DEMICONTINUITY, HEMICONTINUITY AND MONOTONICITY

BY TOSIO KATO

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Recently the notions of monotone, demicontinuous and hemicontinuous functions have been introduced in connection with nonlinear problems in functional analysis (Browder [1; 2; 3; 4; 5], Minty [6; 7; 8]). The object of the present note is to show that under rather general conditions, hemicontinuity is equivalent to demicontinuity for monotone functions.

Let X be a (real or complex) Banach space and X^* its adjoint space as the set of all bounded conjugate-linear functionals on X. The value of $f \in X^*$ at $u \in X$ is denoted by (f, u). We use the notations $\rightarrow$ and $\rightharpoonup$ for strong convergence in X (or in X^* or in the set of real numbers) and weak* convergence in X^*, respectively.

Let G be a function from X to X^* with domain $D = D(G) \subset X$. G is said to be *demicontinuous* if $u_n \in D$, $n = 1, 2, 3, \cdots$, $u \in D$ and $u_n \rightarrow u$ imply $Gu_n \rightharpoonup Gu$. G is *hemicontinuous* if $u \in D$, $v \in X$ and $u + t_n v \in D$, where t_n is a sequence of positive numbers such that $t_n \rightarrow 0$, imply $G(u + t_n v) \rightharpoonup Gu$. We shall say that G is *locally bounded* if $u_n \in D$, $u \in D$ and $u_n \rightarrow u$ imply that Gu_n is bounded. Obviously a demicontinuous function is hemicontinuous and locally bounded.

G is said to be *monotone* if $\operatorname{Re}(Gu - Gv, u - v) \geqq 0$ for $u, v \in D$.

These definitions may be void if D is too arbitrary. In what follows we shall assume that D is *quasi-dense*. By this we mean that for each $u \in D$ there is a dense subset M_u of X such that for each $v \in M_u$, $u + tv \in D$ for sufficiently small $t > 0$ (the smallness of t depending on v). Thus any open subset of X as well as any dense linear manifold of X is quasi-dense.

THEOREM 1. *Let G be a monotone function from X to X^* with a quasi-dense domain D. Then G is demicontinuous if and only if it is hemicontinuous and locally bounded.*

PROOF. By the remark given above, it suffices to prove the "if" part. Suppose G is hemicontinuous and locally bounded. Let $u_n \rightarrow u$, $u_n, u \in D$. We have to show that $Gu_n \rightharpoonup Gu$. Obviously we may assume that $u_n \neq u$.

Let M_u be the dense subset of X used in the definition of D being quasi-dense. Let $v \in M_u$ and $t_n = \|u_n - u\|^{1/2}$. Then $t_n > 0$, $t_n \rightarrow 0$, $w_n = u + t_n v \in D$ for sufficiently large n and

$$Gw_n \rightharpoonup Gu. \tag{1}$$

Now the monotonicity of G implies

$$0 \leqq \operatorname{Re}(Gu_n - Gw_n, u_n - w_n) = \operatorname{Re}(Gu_n - Gw_n, u_n - u - t_n v). \tag{2}$$

Gu_n is bounded since G is locally bounded. Gw_n is bounded by (1). Hence

$$t_n^{-1} \operatorname{Re}(Gu_n - Gw_n, u_n - u) \to 0$$

because $\|t_n^{-1}(u_n - u)\| = t_n \to 0$. Also $(Gw_n, v) \to (Gu, v)$ by (1). Dividing (2) by t_n and letting $n \to \infty$, we thus obtain

$$\liminf \operatorname{Re}(Gu_n - Gu, -v) \geqq 0. \tag{3}$$

(3) is true for any $v \in M_u$. Since M_u is dense in X and Gu_n is bounded in X^*, it follows that (3) is true for every $v \in X$. Replacing v by $-v$ (and also by $\pm iv$ if X is complex) and putting the results together, we obtain

$$\lim (Gu_n - Gu, v) = 0, \qquad v \in X.$$

This proves that $Gu_n \rightharpoonup Gu$, q.e.d.

REMARK 1. Theorem 1 shows that a monotone hemicontinuous function that maps bounded sets into bounded sets is a notion stronger than a monotone demicontinuous function. (Such functions are considered in [2-III] and [5].)

REMARK 2. It is not clear whether the assumption of local boundedness in Theorem 1 can be eliminated. But this is the case if X is finite-dimensional. We have namely

THEOREM 2. *Let X be a finite-dimensional Banach space. Let G be a monotone function from X to X^* with a quasi-dense domain D. Then G is continuous if and only if it is hemicontinuous.*

PROOF. Since continuity and demicontinuity are equivalent when X is finite-dimensional, it suffices to show that G is locally bounded if it is hemicontinuous; then the result follows from Theorem 1.

Suppose that G is hemicontinuous but not locally bounded. Then there is a $u \in D$ and a sequence $u_n \in D$ such that $u_n \to u$ but Gu_n is unbounded. We may assume without loss of generality that $\|Gu_n\| = s_n \to \infty$. Let M_u be as above and let $v \in M_u$. Take a $t > 0$ so small that $u + tv \in D$. Then by monotonicity

$$\begin{aligned} 0 &\leqq s_n^{-1} \operatorname{Re}(Gu_n - G(u + tv), u_n - u - tv) \\ &= \operatorname{Re}(s_n^{-1} Gu_n - s_n^{-1} G(u + tv), u_n - u - tv). \end{aligned} \tag{4}$$

Now $s_n^{-1}G(u+tv)\to 0$, $u_n-u\to 0$ and $s_n^{-1}Gu_n$ is bounded. On dividing (4) by $t>0$ and letting $n\to\infty$, we thus obtain

$$\liminf \operatorname{Re}(s_n^{-1}Gu_n,\ -v) \geqq 0.$$

As in the proof of Theorem 1, this leads to the result that $s_n^{-1}Gu_n \to 0$. But this is a contradiction, for $\|s_n^{-1}Gu_n\|=1$ and weak* convergence is equivalent to strong convergence.

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University of California, Berkeley