Then

$$\frac{F_{n,m}(x)}{[\theta_k(x)]^n}$$
$$= \frac{(x - a_1) \cdots (x - a_{m-1})(x - a_{m+1}) \cdots (x - a_\nu)G_{n,m}(x)}{(a_m - a_1) \cdots (a_m - a_{m-1})(a_m - a_{m+1}) \cdots (a_m - a_\nu)G_{n,m}(a_m)}.$$

From this relation it is easily seen that all the conditions of Corollary 2 are satisfied, and our statement follows.

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DUALISM IN ABELIAN GROUPS*

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It has been proved† that a finite Abelian group contains as many subgroups of a given order n as it contains factor groups of order n ($=$subgroups of index n) and E. Steinitz‡ knew that a finite Abelian group contains as many subgroups of a given structure $\mathbf{n}$ as it contains factor groups of structure $\mathbf{n}$. It is the aim of this note to prove that such a *dualism* exists in the Abelian group G if, and only if, G is a group without elements of infinite order whose primary components are finite. This is remarkable as an exception to the rule that every proposition which is satisfied in finite Abelian groups holds also true in every primary Abelian group such that the orders of its elements are bounded.

Let G and G' be two (additively written Abelian) groups. Then the function $\mathbf{d}$ is a dualism of G upon G' if it has the following properties:

(1) $\mathbf{d}$ is defined for every subgroup S of G and $S\mathbf{d}$ is a uniquely determined subgroup of G';

(2) to every subgroup S' of G' there exists a subgroup S of G such that $S\mathbf{d} = S'$;

(3) $S \leqq T$ $(\leqq G)$ if, and only if, $T\mathbf{d} \leqq S\mathbf{d}$ $(\leqq G')$;

* Presented to the Society, October 31, 1936.

† Garrett Birkhoff, *Subgroups of Abelian groups*, Proceedings of the London Mathematical Society, (2), vol. 38 (1934), pp. 385–401.

‡ E. Steinitz, Jahresberichte der Deutschen Mathematiker-Vereinigung, vol. 9 (1901), pp. 80–85.

(4) the subgroup S of G and $G'/(S\boldsymbol{d})$ are isomorphic.

Conditions (1)-(3) imply that a dualism is a one-one-correspondence between the whole set of subgroups of G and the whole set of subgroups of G' which interchanges cross cut and join of subgroups. In particular $G\boldsymbol{d}=0$, $O\boldsymbol{d}=G'$.

To every dualism $\boldsymbol{d}$ of G upon G' there exists an inverse function $\boldsymbol{d}^{-1}$ which maps the subgroups of G' upon the whole set of subgroups of G, satisfies (1)-(3) and

(4′) is such that the subgroup $S'\boldsymbol{d}^{-1}$ of G and G'/S' are isomorphic.

If $\boldsymbol{d}$ is a dualism of G and G', then the transformation $\mathbf{c}$ of the set of all the subgroups of G upon the set of all the subgroups of G' is a dualism if, and only if, there exists a one-one-correspondence $\boldsymbol{b}$ between the subgroups of G such that

(i) $S\leqq T$ if, and only if, $S\boldsymbol{b}\leqq T\boldsymbol{b}$;*

(ii) S and $S\boldsymbol{b}$ are isomorphic;

(iii) $\mathbf{c}=\boldsymbol{bd}$.

A dualism of G upon G is a *dualism in* G.

THEOREM. *There exists a dualism in the Abelian group* G *if, and only if,* (a) *the order of every element of* G *is finite*; (b) *every primary component*† G_p *of* G *is finite.*

PROOF. 1. Assume that there exists a dualism $\boldsymbol{d}$ in G. The set F of all the elements of finite order in G is a subgroup of G and there exists therefore a subgroup W of G such that $W\boldsymbol{d}=F$. Since W and $A/(W\boldsymbol{d})=A/F$ are isomorphic, and since all the elements $\neq 0$ of A/F have infinite order, all the elements $\neq 0$ of W are of infinite order. If furthermore w is an element of infinite order, and $\overline{w}$ is the (infinite) cyclic subgroup of G generated by w, then $G/(\overline{w}\boldsymbol{d})$ is an infinite cyclic group and consequently $W\boldsymbol{d}=F\leqq\overline{w}\boldsymbol{d}$, or $\overline{w}\leqq W$. Thus it has been proved that W is a subgroup of G which contains all the elements of infinite order and whose elements $\neq 0$ are of infinite order.

If $w\neq 0$ is an element of G, then there exists a positive integer n such that w is not contained in the subgroup V of G, generated by nw. Then G/V contains elements of finite order, that is, $V\boldsymbol{d}^{-1}\leqq G$ contains elements of finite order; in other words, if $G\neq 0$,

* That is, $\boldsymbol{b}$ preserves the situation of the subgroups of G.

† G_p consists of all the elements of G whose order is a power of the prime number p.

then $W \neq G$. Since the sum of an element of infinite order and of an element of finite order is an element of infinite order, $W < G$ implies $W = 0$. Condition (a) is therefore necessary for the existence of a dualism in G.

2. Assume that G does not contain elements of infinite order. Then G is the direct sum of its primary components G_p and a dualism exists in G if, and only if, there exists a dualism in every G_p.

3. Assume that there exists a dualism in the primary Abelian group G of *characteristic* p. Denote by C the greatest subgroup* of G such that $C = pC$. Then $C/S = p(C/S)$ for every subgroup S of C and C/S is therefore either 0 or infinite. If T is a subgroup of G and T' its intersection with C, then C/T' is isomorphic with a subgroup of G/T. Hence $C \leqq T$, if G/T is finite. The cross cut D of all the subgroups of finite index in G contains therefore C. Since G is the join of its finite subgroups, since furthermore the subgroup S of G is finite if, and only if, $S\boldsymbol{d}$ is of finite index in G, it follows that $C \leqq D = G\boldsymbol{d} = 0$, that is, $C = 0$; in other words, the subgroup S of G satisfies $S = pS$ if, and only if, $S = 0$.

If the orders of the elements of the primary group G are not bounded, then there exists a subgroup U of G, such that G/U is countable and the orders of the elements of G/U are not bounded. It is a consequence of the general theory of countable primary groups† that there exists a subgroup V of G satisfying $U \leqq V < G$, $0 \neq G/V = p(G/V)$. The subgroup V' of G such that $V'\boldsymbol{d} = V$ satisfies, therefore, $0 \neq V' = pV'$, and this is impossible as proved above. The orders of the elements of G are therefore bounded.

If G is infinite, then the number $r(G)$ of elements in G, the number of elements of order p, and the number of subgroups of order p are equal. Since the orders of the elements of G are bounded, there exists‡ a basis $b_0, b_1, \cdots, b_v, \cdots$, of G (if $G \neq 0$), containing $r(G)$ elements. One of the elements b_v, say b_0,

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* C is uniquely determined, since the join of subgroups S, satisfying $S = pS$, satisfies also this equation.

† See L. Zippin, *Countable torsion groups*, Annals of Mathematics, vol. 36 (1935), pp. 86–99.

‡ For a proof see, for example, R. Baer, *Der Kern, eine characteristische Untergruppe*, Compositio Mathematica, vol. 1 (1934), pp. 254–283.

has an order which does not exceed the order of any element b_u. Then the elements pb_0, $b_1+e_1b_0, \cdots, b_v+e_vb_0, \cdots$, generate for every choice of the integers e_v a subgroup of index p, and since the number $r(G)$ of the elements b_v is infinite, there exist at least $2^{r(G)}$ subgroups of index p. Since $r(G)<2^{r(G)}$, this is a contradiction to the existence of a dualism in G, that is, G is finite and (b) is therefore a necessary condition for the existence of a dualism.

4. Assume that G is a finite (primary Abelian) group. Denote by Γ the group of all the characters* of G. Then there exists an isomorphism f, mapping Γ upon G. If S is any subgroup of G, then let $\Pi(S)$ be the group of all those characters of G which map S upon 0. Since S, the group of characters of S, and $\Gamma/\Pi(S)$ are isomorphic groups, it follows that $S\boldsymbol{d}=\Pi(S)f$ is a dualism in G.

This dualism $\boldsymbol{d}$ satisfies not only (1)-(4), but also (4′), and the additional condition

(5) there exists an automorphism of G mapping the subgroup S upon T if, and only if, there exists an automorphism of G mapping $S\boldsymbol{d}$ upon $T\boldsymbol{d}$.

Furthermore this dualism interchanges order and height.†

COROLLARY. *There exists a dualism of the Abelian group G upon the group G' if, and only if,* (a) *G and G' are isomorphic*; (b) *there exists a dualism in G.*

For if $\boldsymbol{d}$ is a dualism of G upon G', then G and $G'/(G\boldsymbol{d})=G'$ are isomorphic. If furthermore $\boldsymbol{e}$ is an isomorphism of G' upon G, then $\boldsymbol{de}$ is a dualism in G. If, conversely, $\boldsymbol{d}$ is a dualism in G and $\boldsymbol{e}$ an isomorphism of G upon G', then $\boldsymbol{de}$ is a dualism of G upon G'.

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* For the theory of characters of finite abelian groups, see, for example, E. Heike, *Vorlesungen über die Theorie der algebraischen Zahlen*, 1923.

† Since the subgroup of the elements x which satisfy $p^ix=0$ and the subgroup of the elements $x=p^iy$ for y in G are mapped upon each other.