ON SINGULAR POINTS OF LINEAR DIFFERENTIAL EQUATIONS WITH REAL COEFFICIENTS.*

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LET us consider the equation

$$(1) \qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + p_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + p_n y = 0,$$

in which the coefficients $p_1, p_2, \cdots, p_n$ are throughout a certain interval $a < x < b$ continuous real (but not necessarily analytic) functions of the real variable x. By a solution of (1) we shall understand any function of x which together with its first $n-1$ derivatives is single valued and continuous throughout the interval $a < x < b$ and at every point of this interval satisfies (1). It is well known that there is one and only one solution of (1) which at an arbitrarily chosen point of the interval in question has together with its first $n-1$ derivatives arbitrarily chosen values. The object of the present paper is to consider the behavior of these solutions as we approach one end of the interval. It will clearly be sufficient if we confine our attention to the point a.

The simplest case would be that in which all the coefficients $p_1, p_2, \cdots, p_n$ approach finite limits as x approaches a. Then would come the case in which, although this is not true, none of these coefficients become infinite as x approaches a. Without considering separately these possibilities we will go on at once to a more general case which includes them as special cases.

We will say of a function $f(x)$ that it is integrable up to the point a if $\int_c^a f(x)dx$ (where $a < c < b$) converges; *i. e.*, if

*The only investigations with which I am acquainted concerning the singular points of linear differential equations whose coefficients are not assumed to be analytic, are contained in two papers by Kneser, *Crelle*, vols. 116, 117. These papers deal with a certain class of *irregular* points, to use the terminology suggested in the present paper. The singular point in question is taken at infinity.

$\int_c^{a+\epsilon} f(x)dx$ approaches a finite limit as ϵ approaches zero. We can now state the following theorem :

I. *If the absolute values* $|p_i|$ $(i = 1, 2, \cdots, n)$ *of all the coefficients of* (1) *are integrable up to the point* a, *there exists one and only one solution of* (1) *such that it and its first* $n - 1$ *derivatives approach arbitrarily chosen finite limits* $\alpha, \alpha', \alpha'', \cdots, \alpha^{(n-1)}$ *as* x *approaches* a. *This solution may be written in the form*

$$\alpha + \alpha'(x - a) + \frac{\alpha''}{2!}(x - a)^2 + \cdots$$
$$+ \frac{\alpha^{(n-1)}}{(n-1)!}(x - a)^{n-1} + (x - a)^{n-1}\varphi(x)$$

where $\varphi(x)$ *is single valued and continuous when* $a \leqq x < b$ *and* $\varphi(a) = 0$.

As this proposition is merely a special case of theorem III we omit its proof here.

If we assign to the n quantities $\alpha^{(i)}(i = 0, 1, \cdots, n - 1)$ in succession the values $\alpha_1^{(i)}, \alpha_2^{(i)}, \cdots, \alpha_n^{(i)}$ we determine by theorem I in succession n solutions of (1) which we will call $y_1, y_2, \cdots, y_n$. Now it is readily seen (just as in the case in which a is a non-singular point*) that the necessary and sufficient condition that $y_1, y_2, \cdots, y_n$ are linearly independent is that the determinant

$$\triangle = \begin{vmatrix} \alpha_1 & \alpha_1' & \cdots & \alpha_1^{(n-1)} \\ \alpha_2 & \alpha_2' & \cdots & \alpha_2^{(n-1)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_n & \alpha_n' & \cdots & \alpha_n^{(n-1)} \end{vmatrix}$$

is different from zero. It is, therefore, possible to give to the n^2 quantities α such values that $y_1, y_2, \cdots, y_n$ are linearly independent of each other, so that any solution of (1) can be expressed in the form

$$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n.$$

From this we infer at once the following theorem :

II. *If all the functions* $|p_i|$ $(i = 1, 2, \cdots, n)$ *are integrable up to* a, *every solution of* (1) *approaches, as does also each of its first* $n - 1$ *derivatives, a finite limit as* x *approaches* a.

It will be seen that the class of singular points we have just considered has all the characteristics of non-singular

* Cf. for instance Schlesinger's Handbuch, vol. 1, part 1, chap. 2.

points. We pass now to another class of singular points and we will begin with the following theorem:

III. *If the functions*

$$|p_1|, |p_2|, \cdots, |p_k|, (x-a)|p_{k+1}|, (x-a)^2|p_{k+2}|, \\ \cdots, (x-a)^{n-k}|p_n|,$$

are integrable up to the point a, there exists one and only one solution of (1) which, together with its first $n-k-1$ derivatives, approaches the limit zero as x approaches a, while the k succeeding derivatives approach arbitrarily assigned finite limits

$$\alpha^{(n-k)}, \alpha^{(n-k+1)}, \cdots, \alpha^{(n-1)}.$$

This solution may be written in the form

$$\frac{\alpha^{(n-k)}}{(n-k)!}(x-a)^{n-k} + \cdots + \frac{\alpha^{(n-1)}}{(n-1)!}(x-a)^{n-1} \\ + (x-a)^{n-1}\varphi(x)$$

where $\varphi(x)$ is single valued and continuous when $a \leqq x < b$ and $\varphi(a) = 0$.

This reduces to theorem I when $k = n$.

In proving this theorem it will clearly be sufficient if we confine our attention to any portion ac, however small, of the interval ab. In order to facilitate the proof we will then take c so near to a that

1. $c - a < 1$; and

2. $M < 1$, where M is the greatest value which the function

$$F(x) = \int_a^x [|p_1| + \cdots + |p_k| + (x-a)|p_{k+1}| \\ + \cdots + (x-a)^{n-k}|p_n|]\, dx$$

takes on when $a \leqq x \leqq c$. That this is possible is clear from the fact that $F(x)$ is continuous and that $F(a) = 0$.

In order to prove our theorem let us use the method of successive approximations. We start with the function

$$Y_0 = \frac{\alpha^{(n-k)}}{(n-k)!}(x-a)^{n-k} + \cdots + \frac{\alpha^{(n-1)}}{(n-1)!}(x-a)^{n-1}$$

and compute the functions $Y_1, Y_2, \cdots$ by means of the formulæ

$$Y_m^{(n-1)} = -\int_a^x [p_1 Y_{m-1}^{(n-1)} + p_2 Y_{m-1}^{(n-2)} + \cdots + p_n Y_{m-1}]\, dx,$$

$$(2) \quad Y_m^{(n-2)} = \int_a^x Y_m^{(n-1)}\, dx, \quad Y_m^{(n-3)} = \int_a^x Y_m^{(n-2)}\, dx,$$

$$\cdots\cdots, \quad Y_m = \int_a^x Y_m'\, dx.$$

We must first of all prove that these integrals converge. In order to do this let us introduce a quantity C so chosen that when $a \leqq x \leqq c$

$$|Y_0| < C(x-a)^{n-k}, \quad |Y_0'| < C(x-a)^{n-k-1}, \cdots\cdots,$$
$$|Y_0^{(n-k)}| < C, \quad |Y_0^{(n-k+1)}| < C, \cdots\cdots, \quad |Y_0^{(n-1)}| < C.$$

We will now prove the convergence of the integrals (2) and at the same time show that they satisfy the following inequalities

$$|Y_m| < (x-a)^{n-k} CM^m, \quad |Y_m'| < (x-a)^{n-k-1} CM^m,$$

$$(3) \quad \cdots, |Y_m^{(n-k-1)}| < (x-a) CM^m, \quad |Y_m^{(n-k)}| < CM^m,$$

$$\cdots, |Y_m^{(n-1)}| < CM^m.$$

These inequalities hold when $m = 0$ and the integrals (2) converge when $m = 1$. Now let m_1 be a positive integer and assume that the integrals (2) converge when $m \leqq m_1$ and that the inequalities (3) hold when $m \leqq m_1 - 1$. From this assumption it follows at once, when we remember that at every point of ac we have $x - a < 1$, that the inequalities (3) hold when $m = m_1$ and that therefore the integrals (2) converge when $m = m_1 + 1$. Thus the convergence of (2) and the truth of (3) are established by the method of mathematical induction.

We can now prove at once by means of the inequalities (3) that the series

$$(4) \qquad Y_0 + Y_1 + Y_2 + \cdots$$

converges throughout the interval ac and represents the desired solution of (1). As the proof to be used does not differ from that always used in such cases it seems unnecessary to give it here.

In order to prove that the series (4) really gives a function of the form mentioned in the closing lines of theorem III,* it will clearly be sufficient to prove that each of the

* This also follows at once from the formula for the remainder in Taylor's theorem.

functions Y_i $(i = 1, 2, \cdots)$ is of the form $(x-a)^{n-1}\varphi_i(x)$ where $\varphi_i(x)$ is continuous throughout the interval $a \leqq x \leqq c$ and $\varphi_i(a) = 0$. This follows at once when we remember that $Y_i^{(n-1)}$ is continuous throughout the interval ac and vanishes when $x = a$ and that Y_i is obtained from $Y_i^{(n-1)}$ by means of $n - 1$ integrations from a to x.*

In order then to establish theorem III completely, it remains merely to show that (1) cannot have two different solutions satisfying the conditions there stated. This, as in other similar cases, will be accomplished if we can prove that a solution of (1) which together with its first $n - 1$ derivatives approaches the limit zero as x approaches a must be identically zero. This can be proved by the method of Lindelöf (*Liouville's Journal*, 1894, p. 118). For let y be such a solution, and consider the functions

$$(5)\qquad \begin{gathered} y^{(n-1)},\ y^{(n-2)}, \cdots, y^{(n-k)},\ (x-a)^{-1}y^{(n-k-1)}, \\ (x-a)^{-2}y^{(n-k-2)}, \cdots (x-a)^{k-n}y. \end{gathered}$$

It is readily seen that all these functions approach zero as x approaches a. We will denote by m the greatest value which the absolute value of any of these functions takes on in the interval ac. If we can prove that $m = 0$, it will evidently follow that y is identically zero, as was to be proved. Now we have

$$y^{(n-1)} = -\int_a^x [p_1 y^{(n-1)} + p_2 y^{(n-2)} + \cdots + p_n y]\,dx,$$

and therefore denoting by $F(x)$ the same function as is thus denoted on p. 277, and by M, its greatest value in ac

$$\begin{gathered} |y^{(n-1)}| \leqq mF(x) \leqq mM, \quad |y^{(n-2)}| \leqq mM\,(x-a) \leqq mM, \\ \cdots, |y^{(n-k)}| \leqq mM, \quad |y^{(n-k-1)}| \leqq mM\,(x-a), \\ |y^{(m-k-2)}| \leqq mM\,(x-a)^2, \cdots, \quad |y| \leqq mM\,(x-a)^{n-k}. \end{gathered}$$

From this it follows that the absolute values of none of the functions (5) can exceed the value mM in the interval $a \leqq x \leqq c$, while by hypothesis one of them, at least, has the value m at a point of this interval. Now, since we know

* We have here merely to use the easily established formula

$$\lim_{x=a}\left[\frac{1}{(x-a)^k}\int_a^x (x-a)^{k-1}\psi(x)\,dx\right] = 0$$

where k is a positive integer while $\psi(x)$ is continuous throughout ac and $\psi(a) = 0$.

that $M < 1$, it is seen at once that we have a contradiction, unless $m = 0$.

We will now confine our attention to differential equations of the second order

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0, \tag{6}$$

and we will assume that

$$p = \frac{\mu}{x-a} + p_1(x), \quad q = \frac{\nu}{(x-a)^2} + q_1(x)$$

where μ and ν are real constants and p_1 and q_1 are continuous throughout the interval $a < x < b$ and $|p_1|$ and $(x-a)|q_1|$ are integrable up to a. When these conditions are satisfied we will speak of a as a *regular point* of (1) from analogy with the case in which p and q are single valued analytic functions of the complex variable x. We may note that the case here considered includes as a special case (viz. $\mu = \nu = 0$) the case considered in theorem III in so far as that theorem refers to equations of the second order. Let us now form the equation

$$\rho(\rho - 1) + \mu\rho + \nu = 0$$

which (from analogy as before) we will call the *indicial equation* of the point a. The roots of this equation we will denote by $\varkappa'$ and $\varkappa''$ and we will call them the *exponents* of the point a. The following theorems may now be established by computation so simple that it is unnecessary to give it here:

IV. *If a is a regular point of the equation* (6) *with exponents $\varkappa'$ and $\varkappa''$, and if we introduce a new dependent variable by means of the relation $y = (x-a)^c\bar{y}$, then $\bar{y}$ will satisfy a homogeneous linear differential equation of the second order for which a is a regular point with exponents $\varkappa' - c$ and $\varkappa'' - c$.*

V. *If a is a regular point of the equation* (6) *with exponents $\varkappa'$ and $\varkappa''$, and if we introduce the new independent variable $\bar{x}$ by means of the relation $x - a = \bar{x}^c$ where c is a real positive constant, the new linear differential equation of the second order thus obtained will have the point $\bar{x} = 0$ as a regular point with exponents $c\varkappa'$ and $c\varkappa''$.*

We shall now leave out of consideration the case in which the exponents of a regular point a of (6) are conjugate imaginaries and also the case in which they are real and equal. There remains then merely the case in which the exponents $\varkappa'$ and $\varkappa''$ are real and unequal. Let $\varkappa'$ be the larger of these two exponents so that $\varkappa = \varkappa' - \varkappa''$ is positive. Let us then introduce into (6) the new dependent variable $\bar{y}$ by means of the relation $y = (x - a)^{\kappa''}\bar{y}$. The equation $(\bar{6})$ thus obtained has by theorem IV the point $x = a$ as a regular point with exponents 0 and $\varkappa$. Let us now introduce into $(\bar{6})$ the new dependent variable $\bar{x}$ by means of the relation $x - a = \bar{x}^{1/\kappa}$. The equation $(\bar{\bar{6}})$ thus obtained has the point $\bar{x} = 0$ as a regular point with the exponents 0 and 1. It must therefore have the form

$$(\bar{\bar{6}}) \qquad \frac{d^2\bar{y}}{d\bar{x}^2} + \bar{p}\frac{d\bar{y}}{d\bar{x}} + \bar{q}\,\bar{y} = 0$$

where $\bar{p}$ and $\bar{x}\,\bar{q}$ are integrable up to the point 0.

Now we know from theorem III that $(\bar{\bar{6}})$ has a solution of the form

$$\bar{y}_1 = \bar{x}\,\bar{E}_1\,(\bar{x})$$

where $\bar{E}_1\,(\bar{x})$ approaches a finite limit different from zero as $\bar{x}$ approaches zero. By using the formula

$$\bar{y}_2 = \bar{y}_1 \int \frac{e^{-\int \bar{p}\,d\bar{x}}}{\bar{y}_1^{\,2}}\,d\bar{x}$$

we readily see that every solution of $(\bar{\bar{6}})$ which is linearly independent of $\bar{y}_1$ approaches a finite limit different from zero as x approaches zero.* Let $\bar{y}_2 = \bar{E}_2\,(\bar{x})$ be such a solution. Going back now to equation (6) we get at once the following theorem:

VI. *If a is a regular point of* (6) *with real unequal exponents $\varkappa'$ and $\varkappa''$, then* (6) *has two linearly independent solutions of the form*

$$y_1 = (x - a)^{\kappa'}E_1(x), \quad y_2 = (x - a)^{\kappa''}E_2(x)$$

where the functions E_1 and E_2 are continuous throughout the interval $a \leqq x < b$ and where $E_1(a)$ and $E_2(a)$ are different from zero.

This theorem fully justifies us in applying the term exponent to the quantities $\varkappa'$ and $\varkappa''$.

In conclusion I will mention that this paper may be regarded as an amplification and generalization of some of the results contained in my third paper on "The Theorems of Oscillation of Sturm and Klein," cf. in particular pp. 25–32 of the present volume of the BULLETIN.

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* For further details cf. p. 32 of the present volume of the BULLETIN.