METRIC SPACES WITH GEODESIC RICCI CURVES. I

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1. **Introduction.** The problem of determining all Riemannian spaces of three dimensions admitting geodesic Ricci curves has been solved by G. Ricci* and P. Walberer† using, however, different methods. Although they obtained all such V_3, the complete explicit determination of all such V_n for $n>3$ does not seem possible because of the increased number and complexity of the differential equations which arise.

In this paper the following two problems related to the above problem will be considered.

In the first problem we suppose given a set of linearly independent vectors‡ $\lambda^i_{a|}$ and wish to determine necessary and sufficient conditions on the $\lambda^i_{a|}$ in order that a set of scalars $\theta_a(\neq 0)$ exist which will define a metric space V_n with a metric determined by

$$g^{ij} = \sum_h e_h \bar{\lambda}^i_{h|} \bar{\lambda}^j_{h|}, \tag{1}$$

where

$$\bar{\lambda}^i_{a|} = \theta_a \lambda^i_{a|}, \tag{2}$$

and e_h $(=\pm 1)$ are arbitrary; and such that the congruences of curves defined by the $\lambda^i_{a|}$ will be geodesics in the V_n thus determined. (The vectors $\bar{\lambda}^i_{a|}$ define the same congruences as do the $\lambda^i_{a|}$, and these congruences form an orthogonal ennuple in the V_n.)

In the second problem we assume that these conditions on the $\lambda^i_{a|}$ have been determined and that the n congruences defined by à set of $\lambda^i_{a|}$ are geodesics in the V_n determined by

$$g^{ij} = \sum_h e_h \lambda^i_{h|} \lambda^j_{h|};$$

we then find necessary and sufficient conditions that, with respect to the metric (1), the congruences be geodesic Ricci curves.

* G. Ricci, *Sulle varieta a tre dimensioni dotate die terne principali di congruenze geodetiche*, Rendiconti della Reale Accademia dei Lincei, (5), vol. 27 (1918), pp. 21–28, 75–87.

† P. Walberer, *Riemannsche Raume mit geodätischen Riccikurven*, Hamburger Abhandlungen, vol. 10 (1934), pp. 152–168.

‡ All indices take the values 1, 2, $\cdots$, n unless otherwise noted.

The method of Walberer is followed, and we obtain generalizations to n dimensions of his conditions for 3 dimensions.*

2. **Geodesic congruences.** In this section we solve the first of the two problems stated in the introduction.

An orthogonal ennuple of unit vectors, $\lambda^i_{a|}$, of a V_n will determine geodesic congruences if†

$$\gamma_{iaa} = 0, \quad a \text{ not summed}, \tag{3}$$

where

$$\gamma_{lhk} \equiv \lambda_{l|i,j}\lambda^i_{h|}\lambda^j_{k|},$$

are the Ricci coefficients of rotation.‡

The congruences are determined from the equations

$$\frac{dx^1}{\lambda^1_{a|}} = \frac{dx^2}{\lambda^2_{a|}} = \cdots = \frac{dx^n}{\lambda^n_{a|}}, \tag{4}$$

and are defined by the $\lambda^i_{a|}$ to within a scalar factor, that is, the $\bar{\lambda}^i_{a|}$ given by (2) will define the same congruences as will the $\lambda^i_{a|}$. If then we wish to determine our conditions on the $\lambda^i_{a|}$ which make these congruences geodesics with respect to (1) we form equations (3) with $\lambda^i_{a|}$ replaced by $\bar{\lambda}^i_{a|}$ given by (2). The resulting equations in θ_a as unknowns must have solutions and this requirement leads to the desired conditions on the $\lambda^i_{a|}$.

We begin by replacing (3) by more suitable equations. If the operators Δ_a are defined by

$$\Delta_a \equiv \lambda^i_{a|}\frac{\partial}{\partial x^i},$$

their integrability conditions take the form§

$$(\Delta_a, \Delta_b) \equiv \Delta_a\Delta_b - \Delta_b\Delta_a = \sum_i e_i(\gamma_{iab} - \gamma_{iba})\Delta_i.$$

We write this in the form

$$(\Delta_a, \Delta_b) = c^k_{ab}\Delta_k,$$

* New conditions are also obtained which do not appear for $n=3$.

† L. P. Eisenhart, *Riemannian Geometry*, p. 100. References to this book will be in the form RG.

‡ RG, pp. 97–98.

§ RG, p. 99.

so that

$$\overset{k}{c}_{ij} = e_k(\gamma_{kij} - \gamma_{kji}) = -\overset{k}{c}_{ji}.$$

These equations can be solved to give

$$\gamma_{ijk} = \tfrac{1}{2}(e_i\overset{i}{c}_{jk} + e_j\overset{j}{c}_{ki} + e_k\overset{k}{c}_{ji}). \tag{5}$$

In terms of the c's the conditions (3) for geodesics become

$$\overset{j}{c}_{ij} = 0, \quad j \text{ not summed}. \tag{6}$$

We also remark, for later use, that the Jacobi identities

$$(\Delta_i, (\Delta_j, \Delta_k)) + (\Delta_j, (\Delta_k, \Delta_i)) + (\Delta_k, (\Delta_i, \Delta_j)) = 0$$

are equivalent to the conditions

$$\Delta_i\overset{l}{c}_{jk} + \Delta_j\overset{l}{c}_{ki} + \Delta_k\overset{l}{c}_{ij} = -(\overset{h}{c}_{jk}\overset{l}{c}_{ih} + \overset{h}{c}_{ki}\overset{l}{c}_{jh} + \overset{h}{c}_{ij}\overset{l}{c}_{kh}). \tag{7}$$

Writing

$$\bar{\Delta}_a = \theta_a\Delta_a,$$

we have

$$(\bar{\Delta}_a, \bar{\Delta}_b) = \overset{k}{\bar{c}}_{ab}\bar{\Delta}_k,$$

from which we obtain

$$\overset{k}{\bar{c}}_{ij} = \frac{\theta_i\theta_j}{\theta_k}\overset{k}{c}_{ij}, \qquad k \neq i, j, \tag{8}$$

$$\overset{i}{\bar{c}}_{ij} = \theta_j(\overset{i}{c}_{ij} - \mu_{ij}), \qquad i \neq j, \tag{9}$$

where

$$\mu_i = \log\theta_i, \qquad \mu_{ij} = \Delta_j\mu_i.$$

The conditions (6) for geodesics in the barred quantities are $\overset{j}{\bar{c}}_{ij} = 0$, ($j$ not summed), and from (9) these conditions become

$$\mu_{ij} = \Delta_j\mu_i = \overset{i}{c}_{ij}, \qquad i \neq j. \tag{10}$$

These equations in the θ_a as unknowns must have solutions* and their integrability conditions give us our conditions on the $\lambda^i_{a|}$.

Forming the integrability conditions for (10), we obtain

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* Of course, we do not assume $\overset{i}{c}_{ij} = 0$, (i not summed), as this condition need only be satisfied by $\overset{i}{\bar{c}}_{ij}$.

$$(\Delta_i, \Delta_j)\mu_k = \Delta_i c^k_{kj} - \Delta_j c^k_{ki} = c^h_{ij}\Delta_h\mu_k, \qquad k \neq i, j,$$

or

$$c^k_{ij}\Delta_k\mu_k = \Delta_i c^k_{kj} - \Delta_j c^k_{ki} - c^h_{ij}c^k_{kh}, \qquad k \neq i, j;\ k \text{ not summed.} \tag{11}$$

If in (7) we put $l = k$ (without summing), and transpose appropriate terms, we obtain

$$\Delta_i c^k_{kj} - \Delta_j c^k_{ki} - c^h_{ij}c^k_{kh} = \Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}, \qquad k \text{ not summed.}$$

Substitution from this equation into (11) gives

$$c^k_{ij}\Delta_k\mu_k = \Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}, \qquad k \neq i, j;\ k \text{ not summed.} \tag{12}$$

The equations (12) are the desired integrability conditions of (10).

If $c^k_{ij} = 0$, $(k \neq i, j)$, for every k, equations (12) are satisfied identically. We thus assume that for each k there is at least one $c^k_{ij} \neq 0$. We can then write (12) in the form*

$$\Delta_k\mu_k = \frac{1}{c^k_{ij}}\left(\Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}\right), \qquad k \neq i, j, \tag{13}$$

and equations (13) are to hold for all i, j such that $c^k_{ij} \neq 0$.

Since the left member of (13) is independent of i, j, so must be the right member, that is, we must have

$$\frac{1}{c^k_{ij}}\left(\Delta_k c^k_{ij} + c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh}\right) = \frac{1}{c^k_{ab}}\left(\Delta_k c^k_{ab} + c^h_{bk}c^k_{ah} + c^h_{ka}c^k_{bh}\right) \equiv H^k. \tag{14}$$

Equations (14) are to hold for all i, j, a, b such that

$$c^k_{ij} \neq 0, \qquad c^k_{ab} \neq 0, \qquad k \neq i, j, a, b.$$

We now consider (13) and (10) combined, and form their integrability conditions. The only new conditions are obtained from $(\Delta_l, \Delta_k)\mu_k$, these conditions reducing to

$$H^{kl} \equiv \Delta_l H^k - \Delta_k c^k_{kl} - c^h_{lk}c^k_{kh} + c^k_{kl}H^k = 0, \qquad k \neq l. \tag{15}$$

If for given k, i, j, $(k \neq i, j)$, $c^k_{ij} = 0$, equations (12) reduce to

$$H^k_{ij} \equiv c^h_{jk}c^k_{ih} + c^h_{ki}c^k_{jh} = 0, \tag{16}$$

which must be satisfied for all such k, i, j.

* If for any given k, $c^k_{ij} = 0$, for all i and j, the corresponding equation is identically zero and may be dropped.

Under (8), (9), we find that the various H's transform in the following manner:

$$\overline{H}^k = \theta_k(H^k - \mu_{kk}), \qquad \overline{H}^k_{(ij)(ab)} = \theta_k H^k_{(ij)(ab)},$$
$$\overline{H}^{kl} = \theta_k\theta_l H^{kl}, \qquad \overline{H}^k_{ij} = \theta_i\theta_j H^k_{ij}.$$

The quantities $H^k_{(ij)(ab)}$ are the result of transposing the right member of (14) to the left, giving $H^k_{(ij)(ab)} = 0$.

The quantities H^{kl} are the generalizations to n dimensions of the corresponding quantities for $n=3$ given by Walberer. The other H's do not appear for $n=3$.

We can state the following theorem:

Given a set of independent vectors $\lambda^i_{a|}$, *necessary and sufficient conditions that a set of scalars* θ_a *should exist such that the vectors* $\overline{\lambda}^i_{a|} = \theta_a\lambda^i_{a|}$ *determine an orthogonal ennuple of geodesic congruences in the metric space defined by*

$$g^{ij} = \sum_h e_h \overline{\lambda}^i_{h|}\overline{\lambda}^j_{h|}, \tag{17}$$

are

$$H^{kl} = 0, \qquad k \neq l, \tag{a}$$

$$H^k_{(ij)(ab)} = 0, \qquad k \neq i, j, a, b, \tag{b}$$

$$H^k_{ij} = 0, \qquad k \neq i, j. \tag{c}$$

Conditions (a) *are used for all indices* k *for which* H^k *is defined. Conditions* (b) *are used only for indices* i, j, a, b, $(\neq k)$, *such that* $c^k_{ij}=0$, $c^k_{ab}=0$. *Conditions* (c) *are used only for indices* k, i, j *such that* $c^k_{ij}=0$, $(k\neq i, j)$.

It is to be noted that if $c_{ij}{}^i = 0$, (i not summed), these integrability conditions are satisfied identically (as is to be expected). In this case $H^k=0$, as is seen from (7) on putting $k=l$. Then (a) and (b) follow, and (c) follows from the Jacobi identities (7).

From the definitions of the H's it is seen that the conditions (a), (b), (c) are independent of the e_h.

3. **Geodesic Ricci curves.** We consider now the second of the two problems stated in the introduction. We suppose given a set of $\lambda^i_{a|}$ satisfying conditions (a), (b), (c) of section 2. Then there exists a set of the θ_a such that $\bar{c}^i_{ij}=0$, (i not summed), and thus we obtain the $\overline{\lambda}^i_{a|}$ of (2). We now drop the bars and suppose given a set of λ's such

that $c^i_{ij}=0$, (i not summed), and wish to find a set of scalars θ_a ($\neq 0$) such that $\bar{\lambda}^i_{a|}=\theta_a\lambda^i_{a|}$ will determine the geodesic Ricci curves of a V_n defined by (1).

An orthogonal ennuple of unit vectors, $\lambda^i_{a|}$, of a V_n will define the n Ricci congruences of the V_n if

$$R_{ij}\lambda^{i}_{a|}\lambda^{j}_{b|} = 0, \qquad a \neq b, \tag{18}$$

where R_{ij} are the components of the Ricci tensor.*

From

$$\begin{aligned}\gamma_{lpqr} \equiv R_{hijk}\lambda^{h}_{l|}\lambda^{i}_{p|}\lambda^{j}_{q|}\lambda^{k}_{r|} &= \Delta_r\gamma_{lpq} - \Delta_q\gamma_{lpr} \\ &+ \sum_m e_m\left[\gamma_{lpm}(\gamma_{mqr} - \gamma_{mrq}) + \gamma_{mlr}\gamma_{mpq} - \gamma_{mlq}\gamma_{mpr}\right],\end{aligned} \tag{19}$$

and

$$R_{ij} = g^{hk}R_{hijk},$$

we obtain as conditions equivalent to (18),

$$\sum_a e_a\gamma_{abca} = 0, \qquad b \neq c. \tag{20}$$

If we assume (3) satisfied (condition for geodesics), conditions (20) reduce to

$$\sum_{a,m} e_a(\Delta_a\gamma_{abc} + e_m\gamma_{abm}\gamma_{mca}) = 0, \qquad b \neq c. \tag{21}$$

If in (7) we put $l=k$, and sum, and use (6), we obtain

$$\Delta_k \overset{k}{c}_{ij} = 0. \tag{22}$$

If now we substitute from (5) in (21) and make use of (6) and (22) we eventually reduce (21) to the form

$$\begin{aligned}&\sum_a e_a(e_b\Delta_a\overset{b}{c}_{ac} + e_c\Delta_a\overset{c}{c}_{ab}) \\ &\quad = \sum_{a,m} e_ae_m\left[e_ae_m\overset{a}{c}_{bm}\overset{m}{c}_{ca} + \overset{a}{c}_{bm}\overset{a}{c}_{cm} - \tfrac{1}{2}e_be_c\overset{b}{c}_{am}\overset{c}{c}_{am}\right], \qquad b \neq c.\end{aligned} \tag{23}$$

We must impose conditions (23) and (6) in the $\bar{c}$ to obtain geodesic Ricci curves.

From (10) we now have

$$\mu_{ij} = \Delta_j\mu_i = 0, \qquad i \neq j. \tag{24}$$

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* RG, pp. 110, 114.

Hence,

$$(\Delta_i, \Delta_j)\mu_k = \Delta_i\mu_{kj} - \Delta_j\mu_{ki} = 0 = \overset{m}{c}_{ij}\Delta_m\mu_k = \overset{k}{c}_{ij}\Delta_k\mu_k, \qquad k \neq i, j,$$

or

$$\overset{k}{c}_{ij}\mu_{kk} = 0, \qquad k \neq i, j;\ k \text{ not summed}. \tag{25}$$

The Jacobi conditions in the $\bar{c}$ add no new conditions on the θ_a, for from (7) in the $\bar{c}$ we get (24) and (25) again.

We now form (23) in the $\bar{c}$ and thus get the remaining conditions on the θ. We find on forming (23) in the $\bar{c}$ and then substituting from (8) and (9) that the resulting conditions can be written in the form

$$\begin{aligned} &\sum_a e_a\theta_a^2\left[(e_b\theta_c^2\overset{b}{c}_{ac} + e_c\theta_b^2\overset{c}{c}_{ab})\mu_{aa} + e_b\theta_c^2\Delta_a\overset{b}{c}_{ac} + e_c\theta_b^2\Delta_a\overset{c}{c}_{ab}\right] \\ &\quad = \sum_{a,m} e_a e_m\left[e_a e_m\theta_b^2\theta_c^2\overset{a}{c}_{bm}\overset{m}{c}_{ca} + \frac{\theta_b^2\theta_c^2\theta_m^2}{\theta_a^2}\overset{a}{c}_{bm}\overset{a}{c}_{cm} - \tfrac{1}{2}e_b e_c\theta_a^2\theta_m^2\overset{b}{c}_{am}\overset{c}{c}_{am}\right], \end{aligned} \tag{26}$$

$$b \neq c.$$

The conditions on the θ_i are given by (24), (25), and (26). We shall solve the simplest case here, that is, the one in which all the c's are zero but one, which we may assume to be c_{23}^1. (If all the c's are zero, the space is flat.)

From (25) we see $\mu_{11}=0$, which together with (24) gives

$$\theta_1 = k_1 = \text{const.}$$

The only conditions obtained from (26) are

$$\mu_{33} + \Delta_3 A = 0, \tag{27}$$

$$\mu_{22} + \Delta_2 A = 0, \tag{28}$$

where

$$\overset{1}{c}_{23} = a, \qquad A = \log a.$$

Since now

$$(\Delta_i, \Delta_j) = \overset{1}{c}_{ij}\Delta_1,$$

we have

$$(\Delta_i, \Delta_j) = 0, \qquad i, j \neq 2.$$

Hence the operators Δ_i, $(i\neq 2)$, define an abelian group,* and by a change of coordinate system, we have the canonical form

* L. P. Eisenhart, *Continuous Groups of Transformations*, p. 49.

$$\Delta_i = \frac{\partial}{\partial x^i}, \qquad i \neq 2, \tag{29}$$

$$\Delta_2 = \alpha(x^2, x^3)\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}, \tag{30}$$

and

$$a = -\frac{\partial \alpha}{\partial x^3}.$$

Equations (24), (29), (30) now show that

$$\mu_i = \mu_i(x^i), \qquad \text{or} \qquad \theta_i = \theta_i(x^i),$$

and then from (27) and (28) we obtain $\partial^2 A/\partial x^2 \partial x^3 = 0$. This gives

$$a = B(x^2)C(x^3), \quad \mu_2' = -\frac{B'}{B}, \quad \mu_3' = -\frac{C'}{C}, \quad \alpha = -BD, \quad D = \int C dx^3,$$

or

$$\theta_2 = \frac{k_2}{B}, \qquad \theta_3 = \frac{k_3}{C}.$$

Then $\bar{c}_{23}^1 = k$ (=const.), and the $\bar{\Delta}_a$, by means of the coordinate transformation

$$x'^1 = \frac{x^1}{k_1}, \quad x'^2 = \int \frac{B}{k_2} dx^2, \quad x'^3 = \frac{D}{k_3}, \quad x'^i = \int \frac{dx^i}{\theta^i}, \qquad i > 3,$$

can be reduced to the canonical form (by dropping primes)

$$\bar{\Delta}_i = \frac{\partial}{\partial x^i}, \qquad i \neq 2,$$

$$\bar{\Delta}_2 = -kx^3\frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2}.$$

From (17) we have

$$g^{ij} = e_i\delta_j^i, \ (i, j \neq 1, 2); \ g^{11} = e_1 + e_2(kx^3)^2, \ g^{12} = -ke_2x^3, \ g^{22} = e_2,$$

from which the g_{ij} are easily obtained.

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