ON THE VALUES ASSUMED BY POLYNOMIALS*

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The equation

$$\frac{f(x+h)-f(x)}{h}=k \tag{1}$$

where $f(x)$ is an integral function and k is an independent complex variable, defines h as a many valued function of x and k. The branches of this function $h(x, k)$ possess a simple property when $f(x)$ is a polynomial; and conversely, when the branches of $h(x, k)$ have the property, $f(x)$ must be a polynomial. It is the object of this paper to exhibit the property, prove that it is characteristic for polynomials, and extend the results to polynomials of several variables.

1. **Polynomials in one variable.** For a polynomial

$$f(x) = a_0x^{n+1} + a_1x^n + \cdots + a_{n+1}, \qquad a_0 \neq 0,\ n > 1,$$

equation (1) becomes

$$h^n + [(n+1)x + a_1/a_0]h^{n-1} + \cdots - k/a_0 = 0,$$

and the n branches $h_i(x, k)$ of the function $h(x, k)$ satisfy the equation

$$\sum_1^n h_i(x, k) = -(n+1)x - a_1/a_0.$$

For $n=1$ we readily find that $h=-2x-a_1/a_0+k/a_0$. Hence we have

$$\frac{\partial}{\partial x}\sum_1^n h_i = -(n+1), \qquad n \geqq 1,$$

as a property of polynomials of degree $n+1$. For $n=0$ no function h is defined by (1).

THEOREM I. *Let $f(x)$ be an integral function. If there exist n functions $h_1(x, k), \cdots, h_n(x, k)$ analytic, distinct, and nonzero on a region in the x plane and a region in the k plane, and which on these regions are such that*

$$\frac{f(x+h_i(x,k))-f(x)}{h_i(x,k)}=k, \qquad i=1,\cdots,n,$$

and

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$$\frac{\partial}{\partial x}\sum_{1}^{n} h_i(x, k) = -(n+1), \tag{2}$$

then $f(x)$ *is a polynomial of degree* $n+1$.

The n functions h_i of our hypothesis satisfy

$$\begin{aligned}(h - h_1)(h - h_2)\cdots(h - h_n) &= 0\\ &= h^n - P_1h^{n-1} + \cdots + (-1)^iP_ih^{n-i} + \cdots + (-1)^nP_n.\end{aligned} \tag{3}$$

In order to prove the theorem we establish relations between the elementary symmetric functions P_i and their partial derivatives. Every function h_i satisfies the equation

$$\left(1 + \frac{\partial h}{\partial x}\right) = \frac{f'(x) - k}{h}\frac{\partial h}{\partial k}. \tag{4}$$

We may derive this by differentiating both members of (1) with respect to x, obtaining

$$f'(x + h)\left(1 + \frac{\partial h}{\partial x}\right) = f'(x) + k\frac{\partial h}{\partial x}$$

and with respect to k obtaining

$$f'(x + h)\frac{\partial h}{\partial k} = h + k\frac{\partial h}{\partial k}.$$

Since $h \neq 0$, $\partial h/\partial k \neq 0$, and we may solve this last equation for $f'(x+h)$; and the substitution of the value thus obtained into the preceding equation yields (4). We shall make use of the fact that $\partial h/\partial k \neq 0$ later in the proof. From (4) we see that

$$\sum_{1}^{n}\left(\frac{\partial h_i}{\partial k}\frac{1}{h_i}\right) = \frac{\partial P_n}{\partial k}\frac{1}{P_n} = \frac{1}{f'(x) - k}\sum_{1}^{n}\left(1 + \frac{\partial h_i}{\partial x}\right);$$

and since by hypothesis $\sum_1^n \partial h_i/\partial x = -(n+1)$, this reduces to

$$\frac{1}{P_n}\frac{\partial P_n}{\partial k} = \frac{1}{k - f'(x)}. \tag{I}$$

We now proceed to deduce a second equation connecting the P's and their partial derivatives; namely,

$$\frac{\partial P_{i+1}}{\partial x} = -(n - i + 1)P_i + (k - f'(x))\frac{\partial P_i}{\partial k}, \qquad i = 1, 2, \cdots, n - 1. \tag{II}$$

To accomplish this observe that

$$\frac{\partial P_{i+1}}{\partial x} + (n - i)P_i = \mathfrak{S} h_1 h_2 \cdots h_i \sum_j \left(1 + \frac{\partial h_j}{\partial x}\right)$$

where $\mathfrak{S}$ denotes the sum of the terms arising by choosing the i indices of the h-factors in all possible ways among the numbers $1, 2, \cdots, n$, and then letting j run through the remaining $n-i$ numbers. The right-hand side of this equation can be written in the form

$$\mathfrak{S} h_1 h_2 \cdots h_i \left\{ \sum_{j=1}^{n} \left(1 + \frac{\partial h_j}{\partial x}\right) - \sum_{j=1}^{i} \left(1 + \frac{\partial h_j}{\partial x}\right)\right\}.$$

With the aid of (2) and (4) the expression now becomes

$$\mathfrak{S} h_1 h_2 \cdots h_i \left\{ [n - (n + 1)] + (k - f'(x)) \sum_{j=1}^{i} \frac{1}{h_j} \frac{\partial h_j}{\partial k} \right\}$$
$$= - P_i + (k - f'(x)) \frac{\partial P_i}{\partial k}.$$

Hence II follows at once.

It will be convenient to change the notation, replacing i by $n-i$. Then

$$\text{II}^* \qquad \frac{\partial P_{n-i+1}}{\partial x} = - (i + 1)P_{n-i} + (k - f'(x)) \frac{\partial P_{n-i}}{\partial k}, \qquad i = 1, 2, \cdots, n - 1.$$

From I it follows that

$$P_n = \phi(x)(k - f'(x))$$

where $\phi(x)$ is analytic in the given domain of the x-plane and does not vanish there. Observe that $P_n \neq 0$ since no $h_i(x, k)$ vanishes, and, since the left-hand side of I is analytic in the given domain, $k-f'(x)$ cannot vanish there

We proceed to show that P_{n-i} is a polynomial in k, at most of degree $i+1$, with coefficients which are analytic functions of x in the given domain, or else that P_{n-i} vanishes identically. We use the method of mathematical induction. First, we know that P_n is linear in k. Next, observe that from II* it follows that

$$\frac{\partial}{\partial k} \frac{P_{n-i}}{(k - f'(x))^{i+1}} = \frac{\partial P_{n-i+1}}{\partial x} \frac{1}{(k - f'(x))^{i+2}}.$$

Assume the statement true for $P_n, P_{n-1}, \cdots, P_{n-i+1}$. Then the right-hand side of this last equation can be represented by partial fractions

$$\frac{A_{i+2}}{(k - f'(x))^{i+2}} + \cdots + \frac{A_2}{(k - f(x))^2},$$

where the A_j are functions of x analytic in the given domain. The fraction whose denominator is $k - f'(x)$ is absent, because the degree of the numerator, $\partial P_{n-i+1}/\partial x$ is at most i. Hence on integrating we have

$$\frac{P_{n-i}}{(k - f'(x))^{i+1}} = -\frac{A_{i+2}}{(i+1)(k - f'(x))^{i+1}} - \cdots - \frac{A_2}{k - f'(x)} + \psi(x).$$

This completes the proof.

If we substitute $(f(x+h) - f(x))/h$ for k in (3), and multiply both members by h, we have an equation of the form

$$(5) \qquad h^{n+1} + \sum C_{pqr}(x) h^p f(x)^q f(x+h)^r = 0, \qquad p + q + r \leqq n + 1.$$

This equation is of positive degree in $f(x+h)$. For, if it were of degree zero, h would depend on x alone and $\partial h/\partial k = 0$. We have seen above that $\partial h/\partial k \neq 0$. Thus, for constant x, $f(x+h)$ coincides with a branch of an algebraic function of $x+h$; and since f is integral, the algebraic function is a polynomial, and so is f.

There remains only to show that f is a polynomial of degree $n+1$. Since (1) defines n distinct, nonzero elements h_i for f, the degree of f is not less than $n+1$. In addition, we readily see the degree of f cannot exceed $n+1$. In fact, if we substitute the Taylor expansion of $f(x+h)$ in (5), the resulting polynomial in h must be identically zero, and this can happen only if the degree of $f(x+h)$ in h does not exceed $n+1$.

2. **Polynomials in N variables.** The above characterization of polynomials of a single variable finds its analogue for polynomials of several variables as follows: The equation (1) is replaced by

$$(1') \qquad \frac{f(x_1 + k_1 h, x_2 + k_2 h, \cdots, x_N + k_N h) - f(x_1, x_2, \cdots, x_N)}{h} = k$$

with $x_1, \cdots, x_N, k_1, \cdots, k_N, k$ independent complex variables. If we denote by D the operator $k_1\partial/\partial x_1 + k_2\partial/\partial x_2 + \cdots + k_N\partial/\partial x_N$, we may assert that if $f(x_1, \cdots, x_N)$ is a polynomial in $x_1, \cdots, x_N$, of total degree $n+1$, then there exist n analytic, distinct, nonzero elements

$$h_i(x_1, \cdots, x_N; k_1, \cdots k_N; k)$$

which satisfy (1′) and are such that

$$(2') \qquad D\sum_1^n h_i = -(n+1).$$

This fact is readily established if we expand $f(x_1+k_1h, \cdots, x_N+k_Nh)$ by Taylor's theorem, obtaining

$$f(x_1+k_1h, \cdots, x_N+k_Nh)$$
$$= f(x_1, \cdots, x_N) + hDf + \cdots + \frac{h^i}{i!}D^if + \cdots + \frac{h^{n+1}}{(n+1)!}D^{n+1}f.$$

Thus (1′) may be written

$$h^n + (n+1)\frac{D^nf}{D^{n+1}f}h^{n-1} + \cdots + \frac{(n+1)!}{i!}\frac{D^if}{D^{n+1}f}h^{i-1} + \cdots - \frac{k(n+1)!}{D^{n+1}f} = 0;$$

and since $D^{n+2}f \equiv 0$,

$$D\sum_1^n h = D\left(-(n+1)\frac{D^nf}{D^{n+1}f}\right) = -(n+1)\left(\frac{D^{n+1}f}{D^{n+1}f}\right) = -(n+1).$$

THEOREM II. *Let $f(x_1, \cdots, x_N)$ be an integral function. If there exist n functions $h_i(x_1, \cdots, x_N; k_1, \cdots, k_N; k)$, analytic, distinct, and nonzero on a region in $(x_1, \cdots, x_N; k_1, \cdots, k_N; k)$ space and which on this set are such that they satisfy (1′) and*

$$D\sum_1^n h_i = -(n+1),$$

then f is a polynomial of total degree $n+1$.

The proof of Theorem II is identical with the proof of Theorem I, where the counterpart of equation (4) is

$$(4') \qquad (1+Dh) = \frac{(Df-k)}{h}\frac{\partial h}{\partial k}$$

and those of I and II are, respectively,

$$\mathrm{I}' \qquad \frac{1}{P_n}\frac{\partial P_n}{\partial k} = -\frac{1}{Df-k},$$

$$\mathrm{II}' \qquad DP_{i+1} = -(n-i+1)P_i - \frac{\partial P}{\partial k}(Dj-k).$$

To establish (4′) from (1′) we operate on both members with D, getting

$$
\begin{aligned}
Df+kDh &= Df(x_1+k_1h, \cdots, x_N+k_Nh)\\
&= \sum k_i \frac{\partial}{\partial x_i} f(x_1+k_1h, \cdots, x_N+k_Nh)\\
&= \Bigg(\sum_i k_i \sum_j \frac{\partial}{\partial x_j + k_jh}\\
&\qquad \cdot f(x_1+k_1h, \cdots, x_N+k_Nh)\left(k_j \frac{\partial h}{\partial x_i}\right)\Bigg)\\
&\qquad + \sum_i k_i \frac{\partial}{\partial x_i + k_ih} f(x_i+k_ih)\\
&= \Bigg(\sum_j k_j \frac{\partial}{\partial x_j + k_jh}\\
&\qquad \cdot f(x_1+k_1h, \cdots, x_N+k_Nh)\Bigg)(1+Dh).
\end{aligned}
$$

Differentiating both members of (1′) with respect to k, we have

$$
k\frac{\partial}{\partial k}h + h = \left\{ \sum k_i \frac{\partial}{\partial x_i + k_ih} f(x_1+k_1h, \cdots, x_N+k_Nh)\right\}\frac{\partial h}{\partial k},
$$

and eliminating the term in the bracket yields equation (4′).

Since the operator D behaves for its variables like the operator d/dx for a single variable, all the reasoning of the previous section can, from this point on, be used verbatim, proving Theorem II.

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