BOOK REVIEWS

Abelian categories. An introduction to the theory of functors. By Peter Freyd. (Harper's Series in Modern Mathematics.) Harper & Row, New York, 1964. 11+164 pp. $7.00.

A healthy new seed was planted some twenty odd years ago in the well fertilized soil of the mathematical periodical literature—the notion of a category. It sprouted, took root, flowered, attracted bees, and by now the landscape is dotted with its progeny. It is a beneficent plant: mathematical gardeners have come to appreciate its usefulness in holding down the topsoil and preventing dust storms; indeed, some half dozen books have appeared within the past dozen years putting it to this use. It is a beautiful plant too, whose rapid proliferation has produced many unique and exotic variants; but, perhaps because of its increasingly multiform variety, the book extolling all its loveliness has not yet been written. So it is that the book at hand dwells at length only upon abelian categories, whose features now seem the most regular and symmetrical, and whose uses are at this point the best known. What may be somewhat surprising, therefore, or even somewhat lamentable, is that these uses are discussed not at length but only in passing, in the introduction and in some of the exercises. The book devotes itself, instead, to the proof of a rather miraculous result—the so-called full embedding theorem—according to which every small abelian category admits a full, exact embedding into the full category of modules over a suitable ring.

But now comes an odd turn of events. From the full embedding theorem there follows a strong metatheorem, to which is due much of the importance of the full embedding theorem in applications of abelian categories. According to the metatheorem, an assertion about a diagram is valid in every abelian category as soon as it can be obtained in each category of modules by a diagram chasing argument that, on the basis of given exactness and commutativity conditions, supplies missing maps at required locations and establishes specified additional exactness and commutativity conditions. Such is the strength of this metatheorem that, in the words of the author, "Much of the theory *within* abelian categories is reduced go the theory of modules." Strange, double edged metatheorem. Surely this statement seems better calculated to defeat than to support the encouragement of further study of abelian categories. Surely, indeed; and yet, equally surely, the value of the two principal ideas behind the proof of the full embedding theorem must be great in proportion to the magnitude of the defeat inflicted by its corollary, the metatheorem. The first idea in question is the condensation of the functors between two categories into a category in its own right, and the detailed study of the relations in and among such functor categories. (The existence of his notes entitled "Functor categories and their application to relative homological algebra" serves notice that the author is not so much to be believed when he claims to have the full embedding theorem and consequent metatheorem as his goal as when he subtitles his book *An introduction to the theory of functors*.) The second idea is the notion of injectivity (the common property of the real numbers among Banach spaces, the unit interval among compact spaces, or any complete boolean algebra among all boolean algebras, described, respectively, by the extension theorems of Hahn-Banach, Tietze, or Sikorski). A further demonstration of the usefulness of this notion is coupled with a gentle demolishment of the defeatist interpretation of the metatheorem quoted above in a lovely passel of exercises on module theory (following Chapter six), in which classical theorems on nötherian ideal theory, semisimple rings, and modules over principal ideal domains are developed via injectives and functorial-categorical techniques.

Having already divulged—inadvertently, as it were—some of the material in the book, it may be appropriate now to describe more fully the topics covered. Roughly speaking, the book comes in two halves. The first, consisting of the introduction and first three chapters, unveils abelian categories and additive functors. The second, consisting of the last four chapters, proceeds more or less directly to the full embedding theorem. The well knit introduction defines categories, in a sequence of successive approximations, motivated, through examples of natural transformations and functors—from the top down, it appears—by the need for something functors are defined on. Chapter one presents the fundamental categorical constructibles, such as dual categories, the set valued hom functors, subobjects, zeros, kernels, products, etc., and is followed by exercises dealing mainly with various specific concrete categories. An abelian category is defined in the second chapter as a category with zero, finite sums and products, and kernels and cokernels, in which every monomorphism (epimorphism) is a kernel (cokernel). In easy stages, the more traditional definition in terms of an additive category with unique factorization is recovered, some useful pushout-pullback properties of commutative squares are explored, and a few classic diagram chases are joined. The third chapter discusses additive, exact, full, and faithful functors on abelian categories, along with the concepts, defined in terms of such functors, of injective and projective objects, exact and full subcategories, generators and cogenerators, etc. The exercises to this chapter are undoubtedly the best in the book: they deal mainly with adjoint functors and include several versions (all non-abelian) and even more applications (the most notable of these being to model theory) of the existence theorem for adjoint functors. It is only to be regretted that the author could not bring himself to include these matters in the main text. "I tried to write them as a separate chapter," he explains in the appendix, "but the chapter grew longer than the rest of the book." It would have been worth it. Further applications of adjointness appear in the exercises to Chapter five.

Chapter four marks the beginning of the road to the full embedding theorem. Here the reader will find the metatheorem mentioned above; here too, he will take the first step along the road, with the proof that in a complete abelian category $\mathcal{B}$ having a projective generator, each small, full, exact subcategory admits a full, exact embedding into the full category of modules over some ring. The full embedding theorem will follow as soon as it is known that every small abelian category $\mathcal{A}$ is a full, exact subcategory of such a category $\mathcal{B}$. The whole purpose of the rest of the book is to establish this fact. As it happens, $\mathcal{B}$ can be taken to be the dual of the category $\mathcal{L}(\mathcal{A})$ of kernel-preserving additive functors from $\mathcal{A}$ to the category of abelian groups. $\mathcal{A}$ is viewed as a subcategory of $\mathcal{B}$ by identifying the object $A\in\mathcal{A}$ with that functor in $\mathcal{B}$ sending $X\in\mathcal{A}$ to the group of all $\mathcal{A}$-morphisms from A to X. That in this way $\mathcal{A}$ becomes a full subcategory of $\mathcal{B}$ follows from the Yoneda theory developed in Chapter five. That $\mathcal{B}$ has a projective generator is a consequence, derived in Chapter seven, of the general considerations on injectives found in Chapter six. The completeness of $\mathcal{B}$ and the exactness of the embedding $\mathcal{A}\rightarrow\mathcal{B}$ are quickly disposed of in the rest of Chapter seven, completing the proof of the full embedding theorem and closing the book.

Since to reveal the mysteries of the index, the appendix, and the pagination would be to rob the prospective reader of a triple source of pleasure, it remains only to tell of the errors, the exercises, and the exposition. In a word, the first are few, the second, many, and the third, well arranged. Specifically, there are eight or ten easily corrected transpositions, substitutions, or omissions of mathematical symbols, a few dropped or added terminal "s"'s, and another dozen or so punctuational misplacements. The exercises, on the other hand, abound: they occupy some sixty pages, nearly forty percent of the book; moreover, they are very well conceived and, with the exception of a few early ones that ask the reader to verify as yet undefined properties, quite appropriately located. Those following Chapters three and six, as has already been pointed out, are particularly lovely; no less so are those following Chapter seven. There are one or two lapses in the otherwise exceptionally smooth, unhesitating, yet comfortably paced exposition, most notably in the last eight lines of page 130, where the author's notation does not convey his intentions (what he has in mind is to be found on pp. 124 ff. of his notes "Abelian categories: the inside theory"), and again, to a lesser extent, in the second five lines of p. 128, where the desire for brevity has all but banished clarity. Everywhere else, the writing is crisp, informative, and unambiguous; and it sweeps the reader gently but unhesitatingly forward to its goal.

In sum, one may quibble about what else should have found room in Freyd's book, but considering it as it comes to us, it is well produced, has worthwhile things to say, and says them clearly and with determination. Any algebraist—indeed, any modern-minded mathematician, whether full-fledged or nestling—will probably want to read at least parts of it at least once.

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Number theory.[1] By Z. I. Borevich and I. R. Shafarevich, Moscow, 1964.

The Russian mathematicians have for many years been partial to number theory, and it is an interesting speculation why this should be so. Can it be due to the influence of Euler who spent many years in the later part of the eighteenth century in what was then called St. Petersburg, or perhaps to some characteristic of the Russian mind? Be it so or not, they have long been known for their numerous contributions to the subject. They have originated many new ideas and made a large number of really important and fundamental discoveries. These have opened up new avenues of investigation and have had far reaching effects and influence on the development of the theory.

One calls to mind the work of Tchebycheff on prime numbers, of Markoff on the minima of indefinite binary quadratic forms, the work of Korkine, Zolotareff, Voronoi on quadratic forms in several variables, the work of Khintchine on the density of sequences of numbers and on Diophantine approximation, Tartowski on Diophantine equations, Tchebotareff on the minimum of the product of linear poly-

[1] An English translation is being prepared by the Academic Press.