightarrow \mathbf{R}^3$ with $\det \nabla\mathbf{f}(\mathbf{x}) > 0$. The point $\mathbf{f}(\mathbf{x})$ is the place occupied by the material point $\mathbf{x}$ in this deformation and the vector $\mathbf{u}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - \mathbf{x}$ is the *displacement* of $\mathbf{x}$. The tensor fields $\mathbf{F} = \nabla\mathbf{f}$ and $\nabla\mathbf{u}$ are called, respectively, the *deformation gradient* and the *displacement gradient*. There are several definitions of strain in the literature but it can be shown (see §32 of [34]) that all the properly invariant choices are, in a certain sense, equivalent. A suitable choice, and the one favoured in [16] is the *finite strain tensor* $\mathbf{D}$ defined by the equation

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

where $\mathbf{F}^T$ denotes the transpose of $\mathbf{F}$ and $\mathbf{I}$ denotes the unit tensor. What is of importance in the linear, or classical, theory of elasticity–and it is the linear theory with which Kupradze *et al.* are concerned–is the *infinitesimal strain tensor*

$$\mathbf{E} = \tfrac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T). \tag{2}$$

Since $\mathbf{D} = \mathbf{E} + \frac{1}{2}\nabla^T\mathbf{u}\nabla\mathbf{u}$ it is obvious that the linear theory models physical situations in which $|\nabla\mathbf{u}|$ is, in some sense, small.

Cauchy introduced the concept of stress into the theory of elasticity to characterize the internal forces. Through the point with position vector $\mathbf{x}$, we consider a surface element with normal $\mathbf{n}$ and area a. The internal forces produced by the interaction of the parts of the body on the opposite sides of this element are statically equivalent to a force $\mathbf{S}$ and a couple $\mathbf{M}$. Cauchy assumed that

$$\lim_{a\to 0} \mathbf{M}/a = 0 \tag{3}$$

and that the limit $\boldsymbol{\sigma}(\mathbf{x}, t; \mathbf{n}) = \lim_{a\to 0} \mathbf{S}/a$ exists and is independent of the shape of the surface element. It is then easily shown that there exists a tensor field $\mathbf{S}$ such that $\boldsymbol{\sigma}(\mathbf{x}, t; \mathbf{n}) = \mathbf{S}(\mathbf{x}, t)\mathbf{n}$.

If we denote by ρ the density of the body and by $\mathbf{b}(\mathbf{x}, t)$ the force per unit volume exerted on an element of volume centred at the point $\mathbf{x}$, then the principle of the balance of linear momentum leads to the equation

$$\operatorname{div}\mathbf{S} + \mathbf{b} = \rho\ddot{\mathbf{u}} \tag{4}$$

and the principle of the balance of angular momentum leads to the conclusion that $\mathbf{S}$ is a *symmetric* tensor field. In the case of statical equilibrium the term occurring on the right-hand side of equation (4) is replaced by the zero vector.

Cauchy's final assumption is that the relation connecting $\mathbf{S}$ and $\mathbf{E}$ is linear, i.e. that there exists a fourth-order tensor $\mathbf{C}$ such that $\mathbf{S} = \mathbf{CE}$. When the body is homogeneous and isotropic and the process of deformation is isothermal we may write this relation in the form

$$\mathbf{S} = \lambda(\operatorname{tr}\mathbf{E})\mathbf{1} + 2\mu\mathbf{E} \tag{5}$$

where λ and μ are constants, now known as the *Lamé constants*. This is one of the important respects in which Cauchy's results differ from Navier's; the latter's generalization of Hooke's law contains only a single elastic constant.

The equations (2), (4), (5) essentially express the *classical theory of elasticity* as established by Cauchy. If we eliminate $\mathbf{S}$ and $\mathbf{E}$ from these equations we obtain an equation for the displacement field $\mathbf{u}(\mathbf{x}, t)$; this equation is known as *Navier's equation*.

An excellent account of the classical theory and of its extension to the case of anisotropic bodies is given by Gurtin [**16**].

The linear theory of elasticity engaged the attention of many distinguished mathematicians in the nineteenth century, as indicated above. Progress was so rapid that in just over 65 years after the publication of Cauchy's results there appeared not only the first edition of what was to become a classic treatise on the subject [**25**] but also the second volume of a comprehensive history of its development [33]. In the 19th century the main research effort went into the solution of special problems in torsion and flexure and to the study of plane strain and plane stress although important general results such as Betti's reciprocal theorem were derived.

The most exciting aspect of research in elasticity in the present century has been the development of general continuum mechanics and, in particular of nonlinear theories of elasticity. This new era was inaugurated by Rivlin in a series of 13 papers on large elastic deformations published in the years 1947–1949 (see [34]). The publication of the treatises [14], [35], [6] and [34] did much to stimulate research in the nonlinear theory. In this connection reference should also be made to [38].

In the linear theory it could be claimed that in the 20th century the emphasis shifted to the solution of boundary value problems. In the period before World War II research effort was divided between the devising of approximate methods for the solution of problems in two and three dimensions (finite-difference methods, the relaxation method, truncation of infinite series, direct variational methods) and the complex variable techniques devised by Muskhelishvili and his co-workers for the solution of plane problems [28]. Muskhelishvili's methods and the Wiener-Hopf technique have proved to be most effective in the solution of boundary value problems in the plane, but, of course they cannot be directly extended to three-dimensional problems.

It was not until the late 1930's that attempts were made to establish *systematic* methods for the solution of three-dimensional problems. Apart from purely numerical methods (such as the 'finite element' technique) the main procedures put forward have been based on one of the three: (a) the theory of integral transforms ([30], [36], [31], [32]); (b) the theory of the potential and of integral equations [21]; (c) the theory of variational inequalities [5]. In addition there has been great interest in deriving rigorous existence and uniqueness theorems ([7], [9], [19]) which *should* be studied carefully before extensive numerical work is undertaken.

The application of the methods of potential theory and of the theory of linear integral equations to the solution of boundary problems of elasticity was first considered by Fredholm [12] and Lauricella [22], though in some respects this was foreshadowed by Boussinesq whose significant work [2] has been unjustly neglected. Fredholm considered the *first boundary value problem* of the equilibrium of an elastic body–i.e. the problem of determining the displacement field in B when the displacement vector $\mathbf{u}(\mathbf{x})$ is prescribed for all $\mathbf{x} \in \partial B$, the boundary of B. The Navier equations of equilibrium form an elliptic system and the problem of solving them for given boundary values of the displacement vector is a Dirichlet problem. An alternative method of solution to Fredholm's was developed by Boggio [1] and Korn [20]. In his critique [39] Weyl points out that while Fredholm's solution of the first boundary value problem is analogous to the Neumann-Fredholm method in potential theory, Boggio's solution leads to integral equations with complicated, intractable kernels which, contrary to Boggio's assumption, are not regular. Weyl's paper contains an account of the use of his "antennae" method for the reduction of the second boundary value problem to that of solving regular inhomogeneous Fredholm equations. In proving the existence of the solutions of these equations, Weyl assumed, without proof, the possibility of biorthonormalizing the sets of fundamental solutions of a pair of adjoint Fredholm equations of the second kind.

Lichtenstein's method [23] which appears to be applicable only to the first boundary value problem imposes severe restrictions on the nature of the boundary surfaces.

A more recent paper [**18**] dealing with the first and second boundary value problems is interesting in that–as Kupradze [**21**] points out–the authors quite illegitimately use Fredholm's theorems to derive results which happen to be correct!

Papers based on Fredholm methods consider only the first and second boundary value problems of elastostatics and only for the solution of the first problems is the method really successful; no mention is made of mixed boundary values in which displacements and stresses are each partly prescribed (contact problems and crack problems). The difficulties are even more severe when we pass to elastodynamics and when the elastic bodies are inhomogeneous. Even in the case when the elastic body is piecewise-homogeneous, i.e. is made up of parts having different elastic parts, difficulties arise because we have to take into account not only the conditions on the outer boundary but also those at the interfaces between different elastic components.

A method of tackling all the boundary value problems mentioned above, from static problems for homogeneous isotropic bodies to dynamical problems for piecewise-homogeneous bodies has been devised and developed over the last 45 years by Kupradze and his Georgian students and colleagues, of whom three of the most distinguished are co-authors of the book under review. Kupradze's method is based on the theorems of potential theory and of multidimensional singular integral equations. For example, the problem of finding the solution of the simplest of the mixed boundary problems for a homogeneous isotropic half-space–the Boussinesq or contact problem–can be shown to be equivalent to that of solving the integral equation

$$\int_\Omega \frac{p(\mathbf{y})\,d\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|} = \delta - f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \subset \mathbf{R}^2$$

where f is prescribed and the unknown constant δ is determined by the condition $p(\mathbf{x}) \to 0$ as $\mathbf{x} \to \partial\Omega$ from the interior of Ω. The existence of solutions of multidimensional integral equations of this type is guaranteed, for functions f belonging to a wide class, by the Giraud-Mikhlin theory. For more complicated problems in elasticity we have to deal not with a single equation but with systems of singular integral equations; this has motivated Kupradze and his colleagues to extend the Giraud-Mikhlin theory to such systems. An account of these methods is given in the large monograph [**21**] where, also, they are used to prove existence and uniqueness theorems. This book also deals with methods of deriving approximate solutions of the boundary value problems of elasticity. Reference should also be made to Mikhlin's book [**27**] which gives a different approach to the theory of systems of multidimensional integral equations on closed manifolds.

It will be recalled that Cauchy assumed that the deformation process is isothermal. If we abandon this hypothesis and denote by $\theta(\mathbf{x}, t)$ the deviation of the temperature from its value in the reference state, then, in the case of a homogeneous isotropic solid, the stress-strain relation (5) is modified by the addition of a term $-\gamma\theta\mathbf{I}$, with γ a positive constant, to its right-hand side. The classical Fourier equation of the conduction of heat has also to be modified by the addition of a term proportional to $(\partial/\partial t)(\operatorname{tr}\mathbf{E})$. The set of equations so modified provides the basis of the classical theory of *thermoelasticity* which is discussed fully in [3].

A modification of the classical theory was introduced by the brothers Cosserat in 1898, [4], who assigned to each molecule of an elastic body a perfectly rigid trihedron which, during any deformation underwent a rotation as well as a displacement. In this way, was envisaged an elastic solid with whose points was associated an orientation, so that each material element of the body had six degrees of freedom. In this theory therefore, the deformation of an elastic body is described by a displacement vector $\mathbf{u}(\mathbf{x}, t)$ and an independent rotation vector $\boldsymbol{\omega}(\mathbf{x}, t)$ and the condition (3) is replaced by the assumption: the limit $\boldsymbol{\mu}(\mathbf{x}, t; \mathbf{n}) = \lim_{a\to 0} \mathbf{M}/a$ exists and is independent of $\mathbf{n}$. Associated with the vector $\boldsymbol{\mu}(\mathbf{x}, t; \mathbf{n})$ is a tensor $\mathbf{M}(\mathbf{x}, t)$ with the property that $\boldsymbol{\mu} = \mathbf{M}\mathbf{n}$; the components of the tensor are known as *couple-stresses*. In the framework of an infinitesimal theory, it is assumed that there is a linear relation connecting $\mathbf{M}$ and a torsion tensor formed from derivatives of the components of the vector $\boldsymbol{\omega}$. These assumptions then lead to the conclusion that both the strain and the stress tensors are asymmetrical. Despite the novelty of the Cosserats' ideas, their work was unnoticed by their contemporaries and was almost entirely neglected for some 50 years until interest in it was derived by Toupin and Truesdell (see §256 of [35] and §98 of [34]). These concepts have been generalized to produce a theory of micropolar elasticity but, in the book under review it is only the simple couple-stress theory which is considered.

In 1959 Signorini posed a new class of three-dimensional boundary value problems in elastostatics in which on part (or all) of the boundary the unknown functions must satisfy one or the other of two given sets of conditions; it is *not* known which of the two sets is satisfied at any given point of the boundary. In addition, the conditions may not only take the form of equations, but also of *inequalities*. A typical example of such a problem is that of an elastic body B resting on a rigid foundation. If Σ is that part of ∂B on which the body can rest in its equilibrium condition and if $\mathbf{f}(\mathbf{x})$ is the force at a point $\mathbf{x} \in \partial B - \Sigma$, then on $\partial B - \Sigma$ we have, in the usual notation, $\mathbf{S}\mathbf{n} = \mathbf{f}$. Obviously the surface force cannot be chosen arbitrarily; $\mathbf{f}$ and the body force $\mathbf{b}$ must be such as to ensure the equilibrium of the body as a whole. As far as Σ is concerned *either* the set of conditions

$$\mathbf{u}\cdot\mathbf{n} = 0, \qquad \mathbf{S}\cdot(\mathbf{n}\otimes\mathbf{n}) > 0, \qquad \mathbf{S}\cdot(\mathbf{n}\otimes\mathbf{t}) = 0 \tag{6}$$

must be satisfied *or* the set

$$\mathbf{u}\cdot\mathbf{n} > 0, \qquad \mathbf{S}\cdot(\mathbf{n}\otimes\mathbf{n}) = 0, \qquad \mathbf{S}\cdot(\mathbf{n}\otimes\mathbf{t}) = 0 \tag{7}$$

where $\mathbf{t}$ denotes any vector tangent to Σ at the point in question. Since it is not known *a priori* which of the set of conditions (6) or (7) is satisfied at any particular point of Σ, Signorini called them *ambiguous boundary conditions*. Another term used in the literature is 'unilateral constraint'; a displacement field $\mathbf{u}$ which is compatible with a given system of constraints is said to be *unilateral* if the displacement field is not compatible with these constraints.

It was in his analysis **[8]** of Signorini's problem that Fichera introduced, in a concrete situation, the concept of a *variational inequality*, later generalized by Lions and Stampacchia **[24]** and the source of many results in mechanics and physics some of which are described in **[5]**. A full discussion of boundary value problems of elasticity with unilateral constraints is given by Fichera in **[10]**.

Although, in the discussion of special problems, reference is made to other methods of solution of three-dimensional problems, the book under review is mainly concerned with the use of potential theory and the theory of integral equations. This approach to proving existence theorems differs from that of Fichera **[9]** which is based on recent results in the theory of elliptic partial differential equations. Although Fichera considers only existence theorems in classical elasticity, his proofs seem more elegant; also in an accompanying article he considers problems of elasticity defined by unilateral constraints.

No reference is made in the present book either to Signorini's problem or to the subsequent work on variational inequalities. This is due no doubt to the fact that Fichera's seminal paper **[8]** had not come to the authors' attention, and that the subsequent paper **[24]** by Lions and Stampacchia had appeared when the manuscript of the original Russian edition of the present book had already gone to press. However, the authors could have taken the opportunity provided by the publication of this English translation to include additional material on this important subject.

In the latter chapters of the book the authors present an original method, not involving the solution of integral equations, for deriving approximate solutions of the equations of elasticity and also a more detailed discussion of some special problems.

The book suffers from the fact that in some respects it reads like an introductory textbook on the linear theories of elasticity and thermoelasticity and in others like a set of collected research papers. Kupradze's earlier book 21 seems more successful perhaps because its aims are clearer. Even the title of the present book is unfortunate in that it is not a collection of solutions of special problems in three-dimensional elasticity–as is, for instance, Lurie's book **[26]**–but an account of existence and uniqueness theorems for the full three-dimensional theory.

Since the original Russian edition was published in Tbilisi in an edition of only 1500, it has been virtually impossible to obtain a copy outside of the U.S.S.R., so we must be grateful to the North Holland Publishing Company for making a translation available. The translation, though adequate to the needs of someone already familiar with the subject, is not always felicitous–"a force of two unities"–and is sometimes inaccurate–"one may have an infinite number of directions at each (interior) point of a medium".

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Analysis, manifolds and physics, by Yvonne Choquet-Bruhat, Cecile de Witt-Morette, and Margaret Dillard-Bleick, North-Holland, Amsterdam, The Netherlands, 1977, xviii + 544 pp., $19.50.

Physical mathematics has always been an important part of mathematics as a discipline which concerns itself with deepening and uncovering mathematical theorems by interpreting them in the light of applications to physics. Of course, in mathematics one is often faced with the challenge of putting a result in the right perspective ("what does this really mean?"), to look at it the right way; but even more so when it comes to relating the formulas to the "real world". Many an apology has been made on behalf of the cult of pure mathematics (pure almost in the sense of virgin, untouched by any reality but the mathematical), that here is where the beauty of the subject is found. This point of view is in turn still under fire from those advocating less abstraction and more solution in mathematics. I think there is an in-between, indeed I see a genuine interest in the mathematical community in applications of mathematics, in combining abstract beauty with concrete power, and even remote hopes of assisting physics in its many struggles with fields and particles.

Theoretical physics deals with building models of so-called physical systems; speaking of a physical system already breaks down the universe in two parts: the system plus a background (to the neglected or influencing the system in a given way). This jig-saw puzzle approach must add up to our given universe (the only true physical system: "les lois physiques concernent tous les mondes possibles, alors que le monde réel n'est tiré qu'à un seul exemplaire" (H. Poincaré))–a complicated verification by experimental physics.

Perhaps the system to which most attention (and success) has been devoted is that of the Hydrogen atom: a point particle moving in $\mathbf{R}^3$ under the influence of a central force field with potential $-r^{-1}$, $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.