THE BOUNDEDNESS OF ORTHONORMAL POLYNOMIALS ON CERTAIN CURVES OF THE THIRD DEGREE

DUNHAM JACKSON

1. **Introduction.** Properties of boundedness of systems of orthonormal polynomials, significant because of their relation to the convergence of the corresponding developments of "arbitrary" functions in series, constitute a subject for detailed investigation in themselves. In the case of the orthonormal systems associated with algebraic curves no method appears to be readily available for dealing with the problem as formulated in general terms, while on the other hand special methods of some degree of variety throw light on the facts relating to curves of particular types [6, 7, 4, 5].[1] An earlier paper by the present writer [4] is concerned with loci of the second degree; a more recent one [5] includes reference to the curves of the third degree represented by the equations $y=Ax^3+Bx^2+Cx+D$ and $y^2=x^3$. Although indefinite multiplication of particular instances would be unprofitable, some additional illustrations may serve to suggest a more adequate picture of the general situation. The following paragraphs present extensions of the reasoning to other curves with equations of the form $y^2=F(x)$, where $F(x)$ is a polynomial of the third degree.

2. **The curve** $y^2=x^2(x+1)$. This curve has a double point with two real branches intersecting at the origin. It has a parametric representation in which the coordinates are not merely rational functions but more specifically polynomials in the auxiliary variable,

$$x = t^2 - 1, \qquad y = t(t^2 - 1).$$

Since $x^3=y^2-x^2$, any monomial in terms of the coordinates containing x^3 as a factor can be replaced on the curve by an expression of lower degree. A fundamental sequence of monomials for application of the Schmidt process in constructing the orthogonal system on the curve is

$$(1) \qquad 1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^2y,\ xy^2,\ y^3,\ x^2y^2,\ xy^3,\ y^4,\ \cdots .$$

In terms of the parameter these are respectively

$$(2) \qquad 1,\ t^2 - 1,\ t(t^2 - 1),\ (t^2 - 1)^2,\ t(t^2 - 1)^2,\ t^2(t^2 - 1)^2,\ \cdots .$$

An arbitrary linear combination of a finite number of the above

Received by the editors June 26, 1946.

[1] Numbers in brackets refer to the references cited at the end of the paper.

expressions in t is *an arbitrary polynomial in t taking on equal values for* $t=\pm 1$. Each of the expressions enumerated does assume equal values for $t=\pm 1$, the value being 0 except in the case of the constant which heads the list. Any polynomial can be expressed as a sum of one containing only even powers and one containing only odd powers; the successive even powers of t are expressible in terms of 1 and the second, fourth, $\cdots$ members of the sequence (2); if an odd polynomial takes on equal values for $t=\pm 1$ the common value must be 0, and the polynomial must be divisible by t^2-1; and any odd polynomial divisible by t^2-1 is expressible in terms of the third, fifth, $\cdots$ members of the sequence (2), since the products of the successive odd powers of t by t^2-1 are thus expressible. The orthonormal polynomials in x and y on an arc of the curve marked off by the inequalities $a \leqq t \leqq b$ are, with suitably adjusted weight function, the orthonormal polynomials $q_n(t)$ on the same interval of values for t, subject to the auxiliary condition that $q_n(1)=q_n(-1)$ (see [1]). The form of the auxiliary condition does not carry any implication as to the relative positions of the intervals (a, b) and $(-1, 1)$.

Let $q_0(t), q_2(t), q_3(t), \cdots$ specifically be the orthonormal polynomials for integration with respect to t and weight function unity. Generalization of the weight function by application of Korous's theorem [**8**, p. 157; **2**, pp. 205–208; **3**; **4**, pp. 352–353] will not be discussed here. The orthonormal system contains no polynomial of the first degree, since no such polynomial satisfies the auxiliary condition [**1**, pp. 72–73].

The proof of boundedness of the q's is similar in principle to others which have been given elsewhere [**1**, **5**], but possesses enough individuality in detail to justify brief attention. Let $p_0(t), p_1(t), p_2(t), \cdots$ be the normalized Legendre polynomials for the interval $a \leqq t \leqq b$, orthogonal for weight unity, with all non-negative degrees represented. Let

$$q_n(t) = \sum_{k=0}^{n} c_{nk} p_k(t),$$

with

$$c_{nk} = \int_a^b q_n(t) p_k(t) dt,$$

a representation of this form being possible for any polynomial of the nth degree. Since $q_n(t)$ is orthogonal to every polynomial of lower degree which satisfies the auxiliary condition, and since

$$p_k(t) + 2^{-1}t[p_k(-1) - p_k(1)]$$

is such a polynomial, it follows that

$$c_{nk} = 2^{-1}[p_k(1) - p_k(-1)] \int_a^b tq_n(t)dt$$

for $k<n$.

Inasmuch as both the p's and the q's are normalized

$$1 = \int_a^b [q_n(t)]^2 dt = \sum_{k=0}^{n} c_{nk}^2,$$

in consequence of which 1 is an upper bound for each of the quantities $|c_{nk}|$, while at the same time

$$\sum_{k=0}^{n-1} c_{nk}^2 = 4^{-1}\left[\int_a^b tq_n(t)dt\right]^2 \sum_{k=0}^{n-1} [p_k(1) - p_k(-1)]^2 \leqq 1,$$

$$\left|\int_a^b tq_n(t)dt\right| \leqq 2\left\{\sum_{k=0}^{n-1} [p_k(1) - p_k(-1)]^2\right\}^{-1/2}. \tag{3}$$

In comparison with the quantity in braces, inequalities will be needed presently for $p_k(1)$ and $p_k(-1)$ separately. The recurrence formula for the p's can be written in the form

$$tp_k(t) = \alpha_{k,k+1}p_{k+1}(t) + \alpha_{kk}p_k(t) + \alpha_{k,k-1}p_{k-1}(t),$$

with coefficients α_{kj} each of which has G as an upper bound for its absolute value, if G is the larger of the numbers $|a|$, $|b|$ [5, p. 178]. For $t = \pm 1$ the formula reads

$$\begin{aligned} p_k(1) &= \alpha_{k,k+1}p_{k+1}(1) + \alpha_{kk}p_k(1) + \alpha_{k,k-1}p_{k-1}(1), \\ -p_k(-1) &= \alpha_{k,k+1}p_{k+1}(-1) + \alpha_{kk}p_k(-1) + \alpha_{k,k-1}p_{k-1}(-1), \end{aligned}$$

whence by subtraction

$$\begin{aligned} p_k(1) + p_k(-1) = \alpha_{k,k+1}&[p_{k+1}(1) - p_{k+1}(-1)] \\ &+ \alpha_{kk}[p_k(1) - p_k(-1)] \\ &+ \alpha_{k,k-1}[p_{k-1}(1) - p_{k-1}(-1)]. \end{aligned}$$

By further addition or subtraction of $p_k(1) - p_k(-1)$,

$$\begin{aligned} 2p_k(\pm 1) = \alpha_{k,k+1}&[p_{k+1}(1) - p_{k+1}(-1)] \\ &+ (\alpha_{kk} \pm 1)[p_k(1) - p_k(-1)] \\ &+ \alpha_{k,k-1}[p_{k-1}(1) - p_{k-1}(-1)]. \end{aligned}$$

There is consequently a number G_1, independent of k, such that

$$|p_k(\pm 1)| \leqq G_1 \sum_{j=k-1}^{k+1} |p_j(1) - p_j(-1)|.$$

Since furthermore, if $j \leqq n-1$,

$$\begin{aligned} |p_j(1) - p_j(-1)| &= \{[p_j(1) - p_j(-1)]^2\}^{1/2} \\ &\leqq \left\{\sum_{j=0}^{n-1} [p_j(1) - p_j(-1)]^2\right\}^{1/2} \end{aligned}$$

it follows that

$$|p_k(\pm 1)| \leqq 3G_1 \left\{\sum_{j=0}^{n-1} [p_j(1) - p_j(-1)]^2\right\}^{1/2}$$

if $k \leqq n-2$.

In the representation

$$q_n(t) = \sum_{k=n-2}^{n} c_{nk} p_k(t) + \sum_{k=0}^{n-3} c_{nk} p_k(t) \tag{4}$$

the sum from 0 to $n-3$ can be written as

$$2^{-1} \int_a^b t q_n(t) dt \left\{\sum_{k=0}^{n-3} p_k(1) p_k(t) - \sum_{k=0}^{n-3} p_k(-1) p_k(t)\right\}. \tag{5}$$

By the Christoffel-Darboux identity

$$\sum_{k=0}^{n-3} p_k(1) p_k(t) = \alpha_{n-3,n-2} \frac{p_{n-2}(1) p_{n-3}(t) - p_{n-3}(1) p_{n-2}(t)}{1 - t},$$

the coefficient $\alpha_{n-3,n-2}$ being defined in agreement with the notation of the preceding paragraph. This coefficient then has an upper bound independent of n. The quantities $p_{n-2}(1)$, $p_{n-3}(1)$ are subject to the inequality at the end of the preceding paragraph. The Legendre polynomials $p_{n-3}(t)$, $p_{n-2}(t)$ are bounded, uniformly with respect to n, throughout any closed interval interior to (a, b), and $1/(1-t)$ is bounded except in the neighborhood of $t=1$. Similar observations apply in the case of the identity

$$\sum_{k=0}^{n-3} p_k(-1) p_k(t) = \alpha_{n-3,n-2} \frac{p_{n-2}(-1) p_{n-3}(t) - p_{n-3}(-1) p_{n-2}(t)}{-1 - t}.$$

These facts together with (3) show that the whole expression (5) has an upper bound independent of n for its absolute value when t is kept away from the values a, b, ± 1. The first sum in the right-hand member of (4) is similarly bounded, without special reference to the points $t = \pm 1$. *The orthonormal polynomials $q_n(t)$ are uniformly bounded throughout any closed interval which is interior to (a, b) and excludes the points $t = \pm 1$*, the last restriction being naturally redundant if the values $t = \pm 1$ do not belong to the interval (a, b).

3. **The curve** $y^2 = x^2(x-1)$. Except for a conjugate point at the origin, which is of no concern for the orthonormal system, this curve is real only for $x \geqq 1$. It has the parametric representation

$$x = t^2 + 1, \qquad y = t(t^2 + 1).$$

Fundamental monomials in terms of x and y are the same as in (1). As functions of t these become

$$1,\ t^2 + 1,\ t(t^2 + 1),\ (t^2 + 1)^2,\ t(t^2 + 1)^2,\ t^2(t^2 + 1)^2, \cdots .$$

An arbitrary linear combination of a finite number of them is an arbitrary polynomial taking on equal values for $t = \pm i$, or, in alternative characterization, an arbitrary even polynomial plus the product of an arbitrary odd polynomial by t^2+1.

Let $p_0(t)$, $p_1(t)$, $p_2(t)$, $\cdots$ as before be the normalized Legendre polynomials on an interval (a, b), and let $q_0(t)$, $q_2(t)$, $q_3(t)$, $\cdots$ be the orthonormal polynomials on the same interval associated with the auxiliary condition $q_n(i) = q_n(-i)$. The polynomial

$$p_k(t) + 2^{-1}it[p_k(i) - p_k(-i)],$$

which is real for real values of t, satisfies the auxiliary condition, and $q_n(t)$ is orthogonal to it for $n > k$. Consequently, in the representation

$$q_n(t) = \sum_{k=0}^{n} c_{nk}p_k(t), \qquad c_{nk} = \int_a^b q_n(t)p_k(t)dt,$$

the form

$$c_{nk} = 2^{-1}i[p_k(-i) - p_k(i)]\int_a^b tq_n(t)dt$$

can be used for the general coefficient when $k < n$. In analogy with the reasoning of the preceding section it is found that

$$\left|\int_a^b tq_n(t)dt\right| \leqq 2\left\{-\sum_{k=0}^{n-1}[p_k(-i) - p_k(i)]^2\right\}^{-1/2},$$

the quantity in braces being real and positive; with regard to the question whether it might conceivably vanish it is sufficient to note that $p_1(-i)-p_1(i)\neq 0$. The reasoning of the preceding section can then be adapted step by step to the discussion of the boundedness of the present polynomials $q_n(t)$, in spite of the degree of novelty associated with the appearance of imaginaries in the formulas; the only material difference is that since the denominators $\pm i-t$ in the Christoffel-Darboux identity can not vanish on the real interval of values of t considered, there is nothing corresponding to the exceptional status of the values $t=\pm 1$ in the earlier passage. The significance of this difference is of course that the singular point of the curve is now definitely excluded from the range of integration. *The orthonormal polynomials $q_n(t)$ on the interval (a, b) are uniformly bounded except near the ends of the interval.*

4. **The curve** $y^2=x^3-x=(x+1)x(x-1)$. This curve has no singular point, and does not admit the sort of parametric representation used above. The discussion of the orthonormal system will be based on properties of geometric symmetry, and will be restricted to particular domains of integration for which the properties of symmetry in question are realized.

Fundamental monomials in x and y for the application of Schmidt's process are still given by (1). In setting up the orthogonal system for a portion of the curve which is symmetric with respect to the x-axis, with arc length as variable of integration and with unity or any other even function of y as weight function, the monomials containing even powers of y and those containing odd powers of y can be considered separately, since under these conditions any even function of y and any odd function of y are orthogonal to each other.

The even monomials in y from the sequence (1) are

$$1,\ x,\ x^2,\ y^2,\ xy^2,\ x^2y^2,\ y^4,\ xy^4,\ \cdots .$$

In terms of x these are equal on the curve to

$$1,\ x,\ x^2,\ x^3-x,\ x(x^3-x),\ x^2(x^3-x),\ (x^3-x)^2,\ x(x^3-x)^2,\ \cdots ,$$

and linear combinations of the first $n+1$ of them are linear combinations of the first $n+1$ members of the sequence

$$1,\ x,\ x^2,\ x^3,\ x^4,\ x^5,\ x^6,\ x^7,\ \cdots ,$$

that is, the polynomials of the orthogonal system on the curve which are even functions of y are merely orthogonal polynomials in x on the corresponding interval with the appropriate weight function. Integration with respect to arc length s and with unit weight function over portions of the curve symmetric to each other above and below the x-axis is equivalent to a single integration with respect to x and with weight

$$2\frac{ds}{dx} = 2\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} = \frac{[4y^2 + (3x^2 - 1)^2]^{1/2}}{(x^3 - x)^{1/2}}.$$

Since y and $3x^2-1$ do not vanish simultaneously on the curve, the numerator in the last expression is a positive differentiable function of x. If the domain of integration is the entire closed loop of the curve, corresponding to the range $-1 \leqq x \leqq 0$, the weight function is equivalent [4, pp. 352–353] from the point of view of Korous's theorem to $[-x(1+x)]^{-1/2}$, and the orthonormal polynomials are uniformly bounded over the entire range, those which belong to the last-named weight function being reducible by a linear transformation of the independent variable to the cosine polynomials on the interval $(-1, 1)$ (see, for example [2, p. 191]). If the domain is a symmetric portion of the loop designated by the inequalities $-1 \leqq x \leqq a < 0$ or by the inequalities $-1 < a \leqq x \leqq 0$, or a symmetric arc of the infinite branch marked off by the specifications $1 \leqq x \leqq b$, the weight function is equivalent to $(1+x)^{-1/2}$ or $(-x)^{-1/2}$ or $(x-1)^{-1/2}$ in the various cases respectively, which means that the orthonormal polynomials are uniformly bounded on the interval of the x-axis except near the end of the interval at which the weight function remains finite (see [2, pp. 200–201]), or, on the curve, uniformly bounded except near the ends of the arc.

The monomials in (1) containing odd powers of y are

$$y,\ xy,\ x^2y,\ y^3,\ xy^3,\ x^2y^3,\ y^5,\ xy^5,\ \cdots,$$

expressible on the curve in the form

$$y,\ yx,\ yx^2,\ y(x^3 - x),\ yx(x^3 - x),\ yx^2(x^3 - x),$$
$$y(x^3 - x)^2,\ yx(x^3 - x)^2,\ \cdots.$$

Linear combinations of these are linear combinations of the successive terms

$$y,\ yx,\ yx^2,\ yx^3,\ yx^4,\ yx^5,\ yx^6,\ yx^7,\ \cdots.$$

Orthonormal combinations for unit weight with respect to x are orthonormal polynomials in x for weight y^2, each multiplied by y. For unit weight with respect to s and for integration over a pair of arcs symmetric with respect to the x-axis, y^2 as weight function is to be replaced by

$$2y^2 \frac{ds}{dx} = (x^3 - x)^{1/2}[4y^2 + (3x^2 - 1)^2]^{1/2}$$

if the formulas are still written with x as variable of integration.

For the oval as domain of orthogonality the weight is equivalent to $[-x(1+x)]^{1/2}$ for the purposes of Korous's theorem. The orthonormal polynomials in terms of x are essentially the sine polynomials (see [2, p. 192]) and on multiplication by $[-x(1+x)]^{1/2}$ or by

$$y = \pm [(1 - x)(-x)(1 + x)]^{1/2}$$

give products which are uniformly bounded on the entire domain. For the symmetric arcs limited above and below the x-axis by the specifications $-1 \leqq x \leqq a < 0$, $-1 < a \leqq x \leqq 0$, $1 \leqq x \leqq b$, the weight functions for which orthonormal polynomials in x are to be constructed are equivalent respectively to $(1+x)^{1/2}$, $(-x)^{1/2}$, $(x-1)^{1/2}$, and the products obtained on multiplication by y, regarded as polynomials in x and y, are uniformly bounded near the points where $y=0$ and elsewhere on the range of integration except near the ends of the respective arcs (see [4, pp. 352–353, 356]).

The complete orthonormal system being composed in each case of the polynomials which are even in y and those which are odd in y taken together, *the orthonormal polynomials on the closed loop of the curve, for arc length as variable of integration and unit weight function, are uniformly bounded on the entire loop; the orthonormal polynomials on each of the other symmetric arcs specified are uniformly bounded except near the ends of the arc.*

5. **The curve** $y^2 = x^3 + x = x(x^2+1)$. This curve has a single real branch, of infinite extent, for all points of which $x \geqq 0$. Fundamental monomials in x and y are once more those containing no power of x above the second. Analysis closely patterned after that of the preceding section shows that *the orthonormal polynomials on an arc of the curve symmetric with respect to the x-axis, for arc length as variable of integration and unit weight function, are uniformly bounded except near the ends of the arc.*

References

1. D. Jackson, *Orthogonal polynomials with auxiliary conditions*, Trans. Amer. Math. Soc. vol. 48 (1940) pp. 72–81.

2. ———, *Fourier series and orthogonal polynomials*, Carus Mathematical Monographs, no. 6, 1941.

3. ———, *Generalization of a theorem of Korous on the bounds of orthonormal polynomials*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 602–608.

4. ———, *Boundedness of orthonormal polynomials on loci of the second degree*, Duke Math. J. vol. 11 (1944) pp. 351–365.

5. ———, *The boundedness of certain sets of orthonormal polynomials in one, two, and three variables*, Trans. Amer. Math. Soc. vol. 58 (1945) pp. 167–183.

6. F. Koehler, *Systems of orthogonal polynomials on certain algebraic curves*, Bull. Amer. Math. Soc. vol. 46 (1940) pp. 345–351.

7. Margaret Martin, *Some new systems of orthogonal polynomials on algebraic curves*, Thesis, University of Minnesota, 1944, not yet published.

8. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939.

University of Minnesota