THE FUNDAMENTAL REGION FOR A FUCHSIAN GROUP*

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1. *Introduction.* The present paper is an attempt to lay the groundwork of the theory of Fuchsian groups by basing the treatment on concepts of a very simple sort. The fundamental region to which we are led is not new. It is given by the Fricke-Klein method† under certain circumstances and is identical with that given by Hutchinson‡ in an important paper. However, we make use neither of non-euclidean geometry nor of quadratic forms, and we are able to derive the major results of the theory of Fuchsian groups in an unexpectedly simple manner.

2. *The Group.* Given a group of linear transformations with an invariant circle or straight line K', the interior of K' (or the half-plane on one side of K') being transformed into itself by each transformation of the group. We shall assume that there exists a point A, not on K', such that there are no points congruent to A in a sufficiently small neighborhood of A.

Let G be a linear transformation carrying K' into the unit circle K with center at the origin and carrying A to the origin. Let S be any transformation of the group; then the set of transformations

$$T = GSG^{-1}$$

is a group with K as principal circle.§ Configurations

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† Fricke-Klein, *Vorlesungen über die Theorie der automorphen Funktionen*, vol. I, Chap. II.

‡ J. I. Hutchinson, *A method for constructing the fundamental region of a discontinuous group of linear transformations*, TRANSACTIONS OF THIS SOCIETY, vol. 8 (1907), pp. 261–269.

§ We use this order to mean the transformation G^{-1}, followed by S, followed by G. That is, writing $z' = S(z)$, $z' = G(z)$, etc., as the combining transformations, the new transformation is $z' = T(z) = G\{S[G^{-1}(z)]\}$.

which are congruent by transformations of the new group are carried by G^{-1} into configurations which are congruent by transformations of the original group. It will suffice, then, to find a fundamental region for the new group.

The condition that

$$z' = T(z) = \frac{az+b}{cz+d}$$

leave the unit circle unchanged is that $b = \bar{c}$, $d = \bar{a}$, where bars indicate conjugate imaginaries. Then

$$T = \frac{az+\bar{c}}{cz+\bar{a}}.$$

Since the origin, 0, must transform into an interior point, and $T(0) = \bar{c}/\bar{a}$, we must have $|c| < |a|$. Hence the determinant $a\bar{a} - c\bar{c}$, which is real, must be positive. We shall insert such a positive factor in numerator and denominator that

$$a\bar{a} - c\bar{c} = 1.$$

There is no point congruent to 0 in a suitably small neighborhood of 0. In particular, 0 is not a fixed point for any transformation; hence $c \neq 0$, unless T be the identical transformation.

3. *Two Locus Problems.* Excluding the identical transformation, we shall solve the following two locus problems for the transformation T.

I. *Find the locus of a point in the neighborhood of which lengths and areas are unchanged in magnitude.*

Infinitesimal lengths are multiplied by $|T'(z)|$ and infinitesimal areas are multiplied by $|T'(z)|^2$. Since

$$T'(z) = \frac{1}{(cz+\bar{a})^2},$$

we have as the required locus the circle

$$C: \quad |cz+\bar{a}| = 1,$$

or

$$|z + \bar{a}/c| = 1/|c|.$$

This circle, which will be called the C-circle of the transformation T, has its center at the point $-\bar{a}/c$; its radius is $1/|c|$.

Writing the relation $a\bar{a}-c\bar{c}=1$ in the form

$$1+\frac{1}{|c|^2}=\left|\frac{\bar{a}}{c}\right|^2,$$

we see that the sum of the squares of the radii of K and C is equal to the square of the distance between their centers. Hence C is orthogonal to K. It follows that 0 is outside C.

If z is within C then

$$|z+\bar{a}/c|<1/|c|, \qquad |cz+\bar{a}|<1, \qquad |T'(z)|>1,$$

and lengths and areas in the neighborhood of z are increased in magnitude when transformed by T. Similarly, if z is outside C lengths and areas are decreased in magnitude when transformed by T.

II. *Find the locus of a point whose distance from* 0 *is unchanged.*

The required locus is

$$|z|=\left|\frac{az+\bar{c}}{cz+\bar{a}}\right|$$

or

$$|cz^2+\bar{a}z|=|az+\bar{c}|,$$

or

$$(cz^2+\bar{a}z)(\bar{c}\bar{z}^2+a\bar{z})=(az+\bar{c})(\bar{a}\bar{z}+c).$$

On expanding and making use of the relation $c\bar{c}=a\bar{a}-1$ this factors into

$$\text{(a)} \qquad [z\bar{z}-1][(cz+\bar{a})(\bar{c}\bar{z}+a)-1]=0,$$

whence

$$|z|=1, \quad \text{or} \quad |cz+\bar{a}|=1.$$

The complete locus, then, consists of the two circles C and K.

If z is inside both C and K or outside both, the first member of (a) is greater than 0, and we find, on retracing our steps, that

$$|z|>\left|\frac{az+\bar{c}}{cz+\bar{a}}\right|.$$

The transform of z is nearer 0 than z is. Similarly, if z is inside one circle and outside the other the transform is farther from 0 than z is.

4. *Geometric Interpretation of T.* The inverse of T,

$$T^{-1} = \frac{-\bar{a}z+\bar{c}}{cz-a},$$

has the C-circle

$$C': \qquad |z-a/c| = 1/|c|.$$

It is clear that T carries C into C'. For, T carries C into a circle C_0 without alteration of lengths; then T^{-1} transforms C_0 without alteration of lengths; whence C_0 coincides with C'.

The region exterior to both C and C' is transformed by T into the interior of C' and by T^{-1} into the interior of C. Let z be a point of the region, and let z be carried by T and T^{-1} into z' and z'' respectively. In both cases there is diminution of lengths and areas near z since z is exterior to both C-circles. Then the inverses, T^{-1} and T, carry z' and z'' respectively back to z with increase of lengths and areas. Hence z' is in C' and z'' is in C.

Let A, B and A', B' be the intersections of C and C' with K, the points being so designated that motion around C from A to B inside K is counter-clockwise about C, and motion from B' to A' around C' in K is counter-clockwise about C'. Now, on applying T, the interior arc AB of C is carried without alteration of length into either $A'B'$ or $B'A'$. The latter is impossible, for it is equivalent to a suitable rotation with 0 as fixed point, which is contrary to hypothesis. Hence A is transformed into A' and B into B'.

We can now give simple geometric interpretations of T. Let L be the perpendicular bisector of the line joining the centers of C and C'. L passes through 0. (In the special case that C and C' coincide let L be the line joining 0 to the center of C.) Either of the following pairs of inversions transforms the points of C exactly as T does and hence is identical with it:

(1) A reflection in L followed by an inversion in C';

(2) An inversion in C followed by a reflection in L.

We can show from these inversions that T is hyperbolic, elliptic, or parabolic according as C and C' are exterior to one another, intersect, or are tangent. The fixed points of the transformation are easily found geometrically.

5. *The Arrangement of the C-Circles.* We shall show first that there is an upper bound of the radii of the C-circles defined by the transformations of the group. There exists, by hypothesis, a circle of radius $\varepsilon < 1$ with 0 as center having no point congruent to 0 in its interior. We have then for any transformation of the group

$$|T(0)| = \left|\frac{\bar{c}}{\bar{a}}\right| \geqq \varepsilon, \text{ whence } \left|-\frac{\bar{a}}{c}\right| \leqq \frac{1}{\varepsilon};$$

that is, the distance from 0 to the center of C is not greater than $1/\varepsilon$. Since 0 is outside C it follows that the radius of C is less than $1/\varepsilon$.

Let

$$T_1 = \frac{\alpha z + \bar{\gamma}}{\gamma z + \bar{\alpha}}$$

be a second transformation of the group. Designate by C_1, C_1' the C-circles of T_1 and T_1^{-1}. Thus C_1 is $|z + \bar{\alpha}/\gamma| = 1/|\gamma|$. Let us now make the transformation

$$TT_1^{-1} = \frac{(-a\bar{\alpha} + \bar{c}\gamma)z + a\bar{\gamma} - \bar{c}\alpha}{(-c\bar{\alpha} + \bar{a}\gamma)z + c\bar{\gamma} - \bar{a}\alpha},$$

which, since $T_1 \neq T$, is not the identical transformation. Designating the radii of C, C_1, and the C-circle of TT_1^{-1} by r, r_1, r_2, respectively, we have

$$r_2 = \frac{1}{|-c\bar{\alpha} + \bar{a}\gamma|} = \frac{1}{|c\gamma| \cdot \left|-\frac{\bar{\alpha}}{\gamma} - \left(-\frac{\bar{a}}{c}\right)\right|} = \frac{rr_1}{d},$$

where d is the distance between the centers of C and C_1. Since $r_2 < 1/\varepsilon$ we have

$$d = \frac{rr_1}{r_2} > \varepsilon rr_1.$$

Consider now all C-circles whose radii equal or exceed some positive number k. The distance between the centers of any two of the circles satisfies the inequality

$$d > \epsilon k^2.$$

Since the centers lie in the finite region bounded by K and by the circle of radius $1/\epsilon$ with center at 0 it follows that the number of such circles is finite.

It follows from the result just found that any closed region lying entirely within K, for example a circle with 0 as center and radius less than 1, is exterior to all but a finite number of C-circles.

Another consequence is that the transformations of the group are denumerable, since the radii of the corresponding C-circles are denumerable.

6. *The Fundamental region.* Let R be the region within K which lies outside all C-circles formed for the transformations of the group. The region is connected, since any of its points can be joined to 0 by a straight line segment which does not cross the boundary. We shall show that R is a fundamental region for the group; that is, (1) that no two interior points of R are congruent, and (2) that no region adjacent to R and lying in K can be added to R without the inclusion of points congruent to points of R.

The proof of the first property is immediate. The transform of any interior point of R by any transformation of the group, the identical transformation excepted, lies within some C-circle and hence is exterior to R.

To establish the second property we shall show first that if P, a point of C, the C-circle of some transformation T of the group, lies on the boundary of R, then P', the transform of P by T, also lies on the boundary of R. P' lies on C', the C-circle of T^{-1}.

Suppose P' does not lie on the boundary of R. Then P' lies within the C-circle of some transformation T_1 of the group. Consider the transformation $T_1 T$. By the transformation T lengths in the neighborhood of P are carried without alteration of magnitude into the neighborhood of P'. By T_1 lengths in the neighborhood of P' are magnified. Hence T_1T magnifies lengths in the neighborhood of P; consequently P is within the C-circle of T_1T. This is contrary to hypothesis; hence P' is on the boundary of R.

It follows from the preceding that if an arc ab of C forms part of the boundary of R the congruent arc $a'b'$ of C' is a part of the boundary. The transformation T^{-1} carries R into a region abutting R along ab. Any region adjacent to R abuts along some C-circle and contains points congruent by a suitable transformation to points of R. The second property is thus established.

The following properties of the region R are consequences of the preceding analysis:

1. R is bounded by arcs of circles orthogonal to K. The number of bounding arcs in a circle $|z| = r < 1$ is finite.

2. The bounding arcs are arranged in congruent pairs. Two congruent arcs of the boundary are equal in length, and congruent points thereof are equidistant from the center of K.

3. The vertices of a cycle (congruent vertices) lie on a circle concentric with K, since all are equidistant from the center of K. If the vertices of a cycle lie within K their number is finite.

4. R is the fundamental region of maximum area. For, a different fundamental region must contain points congruent to all points of R, and a shift of any part of R to a congruent position effects a diminution of area.

We shall now prove that *R and the regions congruent to it fill up without overlapping the whole interior of K.*

Suppose R_i and R_j, the transforms of R by T_i and T_j, overlap. Let z_1, z_2, z_3 be three points common to R_i and R_j. These are the transforms by T_i of three points z_1', z_2', z_3' of R, and the transforms by T_j of three points z_1'', z_2'', z_3'' of R. If $z_1' = z_1''$, $z_2' = z_2''$, $z_3' = z_3''$ then T_i and T_j are the same transformation, since they transform three points in the same way, and R_i and R_j coincide. Otherwise the points of one pair, say z_1' and z_1'', are unequal. Being both congruent to z_1 they are congruent, which is impossible.

Let K_r be the circle $|z| = r < 1$. Let $a_1b_1, a_1'b_1', \cdots, a_nb_n, a_n'b_n'$ be the sides of R lying wholly or in part within K_r; and let T_i, $i = 1, 2, \cdots, n$, be the transformation carrying a_ib_i into $a_i'b_i'$. The transforms of all remaining sides of R are exterior to K_r, since the distance from 0 of any point on such a side is not decreased by any transformation of the group.

By applying $T_1, \cdots, T_n$ and their inverses we get regions congruent to R abutting on R along the sides $a_1b_1, \cdots, a_n'b_n'$. The sides of the new regions which lie in K_r are congruent to the sides just mentioned. By combinations of $T_1, \cdots, T_n$ and their inverses we can adjoin further regions along sides of these new regions, provided the sides lie in K_r; and the process can be continued as long as there are any free sides in K_r.

This process will end in a finite number of steps; for, each transformation carries R into the interior of a particular C-circle, and there is but a finite number of C-circles intersecting K_r. Hence K_r is covered by a finite number of regions. Since r may be chosen as near 1 as we like, it follows that the whole interior of K is covered.

We note from the preceding that all transformations of the group are formed by combinations of the transformations connecting congruent sides of R. These are therefore called generating transformations of the group. A further interesting fact is that a transformation which carries R into a region lying wholly or in part in a circle K_r concentric with K is a combination of those generating transformations only whose C-circles intersect K_r.

7. *An Important Special Case.* The sides of R may be finite or infinite in number. There are certain groups in which a fundamental region not extending to the principal circle is known to exist; for example, some of the groups

arising in connection with the uniformization problem.* For this case we have the following proposition.

If there exists a fundamental region F lying within $K_r(|z| = r < 1)$, then R lies within K_r, and the number of sides and of generating transformations is finite.

R and a finite number of its transforms, $R_1, \cdots, R_m$, will cover F completely. Carry the portion of F lying in each R_i into R by means of the transformation which carries R_i into R. The totality of these transforms of parts of F, together with the portion of F originally in R, fill up R completely. If this were not so we could construct a region D adjacent to one of these transformed regions and containing no points congruent to points of F. On carrying D back to the boundary of F we should have a region abutting on F and containing no points congruent to points of F, which is contrary to hypothesis.

Finally R is in K_r, for on transforming the parts of F into R the distance of no point from 0 is increased.

Since only a finite number of C-circles intersect K_r, it follows that R has a finite number of sides and that the number of generating transformations is finite.

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* See Osgood, *Lehrbuch der Funktionentheorie*, vol. I, 2d ed., pp. 721–25.