TRANSVERSE SEISMIC WAVES ON THE SURFACE OF A SEMI-INFINITE SOLID COMPOSED OF HETEROGENEOUS MATERIAL*

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In his Adams' Prize Essay of 1910, A. E. H. Love† discussed the propagation of transverse seismic waves in a homogeneous superficial layer covering a homogeneous solid differing from the surface layer in rigidity and density. The analysis has been extended by K. Aichi‡ and E. Meissner§ to the case of a heterogeneous material stratified in horizontal layers. Aichi adopts an exponential law for the variation of secondary wave velocity|| with depth and obtains a solution involving Bessel functions. Meissner considers various laws of variation of density and rigidity, and in one case he also obtains a solution involving Bessel functions. In this paper, a solution is given for a heterogeneous material of such a kind that the functions occurring are all of an elementary nature.

Using the same notations as Aichi, but taking the positive direction of the axis of z upwards, the axis of x being in the direction of propagation, we assume that the transverse displacement η is given by the equation

$$\eta = Y(z) \cos (pt + fx), \tag{1}$$

and that the density ρ and rigidity μ of the material depend only on the coordinate z.

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* Presented to the Society, San Francisco Section, October 29, 1927.

† *Some Problems of Geodynamics*, Cambridge University Press, 1911, p. 160.

‡ Proceedings of the Physico-Mathematical Society of Japan, vol. 4 (1922), pp. 137–142.

§ Proceedings of the Second International Congress for Applied Mathematics, Zürich, 1926.

|| This is the quantity c defined by equation (5).

The components of stress across an area perpendicular to the axis of y are $\mu\partial\eta/\partial x$, 0, $\mu\partial\eta/\partial z$, respectively, and so the equation of motion

$$\rho\frac{\partial^2\eta}{\partial t^2} = \frac{\partial}{\partial x}\left(\mu\frac{\partial\eta}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu\frac{\partial\eta}{\partial z}\right)$$

gives Meissner's equation

$$\frac{d}{dz}\left(\mu\frac{dY}{dz}\right) + (\rho p^2 - \mu f^2)Y = 0. \tag{2}$$

When the rigidity μ is constant, this equation reduces to the equation

$$\frac{d^2Y}{dz^2} + \left(\frac{\rho p^2}{\mu} - f^2\right)Y = 0, \tag{3}$$

used by Aichi.

In addition to the differential equation, Y must satisfy the boundary conditions $Y=0$ when $z=-\infty$, $dY/dz=0$ when $z=a$. The first of these conditions expresses simply that there is no deep penetration of the waves while the second condition expresses that there is no tangential stress at the free surface $z=a$.

Taking first the case of a constant value of μ, we write

$$\frac{\rho p^2}{\mu} = 2n^2 \operatorname{sech}^2 nz + k^2, \qquad f^2 = n^2 \operatorname{ctnh}^2 \omega + k^2, \tag{4}$$

where k^2, ω, and n are positive constants when p is given. Our law of variation of ρ thus apparently depends on the frequency of the waves, and is definite only when this frequency is assigned. We shall therefore regard this particular p as a constant, and we shall use a different symbol P when the frequency is varied.

It is easily seen that the differential equation and the first boundary condition are satisfied by the equation

$$Y = e^{nz \operatorname{ctnh}\omega} \operatorname{sech} nz \cosh (nz - \omega).$$

The second boundary condition is satisfied if

$$\sinh^2 \omega \tanh^2 na - \sinh \omega \cosh \omega \tanh na + 1 = 0.$$

Regarding this as an equation for tanh na when ω is given, we find that if cosh $\omega > 2$ there are two real values of tanh na, both of which are positive. This means that at first ρ increases with the depth. Now c, the velocity of propagation of "S" waves, is given by the formula

$$c^2 = \frac{\mu}{\rho}. \tag{5}$$

Hence at first c decreases downwards reaching a minimum value at the plane $z=0$. Below this plane c increases with the depth and gradually approaches a limiting value which is greater than the value at the surface. The existence of a solution in the present case is particularly interesting because in Love's case a solution exists only when c is greater for the substratum than for the upper layer. Let b denote the value of c at the surface and v the velocity of the transverse waves represented by (1); then we have

$$v = \frac{p}{f}, \quad \frac{p^2}{b^2} = 2n^2 \operatorname{sech}^2 na + k^2, \quad f^2 = n^2 \operatorname{ctnh}^2 \omega + k^2 \, ;$$

but

$$2 \operatorname{sech}^2 na = \operatorname{ctnh}^2 \omega \mp \operatorname{ctnh} \omega (\operatorname{ctnh}^2 \omega - 4 \operatorname{csch}^2 \omega)^{1/2} \, ;$$

therefore

$$\frac{p^2}{b^2} - f^2 = \mp\, n^2 \operatorname{ctnh} \omega (\operatorname{ctnh}^2 \omega - 4 \operatorname{csch}^2 \omega)^{1/2}.$$

This equation tells us that v is greater or less than b according as the lower or upper sign is given to the square root. When the upper sign is taken, the amplitude Y begins to increase downwards until it attains a maximum value at a level z given by the equation

$$2 \sinh \omega \, \tanh nz = \cosh \omega - (\cosh^2 \omega - 4)^{1/2} \, ;$$

beyond this level the amplitude decreases steadily to zero. When the lower sign is taken the amplitude decreases steadily to zero from the surface downwards. To discuss the effect of a change of frequency, we consider the propagation of the disturbance

$$\eta = Y(z) \cos (Pt + Fx),$$

the law of variation of density with depth being the same as before. Writing

$$2P^2 = m(m+1)p^2, \quad 2F^2 = m(m+1)k^2 + 2\sigma^2 n^2,$$

where m and σ depend on P but p is constant as before, we find that Y must now satisfy the differential equation

$$\frac{d^2Y}{dz^2} + n^2 Y\left[m(m+1)\operatorname{sech}^2 nz - \sigma^2\right] = 0. \tag{6}$$

A particular solution is given by the hypergeometric function

$$Y = e^{n\sigma z} F(-m, m+1; 1-\sigma; \xi), \tag{7}$$

where

$$2\xi = 1 - \tanh nz.$$

When m is an integer, this function is a polynomial in ξ except for the exponential factor. If σ is positive, Y will satisfy the boundary condition at $z = -\infty$. The boundary condition at $z = a$ gives a relation between σ and the quantity $\tanh na$, which will be denoted by the symbol τ. In the cases $m = 1, 2, 3$, the relations are

$$\sigma^2 - \sigma\tau + \tau^2 = 1,$$

$$\sigma^3 - 3\sigma^2\tau + 6\sigma\tau^2 - 6\tau^3 = 4\sigma - 6\tau,$$

$$\sigma^4 - 6\sigma^3\tau + 21\sigma^2\tau^2 - 45\sigma\tau^3 + 45\tau^4 = 10\sigma^2 - 39\sigma\tau + 54\tau^2 - 9,$$

respectively. Portions of the curves represented by these equations are drawn roughly in Fig. 1, σ and τ being rectangular coordinates.

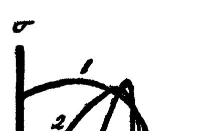

FIG. 1

For large values of m there are several possible values of σ corresponding to a given value of τ; thus when $m=3$ and $\tau=1$, we can have $\sigma=0$, 1, 2, or 3.

For all integral values of m, unity is a possible value of σ corresponding to $\tau=1$, and when τ is very nearly equal to one, there is a value of σ which is very nearly equal to one.

An approximate expression for this value of σ may be derived from equation (7) by differentiating with respect to z and retaining only first powers of $1-\tau$ after z has been put equal to a. The resulting equation,

$$0 = \sigma\left[1 - \frac{m(m+1)}{2(1-\sigma)}(1-\tau)\right] + (1-\tau^2)\left[\frac{m(m+1)}{2(1-\sigma)}\right],$$

gives approximately

$$\sigma = 1 + \frac{m(m+1)}{2}(1-\tau).$$

Using this value of σ, we have approximately

$$\begin{aligned} 2F^2 &= 2n^2 + m(m+1)[k^2 + n^2(1-\tau)] \\ &= 2n^2 + 2\frac{P^2}{p^2}[k^2 + n^2(1-\tau)]. \end{aligned}$$

Now the wave-velocity v and the group-velocity u are given by the formulas,*

$$\frac{1}{v} = \frac{F}{P} = (pP)^{-1}[n^2p^2 + P^2k^2 + P^2n^2(1-\tau)]^{1/2},$$

$$\frac{1}{u} = \frac{dF}{dP} = Pp^{-1}[k^2 + n^2(1-\tau)] \cdot [n^2p^2 + P^2k^2 + P^2n^2(1-\tau)]^{-1/2},$$

$$\frac{u}{v} = 1 + \frac{n^2p^2}{P^2k^2 + P^2n^2(1-\tau)}.$$

Since $1-\tau$ is positive the group-velocity is greater than the

* H. A. Lorentz, *Problems of Modern Physics*, p. 17.

wave-velocity. This result is fundamentally different* from that obtained in Love's case of two adjacent homogeneous layers, and also from that obtained by Meissner for the case of a heterogeneous medium in which the density and rigidity vary continuously and monotonically. Some doubt may be felt with regard to the validity of our approximation and the differentiation with respect to P and it is hoped that the result will be checked by some independent method.

Turning now to the case in which the rigidity varies with the depth, we put

$$Z = \mu^{1/2}Y\ ;$$

the differential equation satisfied by Z is then

$$\frac{d^2Z}{dz^2} + Z\left[\frac{oP^2}{\mu} - F^2 + \frac{1}{2\mu}\frac{d^2\mu}{dz^2} - \frac{1}{4\mu^2}\left(\frac{d\mu}{dz}\right)^2\right].$$

This differential equation may be reduced to the type already considered in the following cases:

(i) $\mu = \nu e^{-2hz}, \qquad \dfrac{\rho p^2}{\mu} = 2\omega^2 \operatorname{sech}^2 nz + k^2,$

$$F^2 = \frac{P^2k^2}{p^2} + \sigma^2 n^2 + h^2,$$

(ii) $\mu = \nu \cosh nz, \qquad \dfrac{\rho p^2}{\mu} = 2\omega^2 \operatorname{sech}^2 nz + k^2,$

$$F^2 = \frac{P^2k^2}{p^2} + \frac{n^2}{4} + \sigma^2 n^2,$$

where ω is an arbitrary constant.

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* The group velocity is generally less than the wave-velocity. This is shown very clearly in Meissner's diagrams but Meissner's definition of the group-velocity is different from ours.