RECURSION AND DOUBLE RECURSION

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1. Introduction. We shall apply the results of PRF[1] to construct by double recursion two functions which are not themselves primitive recursive, but which are related in interesting ways to the class of primitive recursive functions. In a sense, this note is a revised version of a paper by Rózsa Péter,[2] much simplified by the use of PRF.

Let Sx denote the successor of x. We shall say that a function $G_n x$ of two variables n and x is defined by a double recursion from certain given functions, if

(1) $G_0 x$ is a given function of x.

(2) $G_{Sn}0$ is obtained by substitution from $G_n z$ (considered as a function of z) and from given functions.

(3) $G_{Sn}Sx$ is obtained by substitution from the number $G_{Sn}x$, from $G_n z$ (considered as a function of z), and from given functions.

It is clear that if the given functions are primitive recursive, then $G_n x$ is a primitive recursive function of x for each fixed n. However, as we shall see, $G_n x$ need not be a primitive recursive function of n and x.

In §2, we shall show that the double recursion

$$G_0 x = Sx, \qquad G_{Sn}0 = G_n 1, \qquad G_{Sn}Sx = G_n G_{Sn} x$$

defines a function $G_n x$ which *majorizes* all primitive recursive functions of one variable in the following sense: *If Fx is a primitive recursive function of x, then there exists a number n such that*

$$Fx < G_n x$$

for all x. It is also shown that $G_n x$ is an increasing function of n, so that

$$Fx < G_x x$$

for all sufficiently large x. It follows that $G_x x$ is not a primitive recursive function of x, and hence that $G_n x$ is not a primitive recursive

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[1] R. M. Robinson, *Primitive recursive functions*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 925–942.

[2] R. Péter, *Konstruktion nichtrekursiver Funktionen*, Math. Ann. vol. 111 (1935) pp. 42–60.

function of n and x. The above example is essentially the same as that given by Péter, which was a simplification of one previously given by Ackermann.[3]

In §3, we shall determine two primitive recursive functions Ax and $B(x, y)$, such that the double recursion

$$G_0x = Ax, \qquad G_{Sn}0 = 0, \qquad G_{Sn}Sx = G_nB(x, G_{Sn}x)$$

defines a function G_nx which *generates* all primitive recursive functions of one variable in the following sense: A primitive recursive function $H(n, x)$ can be found, such that *if Fx is a primitive recursive function of x, then there exists a number n such that*

$$Fx = G_nH(n, x).$$

It follows that $G_xH(x, x)$ is not a primitive recursive function of x, and hence that G_nx is not a primitive recursive function of n and x. The above double recursion is of a much simpler form than the one given by Péter for a similar purpose. (In a later paper,[4] she showed how all double recursions can be reduced to a standard form, which is however still not as simple as the above.) Also, the functions Ax and $B(x, y)$ which we use are comparatively simple; they can be obtained by substitution from constant and identity functions, and

$$x + y, \quad x \dot{-} y, \quad x^2, \quad [x^{1/2}], \quad [x/2], \quad [x/3].$$

Here $x \dot{-} y = x - y$ if $x \geqq y$ and $x \dot{-} y = 0$ otherwise. The function $H(n, x)$ which we use is a certain quartic polynomial in n and x.

Both of these results may be derived from PRF, §7, Theorem 3, which states that *all primitive recursive functions of one variable can be obtained by starting with the two functions S and E, and repeatedly using any of the formulas*

$$Fx = Ax + Bx, \qquad Fx = BAx, \qquad Fx = B^x0$$

to construct a new function F from known functions A and B. A second form of the result has E replaced by Q, and $Ax + Bx$ by $|Ax - Bx|$. Here $Ex = x - [x^{1/2}]^2$ is the excess of x over a square, and $Qx = 0^{Ex}$ is the characteristic function of squares.

2. **The majorizing function.** Let the function G_nx be defined by the double recursion

$$G_0x = Sx, \qquad G_{Sn}0 = G_n1, \qquad G_{Sn}Sx = G_nG_{Sn}x.$$

[3] W. Ackermann, *Zum Hilbertschen Aufbau der reelen Zahlen*, Math. Ann. vol. 99 (1928) pp. 118–133.

[4] R. Péter, *Über die mehrfache Rekursion*, Math. Ann. vol. 113 (1936) pp. 489–527.

It is clear that $G_n x$ is never zero. We shall show that

$$G_n Sx > G_n x,$$

so that for any n, $G_n x$ is a strictly increasing function of x, hence $G_n x > x$, that is, $G_n x \geqq Sx$. In the first place, this is true for $n=0$. Now assume for any value of n and prove for Sn. We have indeed

$$G_{Sn} Sx = G_n G_{Sn} x > G_{Sn} x.$$

We shall show next that

$$G_{Sn} x \geqq G_n Sx.$$

This may be shown for a fixed n by induction in x. For $x=0$ we have equality by definition. Now assume the inequality for some value of x and prove for Sx. By the inductive hypothesis and the inequality $G_n Sx \geqq SSx$, we have

$$G_{Sn} Sx = G_n G_{Sn} x \geqq G_n G_n Sx \geqq G_n SSx,$$

as was to be shown. In particular, we have

$$G_{Sn} x > G_n x,$$

so that $G_n x$ is a strictly increasing function of n for a fixed x.

The arguments used up to this point are the same as those given by Péter, although we have modified her function slightly. We shall now show that to every primitive recursive function Fx there exists a number n such that

$$Fx < G_n x.$$

We shall use the result quoted in §1 from PRF in the second form.

We must show that the conclusion holds for Sx and Qx, and that if it holds for Ax and Bx, then it also holds for $|Ax - Bx|$, BAx, and $B^x 0$. We have in the first place

$$Qx \leqq Sx = G_0 x < G_1 x.$$

Now suppose that

$$Ax < G_k x, \qquad Bx < G_l x.$$

If we set $n = \max(k, l)$, then

$$Ax < G_n x, \qquad Bx < G_n x.$$

Hence

$$|Ax - Bx| < G_n x, \qquad B^x 0 < G_n^x G_n 1 = G_{Sn} x,$$

and

$$BAx < G_nG_nx < G_nG_{Sn}x = G_{Sn}Sx \leqq G_{SSn}x.$$

3. The generating function. We shall make use of pairing functions $J(u, v)$, Kx, Lx, that is, functions which satisfy

$$KJ(u, v) = u, \qquad LJ(u, v) = v.$$

Such functions establish a one-to-one correspondence between all pairs of numbers and some numbers. We shall want to use functions $J(u, v)$, Kx, Lx, which are primitive recursive, and the conditions $Kx \leqq x$, $Lx \leqq x$ will be needed. We shall also suppose, as in PRF, §4, that $J(0, 0) = 0$, and that if $LSx > 0$, then $KSx = Kx$ and $LSx = SLx$; the interpretation of these conditions is discussed there. Suitable functions are

$$J(u, v) = ((u + v)^2 + u)^2 + v, \qquad Kx = E[x^{1/2}], \qquad Lx = Ex.$$

Since for such pairing functions, $Kx = u$ and $Lx = v$ have infinitely many solutions for x when u and v are given, we can extend the correspondence from one between pairs of numbers and numbers to one between triples of numbers and numbers, by finding two primitive recursive functions $J(u, v, w)$ and Mx, with $J(u, v, 0) = J(u, v)$, and such that

$$KJ(u, v, w) = u, \qquad LJ(u, v, w) = v, \qquad MJ(u, v, w) = w.$$

Since $J(0, 0, 0) = 0$, we see that $K0 = 0$, $L0 = 0$, $M0 = 0$. Suitable functions extending the $J(u, v)$, Kx, Lx given above are

$$J(u, v, w) = ((u + v + w)^2 + u)^2 + v, \qquad Mx = [x^{1/4}] \dot{-} (Kx + Lx).$$

We now define F_nx by the formulas

$$F_0x = Ex, \qquad F_1x = Sx,$$

$$F_{3u+2}x = F_{Ku}x + F_{Lu}x, \qquad F_{3u+3}x = F_{Lu}F_{Ku}x, \qquad F_{3u+4}x = F_u^x0.$$

According to the theorem of PRF quoted in §1, if Fx is a primitive recursive function, then there exists a number n such that $Fx = F_nx$.

We shall now define a function Gx (which is not primitive recursive) such that $G(2v) = F_{Kv}Lv + Mv$. The definition of Gx is completed by supposing that for $v > 0$

$$G(2v - 1) = \begin{cases} F_{Ku}Lv & \text{if} \quad Kv = 3u + 2 \quad \text{or} \quad 3u + 3, \\ F_{Kv}PLv & \text{if} \quad Kv = 3u + 4,\ Lv > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $Py = y \dot{-} 1$ is the predecessor of y.

The definition of $G(2v)$ was so chosen that

$$F_n x = G(2J(n, x)).$$

Thus Gx may be considered as generating all primitive recursive functions. The essential numbers $F_{Kv}Lv$ were put in alternate places and modified by adding Mv, and then such values were interpolated for $G(2v-1)$ that Gx would satisfy a functional equation

$$GSx = GB(x, Gx),$$

where $B(x, y)$ is a primitive recursive function. Indeed, we shall define $B(x, y)$ so that the function determined from suitable initial values by the formula $G_{Sn}Sx = G_nB(x, G_{Sn}x)$ will approximate to Gx in such a way that

$$F_n x = G_n(2J(n, x)).$$

Thus all primitive recursive functions will be generated by a double recursion.

The definition of $B(x, y)$ is the following: For every $v > 0$, let

$$B(2v - 2, y) = \begin{cases} 2J(Ku, Lv, 0) & \text{if } Kv = 3u + 2 \text{ or } 3u + 3, \\ 2J(0, 0, y \dot{-} MPv) & \text{if } Kv = 3u + 4, Lv > 0, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B(2v - 1, y) = \begin{cases} 2v & \text{if } Kv = 0 \text{ or } 1, \\ 2J(Lu, Lv, y + Mv) & \text{if } Kv = 3u + 2, \\ 2J(Lu, y, Mv) & \text{if } Kv = 3u + 3, \\ 2J(0, 0, Mv) & \text{if } Kv = 3u + 4, Lv = 0, \\ 2J(u, y, Mv) & \text{if } Kv = 3u + 4, Lv > 0. \end{cases}$$

We proceed to verify that $GSx = GB(x, Gx)$ in all cases.

$x = 2v - 2$, $Kv = 3u + 2$ or $3u + 3$:

$$GB(x, Gx) = F_{Ku}Lv = GSx.$$

$x = 2v - 2$, $Kv = 3u + 4$, $Lv > 0$:

$$GB(x, Gx) = F_0 0 + (Gx \dot{-} MPv) = F_{KPv}LPv = F_{Kv}PLv = GSx.$$

$x = 2v - 2$, other cases:

$$GB(x, Gx) = G0 = 0 = GSx.$$

$x=2v-1$, $Kv=0$ or 1:

$$GB(x, Gx) = G(2v) = GSx.$$

$x=2v-1$, $Kv=3u+2$:

$$\begin{aligned} GB(x, Gx) &= F_{Lu}Lv + (Gx + Mv) = F_{Lu}Lv + F_{Ku}Lv + Mv \\ &= F_{3u+2}Lv + Mv = GSx. \end{aligned}$$

$x=2v-1$, $Kv=3u+3$:

$$GB(x, Gx) = F_{Lu}Gx + Mv = F_{Lu}F_{Ku}Lv + Mv = F_{3u+3}Lv + Mv = GSx.$$

$x=2v-1$, $Kv=3u+4$, $Lv=0$:

$$GB(x, Gx) = F_0 0 + Mv = Mv = F_{3u+4}0 + Mv = GSx.$$

$x=2v-1$, $Kv=3u+4$, $Lv>0$:

$$\begin{aligned} GB(x, Gx) &= F_u Gx + Mv = F_u F_{3u+4}PLv + Mv = F_u F_u^{PLv} 0 + Mv \\ &= F_u^{Lv} 0 + Mv = F_{3u+4}Lv + Mv = GSx. \end{aligned}$$

Let the function Ax be defined by

$$A(2v) = \begin{cases} ELv + Mv & \text{if } Kv = 0, \\ SLv + Mv & \text{if } Kv = 1, \end{cases}$$

and $Ax=0$ otherwise. Then the double recursion $G_0x=Ax$, $G_{Sn}0=0$, $G_{Sn}Sx=G_nB(x, G_{Sn}x)$ defines a function G_nx, which we shall show approaches Gx as n increases in such a way that

$$G_n(2v) = G(2v) \qquad \text{if } Kv \leqq Sn,$$

and also for $v>0$

$$G_n(2v-1) = G(2v-1) \qquad \text{if } Kv \leqq Sn.$$

We shall prove these two equalities by induction in n. If $n=0$, we must have $Kv\leqq 1$, and hence

$$G_0(2v) = A(2v) = F_{Kv}Lv + Mv = G(2v),$$

and for $v>0$

$$G_0(2v-1) = 0 = G(2v-1).$$

Now assume the result for some value of n and prove for Sn. In the first place, $G_{Sn}0=0=G0$. If $v>0$ but $Kv\leqq 1$, we have

$$G_{Sn}(2v) = G_n(2v) = G(2v),$$
$$G_{Sn}(2v-1) = G_n 0 = 0 = G(2v-1).$$

It remains to consider the case $2 \leqq Kv \leqq SSn$. Now if $Kv = 3u+2$, $3u+3$, or $3u+4$, we see that for any y, $B(2v-2, y)$ and $B(2v-1, y)$ are both of the form $2w$ with $Kw \leqq u < Kv$, hence $Kw \leqq Sn$. It follows that

$$G_nB(2v-2, y) = GB(2v-2, y), \qquad G_nB(2v-1, y) = GB(2v-1, y).$$

Hence

$$G_{Sn}Sx = G_nB(x, G_{Sn}x) = GB(x, G_{Sn}x) \qquad \text{for } x = 2v-2 \text{ or } 2v-1.$$

We shall now prove by induction in v that if $2 \leqq Kv \leqq SSn$ then

$$G_{Sn}(2v-1) = G(2v-1), \qquad G_{Sn}(2v) = G(2v),$$

which will complete the proof. For a given value of v, we shall derive both of these, assuming that

$$G_{Sn}(2v-2) = G(2v-2)$$

provided that $2 \leqq KPv \leqq SSn$. We have indeed

$$\begin{aligned} G_{Sn}(2v-1) &= GB(2v-2, G_{Sn}(2v-2)) \\ &= GB(2v-2, G(2v-2)) = G(2v-1), \end{aligned}$$

since $B(2v-2, y)$ depends on y only if Kv has the form $3u+4$ and $Lv > 0$, in which case $KPv = Kv$, so that the inductive hypothesis may be used. Finally,

$$G_{Sn}(2v) = GB(2v-1, G_{Sn}(2v-1)) = GB(2v-1, G(2v-1)) = G(2v).$$

Let

$$H(n, x) = 2J(n, x).$$

If Fx is any primitive recursive function of x, then there exists an n such that $Fx = F_nx$. From what we have proved, we see that

$$Fx = F_nx = G(2J(n, x)) = G_n(2J(n, x)) = G_nH(n, x).$$

thus establishing the result stated in the introduction.

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