Hurwitz' definition of commutative group employs two postulates;* the reader will find it interesting to compare with that of the present paper.

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ON SOME QUADRATURE FORMULAS AND ON ALLIED THEOREMS ON TRIGONOMETRIC POLYNOMIALS

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1. *Introduction.* We consider the following problem.

Find $2n$ *numbers* $0\leqq\theta_1<\theta_2<\cdots<\theta_{2n-1}<\theta_{2n}<2\pi$ *such that for every trigonometric polynomial*

$$(1)\qquad G(\theta) = \alpha_n + \sum_{k=0}^{n-1}\{\alpha_k \cos (n-k)\theta + \beta_k \sin (n-k)\theta\}$$

of order $\leqq n$ *the equality*

$$(2)\qquad \int_0^{2\pi} F(\theta)G(\theta)d\theta = L\left\{\sum_{i=1}^{n} G(\theta_{2i-1}) - \sum_{i=1}^{n} G(\theta_{2i})\right\}$$

holds true, where $F(\theta)$ *is the given function*

$$(3)\qquad F(\theta) = \sum_{k=n-s}^{\infty} (A_k \cos k\theta + B_k \sin k\theta), \qquad (s \leqq (n-1)/2),$$

and L *is a given positive number.*

Let

$$(4)\qquad \begin{aligned} F_n(\theta) &= \sum_{k=n-s}^{n} \{A_k \cos k\theta + B_k \sin k\theta\}, \\ p_n(\theta) &= -\int F_n(\theta)d\theta, \qquad G^*(\theta) = \text{const.}\prod_{k=1}^{2n} \sin\frac{\theta-\theta_k}{2}. \end{aligned}$$

Then, integrating (2), we get

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* For references, see the first footnote to this paper.

$$\int_0^{2\pi} \left\{ p_n(\theta) - \frac{L}{2} \operatorname{sgn} G^*(\theta) \right\} G'(\theta) d\theta = 0,$$

whence it follows that

$$(5) \quad \frac{L}{2} \operatorname{sgn} G^*(\theta) = \frac{c}{2} + p_n(\theta) + \sum_{k=n+1}^{\infty} (M_k \cos k\theta + N_k \sin k\theta),$$

where c is an arbitrary constant; it is clear that

$$(6) \qquad |c| \leqq L.$$

By the theorem of N. Achyeser and M. Krein† a necessary and sufficient condition for the existence of a function deviating from zero by not more than $L/2$ and having the first members of the Fourier expansion

$$\frac{c}{2} + \Re \sum_{k=1}^{n} \bar{c}_k z^k, \quad c_k = a_k + ib_k, \qquad (k = 1, 2, \cdots, n;\ z = e^{i\theta}),$$

is that *the form*

$$(7) \qquad \sum_{r=0}^{n} \sum_{k=0}^{n} \gamma_{r-k} x_r \bar{x}_k$$

be non-negative, where the γ'*s are to be found as the coefficients of the expansion*

$$(8) \qquad e^{(\pi i c)/(2L)} \cdot e^{(\pi i/L)S} = \gamma + \gamma_1 z + \cdots + \gamma_n z^n + \cdots,$$

where

$$S = \sum_{k=1}^{n} \bar{c}_k z^k,$$

and where $\gamma_0 = \gamma + \bar{\gamma}$, $\gamma_{-k} = \bar{\gamma}_k$, $(k = 1, 2, \cdots, n)$.

It is clear that in our case we have

$$(9) \quad \begin{aligned} c_{n-k} &= \frac{B_{n-k} - iA_{n-k}}{n-k}, \qquad \gamma_{n-k} = \frac{\pi i}{L} e^{(\pi i c)/(2L)} \bar{c}_{n-k}, \\ & \qquad\qquad\qquad\qquad (k = 0, 1, \cdots, s), \\ \gamma_0 &= 2 \cos \frac{\pi c}{2L}, \qquad \gamma_1 = \gamma_2 = \cdots = \gamma_{n-s-1} = 0. \end{aligned}$$

† N. Achyeser and M. Krein, *Über Fouriersche Reihen beschränkter summierbarer Funktionen und ein neuer Extremumproblem* (1 Teil), Transactions of the Kharkow Mathematical Society, vol. 9 (1934), pp. 18–19.

Setting

$$ie^{(\pi i c)/(2L)}x_{n-k} = y_{n-k}, \quad x_k = y_k, \qquad (k = 0, 1, \cdots, s), \tag{10}$$

we may write our condition thus:

$$\frac{2L}{\pi}\cos\frac{\pi c}{2L} + A \geqq 0, \tag{11}$$

where we have put

$$A = \frac{\sum_{r,k} c_{r-k}y_k\bar{y}_r}{\sum_{k=0}^{n} |y_k|^2}, \qquad (n - s \leqq |r - k| \leqq n). \tag{12}$$

It is easy to find that A satisfies the inequality

$$-\delta_0 \leqq A \leqq \delta_0,$$

where δ_0 is the greatest root of the equation

$$\begin{vmatrix} \delta & 0 & \cdots & 0 & c_{n-s} & c_{n-s+1} & \cdots & c_n \\ 0 & \delta & \cdots & 0 & 0 & c_{n-s} & \cdots & c_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \delta & 0 & 0 & \cdots & c_{n-s} \\ \bar{c}_{n-s} & 0 & \cdots & 0 & \delta & 0 & \cdots & 0 \\ \bar{c}_{n-s+1} & \bar{c}_{n-s} & \cdots & 0 & 0 & \delta & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \bar{c}_n & \bar{c}_{n-1} & \cdots & \bar{c}_{n-s} & 0 & 0 & \cdots & \delta \end{vmatrix} = 0. \tag{13}$$

Therefore our form (7) will be non-negative if and only if

$$\cos\frac{\pi c}{2L} = \frac{\pi\delta_0}{2L}, \qquad L \geqq \frac{\pi\delta_0}{2}. \tag{14}$$

Such are the necessary conditions for the existence of the quadrature formula (2); it is not difficult to see that they are sufficient too.

2. *Roots of Trigonometric Polynomials*. Thus we are to find the polynomial $G^*(\theta)$ of order $\leqq n$ from the condition

$$\operatorname{sgn} G^*(\theta) = \frac{c}{L} + \frac{2}{L}\Re\sum_{k=0}^{s} \bar{c}_{n-k}z^{n-k} + \Re\sum_{k=n+1}^{\infty} \mu_k z^k, \qquad (z = e^{i\theta}).$$

Consider the polynomial

$$P(\theta, \alpha) = \Re[q^2(z)z^{n-2s+\nu}] + |q(z)|^2 \cos\alpha, \qquad (z = e^{i\theta}), \tag{15}$$

where $\nu+1$ is the multiplicity of δ_0 and the polynomial $q(z)$ of degree $s-\nu$ is to be found from the condition

$$\delta_0 \frac{q(z)}{z^{s-\nu}\bar{q}(1/z)} = \bar{c}_{n-s} + \bar{c}_{n-s+1}z + \cdots + \bar{c}_n z^s + \cdots, \qquad (|z| \leqq 1). \tag{16}$$

We have

$$\operatorname{sgn} P(\theta, \alpha)$$
$$= \operatorname{sgn}\left\{\cos\frac{(n-2s+\nu)\theta + 2\vartheta + \alpha}{2}\cos\frac{(n-2s+\nu)\theta + 2\vartheta - \alpha}{2}\right\},$$

where $\vartheta = \arg q(z)$, whence we find

$$\operatorname{sgn} P(\theta,\alpha) = \operatorname{sgn}\left\{ e^{-i\alpha}\frac{1 + z^{n-s}e^{i\alpha}\dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}}{1 + z^{n-s}e^{-i\alpha}\dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}}\right\}$$

$$= \Re\left\{ 1 - \frac{2\alpha}{\pi} + \frac{2}{\pi i}\log\frac{1 + z^{n-s}e^{i\alpha}\dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}}{1 + z^{n-s}e^{-i\alpha}\dfrac{q(z)}{z^{s-\nu}\bar{q}(1/z)}}\right\} \tag{17}$$

$$= 1 - \frac{2\alpha}{\pi} + \frac{4\sin\alpha}{\pi\delta_0}\Re\sum_{k=0}^{s}\bar{c}_{n-k}z^{n-k} + \Re\sum_{k=n+1}^{\infty}\lambda_k z^k.$$

On putting

$$1 - \frac{2\alpha}{\pi} = \frac{c}{L}, \qquad (0 \leqq \alpha \leqq \pi), \tag{18}$$

we see that

$$\operatorname{sgn} G^*(\theta) - \operatorname{sgn} P(\theta, \alpha) = \sum_{k=n+1}^{\infty} (M_k \cos k\theta + N_k \sin k\theta).$$

Hence it follows that almost everywhere in the interval $(0, 2\pi)$

$$\operatorname{sgn} G^*(\theta) = \operatorname{sgn} P(\theta, \alpha),$$

for we have

$$\int_0^{2\pi} [\operatorname{sgn} G^*(\theta) - \operatorname{sgn} P(\theta, \alpha)]G^*(\theta)d\theta = 0,$$

and the integrand is non-negative. Hence we have the following theorem.

THEOREM 1. *Being given the function*

$$F(\theta) = \sum_{k=n-s}^{\infty} \{A_k \cos k\theta + B_k \sin k\theta\}, \qquad (s \leqq (n-1)/2),$$

and the positive number L, *we can find* $2m \leqq 2n$ *real numbers* $0 \leqq \theta_1 < \theta_2 < \cdots < \theta_{2m} < 2\pi$ *such that the quadrature formula*

$$\int_0^{2\pi} F(\theta)G(\theta)d\theta = L\left\{ \sum_{i=1}^{m} G(\theta_{2i-1}) - \sum_{i=1}^{m} G(\theta_{2i}) \right\}$$

holds true for every trigonometric polynomial $G(\theta)$ *of order* $\leqq n$; L *must satisfy the inequality* $L \geqq \pi\delta_0/2$, *where* δ_0 *is the greatest root of* (13); $c_{n-k} = (B_{n-k} - iA_{n-k})/(n-k)$, $(k = 0, 1, \cdots, s)$; $m = n - \nu$, *where* $\nu + 1$ *is the multiplicity of* δ_0; *the numbers* $\theta_1, \theta_2, \cdots, \theta_{2m}$ *are the roots of the trigonometric polynomial of order* m

$$
\begin{aligned}
G^*(\theta) &= P(\theta, \alpha) \\
&= \mathcal{R}[q^2(z)z^{n-2s+\nu}] \pm |q(z)|^2\left(1 - \left(\frac{\pi\delta_0}{2L}\right)^2\right)^{1/2},
\end{aligned}
\tag{19}
$$

where the polynomial $q(z)$ *of degree* $s - \nu$ *is to be found from the expansion* (16).

3. *Minima of Polynomials.* Putting

$$1 - \frac{2\alpha}{\pi} = \lambda, \qquad (-1 \leqq \lambda \leqq 1), \tag{20}$$

we obtain the equality

$$\int_0^{2\pi} \operatorname{sgn} G^*(\theta)G(\theta)d\theta = \lambda \int_0^{2\pi} G(\theta)d\theta + \frac{4}{\delta_0}\cos\frac{\pi\lambda}{2}, \tag{21}$$

which is valid for every trigonometric polynomial

$$G(\theta) = \alpha_n + \sum_{k=0}^{n-1}\{\alpha_k \cos (n-k)\theta + \beta_k \sin (n-k)\theta\}$$

satisfying the condition

$$(22) \qquad \omega(G) = \sum_{k=0}^{s} (\alpha_k a_{n-k} + \beta_k b_{n-k}) = 1, \qquad (s \leqq (n-1)/2).$$

Hence follows the inequality

$$(23) \qquad \int_0^{2\pi} | G(\theta) | \, d\theta - \lambda \int_0^{2\pi} G(\theta) d\theta \geqq \frac{4}{\delta_0} \cos \frac{\pi\lambda}{2},$$

where the equality holds true only for

$$(23') \qquad G(\theta) = G^*(\theta) = P(\theta, \alpha), \qquad (\lambda = 1 - 2\alpha/\pi).$$

First put $\lambda = 0$; then we get

$$(24) \qquad \int_0^{2\pi} | G(\theta) | \, d\theta \geqq \frac{4}{\delta_0}.$$

Suppose now that $G(\theta)$ is non-negative; then

$$\int_0^{2\pi} G(\theta) d\theta \geqq \frac{4}{\delta_0} \frac{\cos (\pi\lambda/2)}{1-\lambda}.$$

Supposing that $\lambda \to 1$, we get finally

$$(25) \qquad \int_0^{2\pi} G(\theta) d\theta \geqq \frac{2\pi}{\delta_0};$$

the equality is valid for the polynomial

$$(25') \qquad G(\theta) = P(\theta, 0) = | q(z) |^2 + \Re[q^2(z) z^{n-2s+\nu}].$$

We have thus proved the following theorem.

THEOREM 2. *For every trigonometric polynomial*

$$G(\theta) = \alpha_n + \sum_{k=0}^{n-1}\{\alpha_k \cos (n-k)\theta + \beta_k \sin (n-k)\theta\}$$

satisfying the condition

$$\omega(G) = \sum_{k=0}^{s} \{\alpha_k a_{n-k} + \beta_k b_{n-k}\} = 1, \qquad (s \leqq (n-1)/2),$$

the inequality

$$\int_0^{2\pi} | G(\theta) | \, d\theta \geqq \frac{4}{\delta_0} \tag{26}$$

is valid; the polynomial for which this minimum is actually attained is

$$P\left(\theta, \frac{\pi}{2}\right) = \mathfrak{R}[q^2(z)z^{n-2s+\nu}].$$

If the polynomial is bound to be non-negative, then the minimum will be $\pi/2$ *times* (26) *and this minimum will be attained by the polynomial*

$$P(\theta, 0) = | q(z) |^2 + \mathfrak{R}[q^2(z)z^{n-2s+\nu}],$$

where $q(z)$, δ_0, *and* ν *have the same meaning as in Theorem* 1.

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