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DOUBLE CENTRALIZERS OF PEDERSEN'S IDEAL OF A C^*-ALGEBRA

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1. **Introduction**. The theory of double centralizers was developed for topological algebras by Johnson in [9] and [10] and further investigated in the case of a C^*-algebra by Busby [2]. If A is a commutative C^*-algebra, that is $A = C_0(X)$—the algebra of all complex valued continuous functions which vanish at infinity on a locally compact Hausdorff space X—then the algebra of all double centralizers of A is $C_b(X)$—all the bounded compley continuous functions on X [10]. The noncommutative generalization of the relationship between $C_0(X)$ and $C_b(X)$ was found useful by Busby in his papers on extensions of C^*-algebras ([2], [3]). The purpose of this note is to construct and to investigate a noncommutative analogue of $C(X)$, the algebra of all complex continuous functions on X. Let $C_c(X)$ be the ideal of $C_0(X)$ consisting of all the functions with compact support. Then the algebra of all double centralizers of $C_c(X)$ can be identified with $C(X)$ [10]. Thus, a way to find a generalization of the algebra $C(X)$ is by using an ideal of a C^*-algebra which plays a similar role to that of $C_c(X)$ in $C_0(X)$. Such an ideal was shown to exist in any C^*-algebra by Pedersen [12] and we shall exploit its properties towards the above stated aim. The full details of our discussion will appear elsewhere and we intend to pursue the matter in subsequent papers.

We refer the reader to the papers [2], [9] and [11] for the definitions and the main facts concerning double centralizers and Pedersen's ideal. From now on A will denote a C^*-algebra. We shall denote its Pedersen's ideal by K_A or simply by K if the C^*-algebra under consideration is well understood. The algebra of all double centralizers of A (respectively, K) is denoted by $\Gamma(A)$ (respectively, $\Gamma(K)$). The subalgebra of $\Gamma(K)$ consisting of all double centralizers (S, T) for which S, T are bounded will be denoted by $M(K)$. When convenient, we shall identify A, respectively K, with their canonical images in $\Gamma(A)$, respectively $M(K)$ [9]. If $B \subset A$ then $B^+ = \{a \in B : 0 \leqq a\}$.

2. In this section we shall present a few simple properties of K and $\Gamma(K)$ which are useful in many of our proofs.

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LEMMA 2.1. *If $(S, T) \in \Gamma(A)$, then K is invariant under S and T.*

PROOF. Let $a \in K^+$. By [12], $a^{1/2} \in K$ thus $S(a) = S(a^{1/2})a^{1/2}$ is in K.

COROLLARY 2.2. *The map $(S, T) \to (S|_K, T|_K)$ from $\Gamma(A)$ into $\Gamma(K)$ is an isometric *isomorphism of $\Gamma(A)$ onto $M(K)$.*

From now on we shall identify $\Gamma(A)$ with $M(K) \subset \Gamma(K)$ by means of the above map.

LEMMA 2.3. *Let $\{a_i\}_{i=1}^n \subset K$. Denote by $\mathscr{L}$ (respectively, $\mathscr{R}$) the smallest closed left (respectively, right) ideal of A containing $\{a_i\}_{i=1}^n$. Then $\mathscr{L}, \mathscr{R} \subset K$.*

PROOF. Since K is positively generated we may suppose $\{a_i\}_{i=1}^n \subset K^+$. It follows from [6, 12.4.1] that $\mathscr{L}$ is the closed left ideal generated by $a = \sum_{i=1}^n a_i$. Let B_a be the C^*-algebra of A generated by a. From [12], we infer that $B_a \subset K$. It is easy to check that $\mathscr{L} \cdot B_a \subset \mathscr{L}$ and that any approximate unit for B_a is a right approximate unit for $\mathscr{L}$. The Cohen-Hewitt factorization theorem (see [1], [8]) implies that any $x \in \mathscr{L}$ can be factored as $x = y \cdot z$ where $y \in \mathscr{L}$, $z \in B_a$.

We shall denote by $\mathscr{L}_a$ ($\mathscr{R}_a$, respectively) the closed left (right respectively) ideal of A generated by $a \in A$.

LEMMA 2.4. *Let $a \in K$ and $(S, T) \in \Gamma(K)$, Then $\mathscr{L}_a$ is invariant under S and $\mathscr{R}_a$ is invariant under T. Consequently, $S_{|\mathscr{L}_a}$ and $T_{|\mathscr{R}_a}$ are bounded operators.*

PROOF. If $x \in \mathscr{L}_a$ then $S(x) = \lim T(e_\lambda)x$ where $\{e_\lambda\}_{\lambda \in \Lambda}$ is an approximative unit for A contained in K [6, 1.7.2].

3. In this section we shall define a locally convex topology on $\Gamma(K)$ which will give to $\Gamma(K)$ a structure of topological algebra.

The locally convex topology on $\Gamma(K)$ determined by the family of seminorms $\lambda_x[(S, T)] = \|S(x)\|$ and $\rho_x[(S, T)] = \|T(x)\|$ for all $x \in K$ is called the κ-topology.

PROPOSITION 3.1. *$(\Gamma(K), \kappa)$ is a locally convex topological algebra; that is, $(\Gamma(K), \kappa)$ is a Hausdorff locally convex space and the multiplication is separately continuous.*

PROPOSITION 3.2. *$(\Gamma(K), \kappa)$ is complete and K is κ-dense in $\Gamma(K)$.*

PROPOSITION 3.3. *If $(S, T) \in \Gamma(K)$, then S, T are κ-continuous operators on K.*

If $A = C_0(X)$ then it is easily checked that κ is the topology of compact-open convergence on $C(X)$. For $A = LC(H)$, the algebra of all compact operators on the Hilbert space H, we have, after suitable identifications, $\Gamma(K) = \Gamma(A) = B(H)$, the algebra of all bounded operators on H. A net $\{T_i\}_{i \in I}$ in $B(H)$ κ-converges to $T \in B(H)$ iff $\{T_i\}_{i \in I}$ and $\{T_i^*\}_{i \in I}$ are strongly convergent to T T^* respectively. This example shows that the multiplication in $\Gamma(K)$ is not jointly κ-continuous in general.

The following is an easy consequence of the Cohen-Hewitt factorization theorem.

LEMMA 3.4. *The collection of sets*

$$W_a = \{(S, T) \in \Gamma(K) : \|S(a)\| \leqq 1, \|T(a)\| \leqq 1\}$$

for all $a \in K^+$ is a neighbourhood base at the origin for the κ-topology.

If $f \in A'$ (the Banach space dual of A) and $a \in A$ we denote by $a \cdot f$ and $f \cdot a$ the functionals $x \to f(ax)$, $x \to f(xa)$, respectively.

THEOREM 3.5. $\Gamma(K)' = \{a \cdot f + g \cdot a : a \in K^+, f, g \in A', \|f\|, \|g\| \leqq 1\}$.

SKETCH OF PROOF. If $U_a = \{(S, T) \in \Gamma(K) : \|S(a)\| \leqq 1\}$, $V_a = \{(S, T) \in \Gamma(K) : \|T(a)\| \leqq 1\}$, $W_a = U_a \cap V_a$ and $C = \{f \in A' : |f| \leqq 1\}$, then we have (i) U_a, V_a are weakly closed absolutely convex sets; (ii) $a \cdot C + C \cdot a$ is a w^*-closed absolutely convex subset of $\Gamma(K)'$; (iii) $U_a \subset \{f \cdot a : f \in C\}$, $V_a \subset \{a \cdot f : f \in C\}$. After these facts are established the conclusion follows readily from Lemma 3.4 and the Hahn-Banach theorem.

PROPOSITION 3.6. *If f is a pure state on A, then $f \in \Gamma(K)'$; that is, f is κ-continuous on A thus admits a unique κ-continuous extension to $\Gamma(K)$.*

By using the polar decomposition for functionals on A [6, 12.2.4] and a representation of $\Gamma(K)'$ similar to that given by Theorem 3.5 we obtain

THEOREM 3.7. *If $f \in \Gamma(K)'$ is selfadjoint, then $|f| \in \Gamma(K)'$.*

Here $|f|$ denotes the absolute value of f as a bounded linear functional on A and $|f| \in \Gamma(K)'$ means that $|f|$ is κ-continuous.

COROLLARY 3.8. *$\Gamma(K)'$ can be identified with a dense positively spanned subspace of A'. If $0 \leqq f \leqq g$ and $g \in \Gamma(K)'$, then $f \in \Gamma(K)'$. If $f \in \Gamma(K)'$ and $a \in A$ then $a \cdot f$ and $f \cdot a$ belong to $\Gamma(K)'$.*

REMARK. A positive linear functional on $\Gamma(K)$ need not be κ-continuous. Indeed, let X be a pseudocompact, locally compact noncompact Hausdorff space. If p is in the Stone-Čech compactification X but not in X, then the evaluation at p is a nonzero positive functional on $C(X) = C_b(X)$ but is not κ-continuous since it vanishes identically on $C_0(X)$.

4. For $A = C_0(X)$, X a locally compact Hausdorff, $M(K) = \Gamma(K)$ means precisely that the spectrum of A, i.e. X, is pseudocompact. In this section we shall analyse the phenomenon $M(K) = \Gamma(K)$ in the noncommutative case.

THEOREM 4.1. *The following statements are equivalent:*
(i) $M(K) = \Gamma(K)$;
(ii) *A is a two-sided ideal in $\Gamma(K)$;*
(iii) *the κ-bounded subsets of A are uniformly bounded;*
(iv) *for every sequence $\{x_n\}_{n=1}^{\infty} \subset K^+$ with $\lim_{n\to\infty}||x_n|| = \infty$ and $x_n x_m = 0$ for $n \neq m$, the sequence of partial sums $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is not κ-Cauchy.*

SKETCH OF PROOF. Let M be a κ-bounded subset of A and $\{z_n\}_{n=1}^{\infty} \subset M$ such that $||z_n|| \geqq 16^n$. Set $x_n = z_n^* z_n/||z_n||^{3/2}$. The sequence of partial sums $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is κ-Cauchy. By Proposition 3.2, it converges to some $x \in \Gamma(K)$. Now let $y_n = x_n/||x_n||^{3/2}$. The series $\sum_{n=1}^{\infty} y_n$ is absolutely convergent. Let $y \in A$ be its sum. By (ii) we have $y^{1/2}xy^{1/2} \in A$. On the other hand, $||y^{1/2}xy^{1/2}|| \geqq ||x_n||^{1/2} \geqq 2^n$ for every n.

(iv) $\Rightarrow$ (i). Let $\{d_\lambda\}_{\lambda\in\Lambda}$ be a positive approximative unit for A contained in K. If $M(A) \neq \Gamma(K)$ then there is $x \in \Gamma(K)$ such that $||xd_\lambda|| \to \infty$. By induction one can find a sequence $\{d_{\lambda_n}\}_{n=1}^{\infty}$ such that $||d_{\lambda_{n+1}}d_{\lambda_p} - d_{\lambda_p}|| < 1/n$ for $1 \leqq p \leqq n$, $n = 1, 2, \ldots$ and $||xd_{\lambda_n}|| \to \infty$. Let $A_0 = \{y \in A: \lim_{n\to\infty} yd_{\lambda_n} = \lim_{n\to\infty} d_{\lambda_n}y = y\}$. A_0 is a C^*-subalgebra of A and d_{λ_n} is an approximative unit for A_0. It follows from **[14]** that A_0 has a positive approximative unit $\{e_n\}_{n=1}^{\infty}$ such that $e_{n+1}e_n = e_n$ for every n. We have $\{e_n\}_{n=1}^{\infty} \subset K^+$ and from Lemmas 2.3 and 2.4 it follows that $\liminf||xe_n|| \geqq ||xd_{\lambda_p}||$ for every p. Thus we may suppose $||xe_{2n+1}|| \geqq 2^{n+1} + ||xe_{2n}||$. Now setting

$$y_n = (e_{2n} - e_{2n-1})x^*x(e_{2n} - e_{2n-1}) \quad \text{and} \quad x_n = y_n/2^n$$

we get a sequence $\{x_n\}_{n=1}^{\infty} \subset K^+$ such that $||x_n|| \to \infty$, $x_n x_m = 0$ if $n \neq m$ and $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is κ-Cauchy.

A C^*-algebra which satisfies any one of the conditions (i)—(iv) of Theorem 4.1 will be called a *PCS*-algebra. Since, for every irreducible representation π of a C^*-algebra A, we have, by **[6, 2.8.3]**, $H_\pi = \{\pi(x)h: x \in K, h \in H_\pi\}$, we get immediately

COROLLARY 4.2. *If A is a C^*-algebra and I is a primitive ideal of A, then A/I is a PCS-algebra.*

THEOREM 4.3. *If the spectrum of A is compact, then A is a PCS-algebra.*

The proof relies on

LEMMA 4.4. *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in K^+ such that $x_n x_m = 0$ if*

$n \neq m$. Assume that $||x_n|| \to \infty$ and that the sequence of partial sums $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is κ-Cauchy. Let $\hat{A}$ be the spectrum of A. Then, for any $a \in K$, the sequence $\{\alpha_n(a)\}_{n=1}^{\infty}$, where

$$\alpha_n(a) = \sup\{||\pi(a)|| : \pi \in \hat{A}, ||\pi(x_n)|| > ||x_n||/2\},$$

converges to zero.

Techniques similar to those used in the proof of Theorem 4.1 together with the Dauns-Hofmann theorem ([**5**], [**7**]) lead to

LEMMA 4.5. *Let I be a closed two-sided ideal in A whose spectrum $\hat{I}$ is Hausdorff and dense in $\hat{A}$. If A is not a PCS-algebra then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in K_I^+ such that the following hold: $||x_n|| \to \infty$; the functions $\pi \to ||\pi(x_n)||$ on $\hat{I}$ have pairwise disjoint compact supports; the sequence $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is κ-Cauchy in $\Gamma(K_A)$.*

This lemma is instrumental in proving part of

THEOREM 4.6. *Let A be a C^*-algebra with Hausdorff spectrum $\hat{A}$. Then A is a PCS-algebra iff $\hat{A}$ is pseudocompact.*

SKETCH OF PROOF. We shall outline here the proof of one implication only. Let A be a *PCS*-algebra and assume that $\hat{A}$ is not pseudocompact. By Theorem 4.1, there is a sequence $\{f_n\}_{n=1}^{\infty}$ of nonnegative continuous functions on $\hat{A}$ which have pairwise disjoint compact supports and such that $||f_n|| \to \infty$ and the sequence $\{\sum_{k=1}^{n} f_k\}_{n=1}^{\infty}$ is Cauchy in the compact open topology. Let $V_n = \{\pi \in \hat{A} : f_n(\pi) > ||f_n||/2\}$ and pick $\pi_n \in V_n$, $n = 1, 2, \ldots$. Let g_n be a continuous function on $\hat{A}$ with values in $[0, 1]$ which vanishes off V_n and $g_n(\pi_n) = 1$. Pick $z_n \in K^+$ with $\pi_n(z_n) \neq 0$. By the theorem of Dauns and Hofmann mentioned above there are $x_n, y_n \in K^+$ such that $\pi(y_n) = g_n(\pi)\pi(z_n)$ and $\pi(x_n) = f_n(\pi)\pi(y_n)/||y_n||$ for every $\pi \in \hat{A}$. Then $x_n \cdot x_m = 0$ if $n \neq m$ and $||x_n|| > ||f_n||/2$ so $||x_n|| \to \infty$.

Now let a be an element of A^+ for which there is $b \in A^+$ with $ab = a$. K is the smallest hereditary two-sided ideal of A containing all such elements $a \in A^+$. For a as above, the function $\pi \to |\pi(a)|$ has compact support in $\hat{A}$ and

$$\left\| a\left(\sum_{k=p}^{q} x_k\right)\right\|^2 \leqq ||a||^2 \sup\left\{\left|\sum_{k=p}^{q} f_k(\pi)\right| : \pi \in \hat{A}, \pi(a) \neq 0\right\}.$$

Since $\{\sum_{k=1}^{n} f_k\}_{n=1}^{\infty}$ is Cauchy in the compact-open topology, $\{\sum_{k=1}^{n} x_k\}_{n=1}^{\infty}$ is κ-Cauchy and this contradicts Theorem 4.1.

The next result generalizes the well-known theorem of Phillips [**13**] on the nonexistence of projections from l^{∞} onto c_0. A commutative version of this result is due to Conway [**4**]. The proof relies on Theorem 4.1 and on Phillips' theorem.

THEOREM 4.7. *Let B be a C^*-subalgebra of A containing an approximative unit for A. If A is complemented in $\Gamma(A)$, then B is a PCS-algebra.*

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