LACUNARY TAYLOR AND FOURIER SERIES

J.-P. KAHANE[1]

To the memory of Jacques Hadamard

Introduction. The history of lacunary Fourier and Taylor series goes back to Weierstrass and Hadamard, if not to Riemann.

According to Weierstrass [49], Riemann told his students in 1861 that the continuous function

$$\sum_{1}^{\infty} \frac{\sin n^2 x}{n^2} \tag{1}$$

is nowhere differentiable. As Weierstrass was not able to prove it (and, in fact, until now, it seems to have been neither proved nor disproved), he gave (1872) his famous example

$$\sum_{1}^{\infty} a^n \cos \lambda^n x \tag{2}$$

where λ is an odd integer $\geqq 3$, and a a positive number such that $a<1$ and $a\lambda > 1+3\pi/2$: (2) is a continuous function which is nowhere differentiable [49]. Later on, Weierstrass's result was improved by Hardy: the previous statement holds under the assumption $a\lambda \geqq 1$ instead of $a\lambda > 1+3\pi/2$ [12]. In Hardy's version, that is a rather hard theorem; as we shall see later, it can be made very easy.

Hadamard (1892) proved that the Taylor series

$$\sum_{1}^{\infty} a_n z^{\lambda_n} \qquad \limsup_{n\to\infty} |a_n|^{1/\lambda_n} = 1 \tag{3}$$

has $|z| = 1$ as a natural boundary, whenever there exists a $q>1$ such that

$$\frac{\lambda_{n+1}}{\lambda_n} > q > 1 \qquad (n = 1, 2, \cdots) \qquad [11, \text{p. } 116]. \tag{4}$$

(4) is known as Hadamard's lacunarity condition. We shall see that Hadamard's condition has played quite an important part in many directions. However, it is not what is needed about $\{\lambda_n\}$ to get that

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$|z| = 1$ is a natural boundary for (3). Fabry (1898) proved it under the weaker assumption

$$\lim_{n\to\infty} \frac{n}{\lambda_n} = 0 \tag{5}$$ [9].

Pólya proved in 1942 that (5) is exactly the relevant condition [34]; for another proof, see [5].

The connection between gaps of a Taylor series and singular points of the represented function (or singular directions if the function is an entire one) has been studied extensively by many authors, and often treated in the more general context of Dirichlet series. Results prior to 1936 are reviewed in Mandelbrojt's pamphlet [28]. Newer results and references can be found in Bieberbach's and Schwartz's books [4], [40]. These references can be completed by [45], [8], [47], and [48]. We shall not discuss this aspect here. Instead we shall consider some problems about power series which have been worked out recently, and where Hadamard's lacunarity condition seems to be more relevant.

In the theory of Fourier series, Hadamard's condition appears in many statements; see e.g. [55, Vol. I, pp. 202–212, 215, 230, 379–380]. We shall try to discuss to what extent it is involved. Besides Zygmund's, two classical books at least are concerned with lacunary Fourier series: Mandelbrojt's *Séries de Fourier et classes quasi-analytiques de fonctions* [29], and Levinson's *Gap and density theorems* [27]. Roughly speaking, Zygmund's book is more interested in the properties of lacunary series,

$$\sum_1^\infty r_n \cos(\lambda_n t + \phi_n), \tag{6}$$

as series of almost independent functions or, what is the same, as series of almost independent random variables. Mandelbrojt's and Levinson's books (the latter in the line of Paley-Wiener's *Fourier transforms in the complex domain*) are mainly concerned with properties of quasianalyticity, uniqueness or continuation.

It will be convenient for us to discuss problems and results for Taylor as well as Fourier series by dividing our review into two parts.

In the first one, we consider (3) or (6) as series of almost independent functions. We try to complete the report that Kac made in 1948 on this question [16].

In the second one, we give partial answers to a general question of Mandelbrojt [29]: assume that $\{\lambda_n\}$ is given; suppose we know a property of a function given by (6) on an interval, or in the neighborhood of a point, or on a perfect set without interior point; to what extent does it give information on (6) as a whole?

Some results cannot enter in this review. In particular, we do not discuss many instances of conjectures in analysis which have been proved or disproved by means of lacunary Fourier series. Even so, we wish to give the feeling that the field of lacunary series, though very tiny, is a very fertile one. Therefore it may be better if a few people go on and plow it and are able from time to time to dig out new curious plants that grow therein.

First part. I. Let us point out a few immediate or well-known properties of the series

$$\sum_1^\infty a_n e^{2\pi i\omega_n} \tag{7}$$

where the a_n are complex numbers and the ω_n independent real variables in $[0, 1[$. We denote by Ω the compact set $[0, 1]^\infty$, and write $\omega=(\omega_1, \omega_2, \cdots, \omega_n, \cdots)\in\Omega$ and $f(\omega)=$sum of (7) whenever (7) is convergent.

I.1. Suppose f bounded on Ω; then $\sum_1^\infty |a_n|<\infty$. Precisely,

$$\sum_1^\infty |a_n| = \sup_{\omega\in\Omega} |f(\omega)| = \sup_{\omega\in\Omega} \operatorname{Re} f(\omega). \tag{8}$$

I.2. Suppose $\sum_1^\infty |a_n| = \infty$, $\lim_{n\to\infty} a_n=0$. Given any complex w, there exists $\omega\in\Omega$ such that (7) is convergent in ω and $f(\omega)=w$.

I.3. Suppose $\sum_1^\infty |a_n|<\infty$, and $|a_n|\leqq\sum_{n=1}^\infty |a_j|$ for each n. Given any complex w in the closed disc $|w|\leqq\sum_1^\infty |a_n|$, there exists $\omega\in\Omega$ such that $f(\omega)=w$. In other words, $f(\Omega)$ is the whole disc $|w| \leqq\sum_1^\infty |a_n|$.

These properties are quite elementary. Let us introduce now the Lebesgue ($=$Haar) measure on Ω; then we can define the Fourier series of a $g\in L^1(\Omega)$.

I.4. All following statements are equivalent:

(a) $\sum_1^\infty |a_n|^2<\infty$,

(b) (7) converges a.e. in Ω,

(c) (7) converges a.e. in Ω and $f\in L^p(\Omega)$ (p given$\in[1, \infty[$),

(d) (7) converges a.e. in Ω and $e^{\lambda|f|^2}\in L^1(\Omega)$ for all $\lambda>0$,

(e) (7) is a Fourier series of a $g\in L^1(\Omega)$,

(f) (7) is a Fourier series of a $g\in L^1(\Omega)$, (7) is convergent a.e., and $f=g$ a.e. in Ω.

I.5 (Central limit theorem). Suppose $\sum_1^\infty r_n^2=\infty$ ($r_n\geqq 0$), let

(9) $$S_N(\omega) = \sum_1^N r_n \cos(2\pi i \omega_n)$$

and

$$B_N = \left(\frac{1}{2}\sum_1^n r_n^2\right)^{1/2}.$$

Given an interval I on the line, consider

$$E_N(I) = \left\{\omega \,\middle|\, \frac{S_N(\omega)}{B_N} \in I\right\}.$$

Then $E_N(I)$ is measurable in Ω, and

$$\lim_{N\to\infty} \left| E_N(I) \right| = \frac{1}{\sqrt{2\pi}} \int_I e^{-u^2/2} du.$$

I.6 (Law of the iterated logarithm). Moreover, suppose

$$A_N = \sup_{n=1,2,\ldots,N} \left| a_n \right| = o(B_N(\log\log B_N)^{-1/2}) \qquad (N \to \infty).$$

Then

$$\limsup_{N\to\infty} \frac{S_N(\omega)}{\sqrt{2B_N^2 \log\log B_N}} = 1 \text{ a.e. in } \Omega.$$

II. We ask what remains of these statements if we replace (7) by

(10) $$\sum_1^\infty a_n e^{2\pi i \lambda_n t}$$

and (9) by

(11) $$S_N(t) = \sum_1^N r_n \cos(2\pi\lambda_n t + \phi_n) \qquad (\phi_n \text{ real})$$

where $\{\lambda_n\}$ is a sequence of increasing integers satisfying Hadamard's lacunary condition (4); we accordingly replace Ω by $[0, 1]$, ω by t, $f(\omega)$ by $f(t)$.

II.1. A famous theorem of Sidon [41] asserts that the analog of I.1 is valid if (8) is replaced by

$$\sum_1^\infty \left| a_n \right| < K \sup_{t\in[0,1]} \operatorname{Re} f(t) \qquad (K = K(\{\lambda_n\})).$$

A more precise form is

$$\sum_1^\infty |a_n| < K \sup_{0\leqq t\leqq A/\lambda_1} \operatorname{Re} f(t) \qquad (K = K(q),\ A = A(q)),$$

(q being as in (4)) [50], [23].

II.2. The analog of I.2 is valid. That was stated without proof by Paley [32], and proved by Mary Weiss [50]. An alternative proof can be found in [23]. More generally, given any closed connected set S in the extended complex plane (or Riemann sphere), there exists a t such that the set of limit points of partial sums of (10) is exactly S [23].

II.3. Under the assumptions

$$|a_n| \leqq \gamma \sum_{n=1}^\infty |a_j|, \quad |w| \leqq \delta \sum_1^\infty |a_n| \qquad (\gamma = \gamma(q),\ \delta = \delta(q))$$

there exists a t such that $f(t)=w$ [23]. In other words, the curve $w=f(t)$ fills a disc. With a more stringent hypothesis (in particular, q had to be chosen big enough), that statement had been proved previously by Salem and Zygmund [37].

Each of the statements II.1, II.2, II.3 expresses that "there exists a t in $[0, 1]$ such that . . . ." It can be asked if the same holds when $[0, 1]$ is replaced by a well-chosen totally disconnected set E, depending only on $\{\lambda_n\}$. The first statement in this direction goes back to 1930; Zygmund proved that the real part of (10) is convergent on a set which is everywhere dense and everywhere of the power of the continuum as soon as $a_n=o(1)$ [53]. The question has been studied again recently, and here is an example of the results in [23]: each of the statements II.1, II.2, II.3 holds if we replace $[0, 1]$ by a set E of a Cantor type constructed in the following way. Let us fix a closed interval I and a constant ξ $(0<\xi<1/2)$. Let us remove from I an open subinterval situated in its middle and of length $(1-2\xi)|I|$; we get two closed intervals of length $\xi|I|$ with each of which we repeat the procedure; keeping up this process we get a closed set $E=E(I, \xi)$. If we choose $|I|>A/\lambda_1$ and $\xi>\xi_0$ $(A=A(q),\ \xi_0=\xi_0(q))$, each of the statements II.1, II.2, II.3, holds if t is restricted to $E(I, \xi)$. We shall discuss later the connection between ξ and q, which is not a very simple one.

Let us go back to II.2. A related problem is the following. Consider

$$f(z) = \sum_1^\infty a_n z^{\lambda_n} \tag{12}$$

$\limsup_{n\to\infty} |a_n|^{1/\lambda_n}=1$, $\sum_1^\infty |a_n| = \infty$. Is it true that f maps the *open* disc $|z|=1$ onto the whole complex plane? This was asked by Paley in [32], and appears to be much more difficult than II.2. An affirmative answer, when q is large enough, was obtained recently by Guido and Mary Weiss [52].

II.4. The analog of I.4 holds completely; all proofs can be found in Zygmund's book. This statement contains theorems of Kolmogoroff ((a)$\Rightarrow$(b), [26]), Zygmund ((b)$\Rightarrow$(a), [53] and about (a)$\Rightarrow$(d), [54]), Sidon ((a)$\Leftrightarrow$(e), [42]). Moreover, if E is a set of positive measure on $[0, 1]$, the statement remains true if we consider E instead of $[0, 1]$: that is a reformulation of a theorem of Zygmund [55, pp. 203 and 206].

II.5. The analog of I.5 holds completely with (11) instead of (9); the result is due to Salem and Zygmund [38]; for a history of the topic, see [16]. Recently, Erdös pointed out that it is not possible to relax the condition (4) in the general case; but, restricting himself to the case $r_n = 1$ for all n's, he proved that the conclusion holds whenever

$$\lambda_{n+1} \geqq \lambda_n\left(1 + \frac{c_n}{\sqrt{n}}\right), \qquad c_n = o(1) \ (n \to \infty) \qquad [7].$$

For example, we can take

$$\lambda_n = [e^{n^\alpha}] \quad \text{with} \quad \alpha > \tfrac{1}{2};$$

it is not known if the conclusion remains valid when $\lambda_n = [e^{\sqrt{n}}]$.

II.6. The analog of I.6 holds completely. The first results in this direction are due to Salem and Zygmund [39], and the final statement to Mary Weiss [51].

III. Let us discuss now to what extent we need a lacunary condition on $\{\lambda_n\}$, to get such and such result. In the context of orthogonal systems, this question arises in Kaczmarz-Steinhaus' book [17]. It has deserved a certain amount of attention in the recent past years [44], [1], [19], [14], [36], [13].

III.1. *Sidon sets.* On account of II.1, we say that a set Λ of integers (not necessarily positive) is a Sidon set if and only if the norms

$$\sum |a(\lambda)|, \qquad \sup_t |P(t)| \tag{13}$$

are equivalent for all finite sums $P(t) = \sum_{\lambda\in\Lambda} a(\lambda)e^{i\lambda t}$. Many equivalent definitions can be given, namely,

(a) Λ is a Sidon set,

(b) whenever $\sum_{\lambda\in\Lambda} a(\lambda)e^{i\lambda t}$ is the Fourier series of a bounded function, $\sum |a(\lambda)| < \infty$,

(c) to every bounded function b on Λ, there is a bounded measure μ on $[0, 2\pi[$ such that $b(\lambda) = \int e^{-i\lambda t} d\mu(t)$ $(\lambda \in \Lambda)$,

(d) to every bounded function b on Λ, tending to zero at infinity, there is an $f \in L^1(0, 2\pi)$ such that $b(\lambda) = (1/2\pi)\int e^{-i\lambda t}f(t)dt$ $(\lambda \in \Lambda)$,

(e) to every function χ on Λ, taking only values 0 and 1, there exists a bounded measure μ on $[0, 2\pi[$ such that $\left|\int e^{-i\lambda t}d\mu(t) - \chi(\lambda)\right| \leqq \delta < 1/2$ for every $\lambda \in \Lambda$, where δ may depend on χ.

The proof of (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d) can be found, e.g., in [**36**]; (e)$\Rightarrow$(a) sharpens an analogous statement of [**36**, p. 208]. The use of (e) makes it easy to get all known structural conditions that imply Λ to be a Sidon set.

All finite unions of Hadamard sets (i.e., sets $\{\pm\lambda_n\}_{n=1,2,\cdots}$ satisfying (4)) are Sidon sets, and there exist Sidon sets which are not finite unions of Hadamard sets [**44**], [**14**], [**36**].

If Λ is a Sidon set, there exists a constant $C = C(\Lambda)$ such that, whatever may be the integers $q_1, q_2, \cdots, q_n, s$ $(q_j \lessgtr 0,\ s > 0)$, Λ contains at most Cns of the numbers $\alpha_1 q_1 + \cdots + \alpha_n q_n$ (α_j integer, $|\alpha_1| + \cdots + |\alpha_n| \leqq 2^s$); in particular, Λ contains at most Cs elements between 2^{-s} and 2^s [**19**]. For previous examples of lacunary sets which are not Sidon sets, see [**6**].

It is not known if the union of two Sidon sets is necessarily a Sidon set. In this connection, there is a conjecture of Beurling: suppose that Λ is a Sidon set; does it imply that each characteristic function of a subset of Λ, defined on the integers, can be uniformly approximated by Fourier transforms of measures? An affirmative answer would prove that the union of two Sidon sets is a Sidon set.

III.2. *$\Lambda(s)$-sets*. Following Rudin [**36**], we say that $\Lambda \in \Lambda(s)$ $(s > 1)$, or that Λ is a $\Lambda(s)$-set, if and only if the norms

$$\int_0^{2\pi} |P(t)|dt, \qquad \left(\int_0^{2\pi} |P(t)|^s dt\right)^{1/s} \tag{14}$$

are equivalent for all finite sums $P(t) = \sum_{\lambda\in\Lambda} a(\lambda)e^{i\lambda t}$. As a consequence of II.4, Hadamard sets belong to $\Lambda(s)$ for all s. More generally, each Sidon set belongs to $\Lambda(s)$ for all $s > 1$. In Rudin's paper [**36**] a series of interesting properties of these sets are considered.

The most interesting class is $\Lambda(2)$. An alternative definition of $\Lambda(2)$ is the following: $\Lambda \in \Lambda(2)$ if and only if, given any function b on Λ such that $\sum |b(\lambda)|^2 < \infty$, there exists a continuous f such that $b(\lambda) = (1/2\pi)\int e^{i\lambda t}f(t)dt$ $(\lambda \in \Lambda)$ [**17**], [**14**], [**36**].

Sidon, in 1932 [**43**], proved that, if Λ is a symmetric set ($\lambda \in \Lambda \Rightarrow -\lambda \in \Lambda$) and if the number of solutions of $\lambda_1 - \lambda_2 = n$ (n given, $\lambda_1 \in \Lambda$, $\lambda_2 \in \Lambda$) is uniformly bounded with respect to n, $\Lambda \in \Lambda(2)$ (see also [**14**]). In fact, the hypothesis implies $\Lambda \in \Lambda(4)$. Until now, no $\Lambda \in \Lambda(2)$, $\Lambda \notin \Lambda(4)$, seems to be known. Here are some results and problems of

Rudin [36]. The union of two sets in $\Lambda(4)$ belongs to $\Lambda(4)$, but the analog is not known for $\Lambda(2)$. Given $\Lambda \in \Lambda(4)$, there exists a $C = C(\Lambda)$ such that Λ contains at most Cn points in an arithmetical progression of n^2 terms; the only known fact for $\Lambda(2)$ is that a $\Lambda \in \Lambda(2)$ cannot contain arbitrarily long arithmetic progressions; it is not even known if each $\Lambda \in \Lambda(2)$ has density zero. The set $\{1, 2^2, 3^2, 4^2, \cdots\}$ does not belong to $\Lambda(4)$, and it is an open question if it belongs to $\Lambda(2)$.

III.3. *Zygmund sets.* On account of a theorem of Zygmund quoted after II.3, we shall say that a set $\Lambda = \{\lambda_1, \lambda_2, \cdots\} (\lambda_{n+1} \geqq \lambda_n)$ of positive integers is a Zygmund set if, whenever $\{a_n\}$ is a complex sequence tending to zero at infinity, the series

$$\sum_1^\infty a_n e^{i\lambda_n t}$$

is convergent at least at one point. If (4) holds, Λ is a Zygmund set. In the opposite direction, Kennedy remarked that, to every $\phi(t) \downarrow 0$ when $t \to \infty$, there exists a Λ with

$$\frac{\lambda_{n+1}}{\lambda_n} > 1 + \phi(\lambda_n)$$

which is not a Zygmund set [25]; a slight change in the proof shows that a Zygmund set cannot contain arbitrarily long arithmetic progressions. Nothing more seems to be known.

III.4. Some investigations have been made on the following problem. Given Λ, find a positive function $\alpha(t)$ such that, if

$$\sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda t}$$

is the Fourier series of a continuous function, and $|a(\lambda)| < \alpha(\lambda)$, the series is convergent to f at every point; the partial sums are defined, as usual, as

$$\sum_{\lambda \in \Lambda, -N \leqq \lambda \leqq N} a(\lambda) e^{i\lambda t} \qquad [2], [46].$$

As an example,

$$\alpha(t) = \frac{1}{\sqrt{|t|}}$$

can be associated in this way with every sequence $\Lambda = \{\pm\lambda_n\}$ such that $\lambda_n \geqq n^2$ (Tomić), but not $\alpha(t) = \omega(t)/\sqrt{|t|}$, whatever may be $\omega(t) \uparrow \infty$ (Alpár).

III.5. *Paley sets.* We finally consider a property of Hadamard sets which does not seem to be connected with an analogous property of series of independent random variables. In 1933, Paley proved that, given any $f \in L^1(0, 2\pi)$,

$$f(t) \sim \sum_0^\infty \hat{f}(n) e^{int}$$

(all Fourier coefficients of negative order being zero), $\sum_{\lambda \in \Lambda} |\hat{f}(n)|^2 < \infty$ whenever Λ is a Hadamard set of positive integers [**31**]. Of course, the same works if Λ is a finite union of Hadamard sets. Rudin proved the converse, namely, that every set Λ of positive integers which enjoys this property (Paley set) is a finite union of Hadamard sets. Here are alternative definitions of a Paley set: a set Λ of positive integers is a Paley set if, given

$$\sum_{\lambda \in \Lambda} a(\lambda) e^{i\lambda t}, \qquad \sum |a(\lambda)|^2 < \infty$$

there exists a series $\sum_{n \leq 0} b(n) e^{int}$ such that the sum of both series is the Fourier series of a function which is bounded, resp. continuous, resp. bounded with constant absolute value [**35**].

Second part. I. In this section we shall discuss some problems of uniqueness or continuation for lacunary Fourier series

$$\sum_0^\infty r_n \cos(\lambda_n t + \phi_n) \qquad (r_n \geqq 0) \tag{15}$$

or Fourier-Taylor series

$$\sum_1^\infty a_n e^{i\lambda_n t} \qquad (a_n \text{ complex}). \tag{16}$$

For sake of simplicity, we suppose $\sum_0^\infty r_n^2 < \infty$ resp. $\sum |a_n|^2 < \infty$. We denote $f(t)$ resp. $F(t)$ the square summable function (resp. continuous, if it exists) whose Fourier series is (15) resp. (16).

We first consider the following problem of Mandelbrojt [**29**, p. 142]. Given $\{\lambda_n\}$, assume that f has a property P on an arbitrarily small interval; does it imply that f has the same property everywhere? We only give a few examples.

1. Take as P the property to vanish (a.e.); we get a problem of uniqueness. The solution results from a recent work of Beurling and Malliavin [**3**] (for the derivation, see [**21**]). It is as follows: in order that $f = 0$ on an interval (a.e.) implies $f = 0$ (a.e.), the following con-

dition on $\{\lambda_n\}$ is necessary and sufficient: there exists an infinity of disjoint intervals $[a_i, b_i[$ on $[0, \infty[$ such that

$$\sum_1^\infty \left(\frac{1}{a_i} - \frac{1}{b_i}\right) = \infty, \qquad \lim_{i\to\infty} \frac{n(a_i, b_i)}{b_i - a_i} = 0,$$

$n(a, b) = n(b) - n(a)$ being the number of points λ_n on $[a, b[$. In other words, that means that the "B-M-inner density" of $\{\lambda_n\}$ is zero, if we define B-M-gaugeable sequences of density D by

$$\int^\infty |n(t) - Dt| \frac{dt}{t^2} < \infty$$

and B-M-inner density of Λ as the upper bound of the densities of B-M-gaugeable sequences contained in Λ.

2. Take as P the property to be represented as an absolutely convergent Fourier series; it is known (Wiener) that, if f has the property P in a neighborhood of each point, the Fourier series of f is absolutely convergent. Given $\{\lambda_n\}$, it is the same to say (a) that every continuous f, given by (15), and satisfying P on an interval, satisfies P everywhere, or (b) that

$$\lim_{h\to\infty} \sup_t \frac{n(t+h) - n(t)}{h} = 0.$$

Remark that (b) expresses that the "uniform outer density" of $\{\lambda_n\}$ is zero, if we now define gaugeable sequences of density D by

$$n(t) - D(t) = O(1) \qquad (t \to \infty)$$

and outer density as the infimum of the densities of gaugeable sequences containing the given sequence. There are several properties P for which the solution is the same: for example, to have absolutely continuous derivatives of order 1, 2, $\cdots$, $p-1$, and derivative of order f (defined almost everywhere) in L^2 [**19**], [**20**]. Previous investigations in this direction are due to Noble [**30**] and Kennedy [**24**].

3. Take as P the property to be infinitely differentiable, resp. analytic. We only know the following: if $\{\lambda_n\}$ has outer uniform density zero (for example $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = \infty$), and if f satisfies P on an interval, f satisfies P everywhere. In the case $\lim(\lambda_{n+1} - \lambda_n) = \infty$, the theorem is due to Wiener [**33**, p. 124].

II. We now consider two more problems of Mandelbrojt, which are problems of uniqueness, or quasianalyticity.

First, let $\mathcal{C}(\{M_n\}, \{\lambda_n\})$ be the class of infinitely differentiable functions f satisfying (15) and

$$\sup_t |f^{(n)}(t)| \leqq KM_n \qquad (n = 0, 1, \cdots; K = K_f). \tag{17}$$

We ask for a condition under which, if $f \in \mathcal{C}(\{M_n\}, \{\lambda_n\})$ and $f, f', f'', \cdots$ vanish at the same point, it results in $f=0$.

To state the known results, it is convenient to suppose either $M_{2n+1}=\infty$ $(n=1, 2, \cdots)$ or M_{n+1}/M_n increasing. A sufficient condition is

$$\liminf_{n\to\infty} \frac{M_{2n}}{\lambda_1^2 \cdots \lambda_n^2} = 0.$$

If $\{\lambda_n\}$ is a Hadamard sequence $(\lambda_{n+1}/\lambda_n \geqq q > 1)$, it is necessary too, for there exists $f_0 \in \mathcal{C}(\{M_n\}, \{\lambda_n\})$, such that $f_0(0) = f_0'(0) = \cdots = 0$, and $0 < \sup_t |f^{(2n)}(t)| \leqq \lambda_1^2 \cdots \lambda_n^2$ [**18**].

Secondly, let $J(t)$ be an even function defined on $[-\pi, \pi]$, $\geqq 0$ and vanishing at 0, and let $\mathcal{C}(J(t), \{\lambda_n\})$ the class of functions f satisfying (15) and

$$|f(t)| \leqq KJ(t) \quad (K = K_f, |t| \leqq \pi). \tag{18}$$

We ask if $\mathcal{C}(J(t), \{\lambda_n\})$ contains only the function 0, or not. It has been proved recently that, if $\{\lambda_n\}$ is a Hadamard sequence, and $J(t)$ satisfies a regularity condition (log $J(t)$ convex function of log t),

$$\mathcal{C}(J(t), \{\lambda_n\}) \neq \{0\} \Leftrightarrow f_0 \in \mathcal{C}(J(t), \{\lambda_n\}),$$

f_0 being the same as before. In other words, either f_0 satisfies (18), or no $f \neq 0$ given by (15) may satisfy (18) [**15**]. A former statement needed the stronger hypothesis

$$\sum_1^\infty \frac{\lambda_n}{\lambda_{n+1}} < \infty \qquad [\mathbf{18}].$$

In the last statements, we do not know how far Hadamard's lacunarity condition can be relaxed.

III. A problem of interest is the behavior of f on closed sets without interior points. Let us first recall a theorem of Zygmund: if $\{\lambda_n\}$ is a Hadamard sequence and if, on a set E of positive measure, f (given by (15)) coincides with a function g defined in a neighborhood G of E and analytic on G, f is analytic [**55**, I, p. 206]. In particular, if $f=0$ on E, $f=0$.

We now direct our attention to sets $E(I, \xi)$ of a Cantor type, with ratio of dissection ξ, as defined in §I. All these sets have measure 0, and $E(I, \xi)$ has Hausdorff dimension $(\log 1/\xi)/\log 2$. We have seen already that, if $\{\lambda_n\}$ is a Hadamard sequence satisfying (4), there exists a $\xi_0 = \xi_0(q)$ such that, if $\xi > \xi_0$, the only continuous f, given by (15), vanishing on $E(I, \xi)$, is $f = 0$. We shall review a few results here about [20].

If $|I| > \pi$ and $\xi > 1/3$, there exists a $q > 1$ such that, whenever $\{\lambda_n\}$ satisfies (4), the only continuous f given by (15) and vanishing on $E(I, \xi)$ is $f = 0$.

In the opposite direction, suppose $\xi = 1/N$, N integer $\geqq 7$, and fix I. Then, whatever may be the sequence $\{q_n\}$, there exists a sequence $\{\lambda_n\}$, satisfying $\lambda_{n+1} \geqq q_n \lambda_n$ $(n = 1, 2, \cdots)$, and a continuous function F, given by (16), which vanishes on $E(I, \xi)$.

Some results depend strongly on number theoretical properties of ξ, as in the problem of uniqueness for trigonometric series ([55, Chapter XII], or [22, Chapter VI]). We say that $\theta \in S$ if and only if θ is a positive algebraic integer, $\neq 1$, whose conjugates $(\neq \theta)$ are contained in the open disc $|z| < 1$. We say that $\{\lambda_n\}$ is $\{\omega_p\}$-scarce if $\lambda_{n+p} - \lambda_n \geqq \omega_p$ for each n and p (positive integers). Then $1/\xi \in S$ is the necessary and sufficient condition in order that, whatever may be $\{\omega_p\}$, there exist an $\{\omega_p\}$-scarce $\{\lambda_n\}$ and a continuous $f \neq 0$ given by (15), vanishing on $E([0, 2\pi], \xi)$.

Moreover, if $1/\xi \in S$, there exist a $q > 1$, a sequence $\{\lambda_n\}$ satisfying (4), and a continuous $F \neq 0$, given by (16), vanishing on $E([0, 2\pi], \xi)$.

Some more statements of this kind can be found in [20] and [22].

IV. Lastly, we shall indicate a few results which are both easy to state and to obtain. Let us consider functions f given by (15).

1. If f is differentiable at some point, and $\{\lambda_n\}$ is a Hadamard sequence, $r_n = o(\lambda_n^{-1})$ $(n \to \infty)$.

2. If $f(t) = O(|t|^\alpha)$ when $t \to 0$ $(0 < \alpha < 1)$, and $\{\lambda_n\}$ is a Hadamard sequence, f belongs to the class Λ_α, i.e.,

$$\sup_{t,h} \frac{|f(t+h) - f(t)|}{|h|^\alpha} < \infty.$$

3. If $\{\lambda_n\}$ is not a Hadamard sequence, there exists an $f \notin \Lambda_\alpha$ such that $f(t) = O(|t|^\alpha)$ when $t \to 0$.

4. If $f(t) = O(|t|^n)$ when $t \to 0$ $(n = 1, 2, \cdots)$, and $\lambda_{n+1} - \lambda_n > \lambda_1^\alpha$ $(n = 1, 2, \cdots)$ $(\alpha > 0)$, f is infinitely differentiable.

Statement 1 is due to G. Freud [10, Theorem 5]; it results immediately that Hardy-Weierstrass series are nowhere differentiable. Both statements 1 and 2 can be proved as follows [15]: consider a trigonometric polynomial T_n of degree l_n less than $\inf(\lambda_{n+1} - \lambda_n, \lambda_n - \lambda_{n-1})$; then

$$r_n = 2\int_{-\pi}^{\pi} f(t)T_n(t)\cos(\lambda_n t + \phi_n)dt \Big/ \int_{-\pi}^{\pi} T_n(t)dt$$

and the proof reduces to finding a positive T_n such that

$$\int_{-\pi}^{\pi} |t|^{\alpha} T_n(t)dt < Kl_n^{-\alpha}\int_{-\pi}^{\pi} T_n(t)dt;$$

it is enough to choose as T_n the square of a Fejer kernel.

Finally, remark that 2 and 3 give an example of a statement where Hadamard's lacunarity condition is both necessary and sufficient.

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Université de Paris and
Institute for Advanced Study