NOTE ON HYPERELLIPTIC INTEGRALS.

BY PROFESSOR ALEXANDER S. CHESSIN.

(Read before the American Mathematical Society at the Meeting of October 30, 1897.)

LET X_r denote a polynomial in x of degree r; $P_m(x)$, $Q_n(x)$, $\cdots$ polynomials in x of degrees m, n, $\cdots$. We know that the integration of

$$\int f(x, \sqrt{X_r})\, dx,$$

where $f(x, \sqrt{X_r})$ is a rational function of x and $\sqrt{X_r}$, is reduced to the integration of

$$\int \frac{R(x)\,dx}{\sqrt{X_r}} \tag{1}$$

where $R(x)$ is a rational function of x. This note is intended to give a practical rule for the integration of (1).

Let

$$R(x) = \frac{P_m(x)}{\prod\limits_{x=1}^{k=s} (x - a_k)^{n_k}} \tag{2}$$

We may assume that $P_m(x)$ has no factor $x - a_k$, otherwise the common factors may be cancelled. We also assume that all the factors of X_r are simple, for double factors could be taken outside the radical.

Suppose first that none of the a_k are roots of $X_r = 0$. Then we have the equality

$$\int \frac{P_m(x)\,dx}{\prod\limits_{k=1}^{k=s} (x - a_k)^{n_k} \sqrt{X_r}} = \frac{Q_p(x)\sqrt{X_r}}{\prod\limits_{k=1}^{k=s} (x - a_k)^{n-1}} \tag{3}$$

$$+ \sum_{k=0}^{k=r-2} \lambda_k \int \frac{x^k dx}{\sqrt{X_r}} + \sum_{k=1}^{k=s} \mu_k \int \frac{dx}{(x - a_k)\sqrt{X_r}}$$

where

$$(4)\qquad \begin{cases} p = m - s - r + 1 & \text{if} \quad m > \sum_{k=1}^{k=s} n_k + r - 2 \\ p = \sum n_k - s - 1 & \text{if} \quad m \leqq \sum_{k=1}^{k=s} n_k + r - 2 \end{cases}$$

In fact after differentiating formula (3) and multiplying the result by $\sqrt{X_r} \prod_{k=1}^{k=s} (x - a_k)^{n_k}$ we have a polynomial of the degree m in the left hand side, and a polynomial of the degree $p + s + r - 1$

or
$$\sum_{k=1}^{k=s} n_k + r - 2$$
on the right hand side according as
$$m > \sum_{k=1}^{k=s} n_k + r - 2$$
or
$$m \leqq \sum_{k=1}^{k=s} n_k + r - 2$$

In the first case by taking for p the value $m - s - r + 1$ we obtain a polynomial of degree m on the right hand side; in the second case it will be a polynomial of degree $\leqq m$. In either case we have as many equations to find the indeterminate coefficients of $Q_p(x)$ and the λ_k and μ_k as there are coefficients, namely $m + 1$ in the first case and $\sum_{k=1}^{k=s} n_k + r - 1$ in the second.

Suppose now that $a_1, a_2, \cdots, a_\rho$ are roots of $X_r = 0$. Then we have the equality

$$(5)\quad \int \frac{P_m(x)\,dx}{\prod_{k=1}^{k=s} (x - a_k)^{n_k} \sqrt{X_r}} = \frac{Q_q(x)\sqrt{X_r}}{(x - a_1) \cdots (x - a_\rho) \prod_{k=1}^{k=s} x - a_k)^{n_k - 1}} + \sum_{k=0}^{k=r-2} \lambda_k \int \frac{x^k dx}{\sqrt{X_r}} + \sum_{k=\rho+1}^{k=s} \mu_k \int \frac{dx}{(x - a_k)\sqrt{X_r}}$$

where

$$(6)\qquad \begin{cases} q = m - s - r + \rho + 1 \text{ if } m > \sum_{k=1}^{k=s} n_k + r - 2 \\ q = \sum_{k=1}^{k=s} n_k - s + \rho - 1 \text{ if } m \leqq \sum_{k=1}^{k=s} n_k + r - 1 \end{cases}$$

In fact after differentiating formula (5) and multiplying the result by $\sqrt{X_r} \prod_{k=1}^{k=s} (x - a_k)^{n_k}$ we obtain a polynomial of degree m on the left hand side, and a polynomial of the degree $q+s+r-\rho-1$ or $\sum_{k=1}^{k=s} n_k + r - 2$ on the right hand side according as $\sum_{k=1}^{k=s} n_k + r - 2 < m$ or $\geqq m$. In the first case by taking for q the value $m-s-r+\rho+1$ we obtain a polynominal of degree m on the right hand side; in the second case the degree will be $\geqq m$. In either case the number of equations to determine the coefficients of $Q_q(x)$ and the λ_k and μ_k is equal to the number of these coefficients, namely $m+1$ in the first case and $\sum_{k=1}^{k=s} n_k + r - 1$ in the second.

That formula (3) does not hold in general when $X_r(a_k)=0$ can be easily seen by substituting for x the value a_k in the result of the differentiation of (3). In fact it will be found that unless $n_k = 1$ formula (3) involves the equality $P_m(a_k) = 0$ which is contrary to our assumption that $P_m(x)$ has no factor $x - a_k$. *But formula (3) still holds if a_k is a root of $X_r = 0$ provided $n_k = 1$.* And indeed, in this case formulas (3) and (5) can be brought to the same form if we remember that the integral

$$\int \frac{dx}{(x-a_k)\sqrt{X_r}}$$

is reducible to the integral

$$\int \frac{xdx}{\sqrt{X_r}}$$

when a_k is a root of $X_r = 0$.

Remark. In order to determine the algebraic part of formulas (3) and (5) it is not necessary to break up the denominator of $R(x)$ into factors. Formula (3) may be written as follows:

$$\int \frac{R(x)dx}{\sqrt{X_r}} = \frac{Q_p(x)\sqrt{X_r}}{\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}}$$

$$+ \sum_{k=0}^{k=r-2} \lambda_k \int \frac{x^k dx}{\sqrt{X_r}} + \int \frac{T_{s-1}(x)dx}{\prod_{k=1}^{k=s}(x-a_k)\sqrt{X_r}}$$

and to find $\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}$ and $\prod_{k=1}^{k=s}(x-a_k)$ we only need to find the greatest common measure of the denominator of

$R(x)$ and its derivative. As to formula (5) it may be written thus

$$\int \frac{R(x)dx}{\sqrt{X_r}} = \frac{Q_q(x)\sqrt{X_r}}{\prod_{k=1}^{k=\rho}(x-a_k)\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}}$$

$$+\sum_{k=0}^{k=r-2}\lambda_k\int\frac{x^k dx}{\sqrt{X_r}}+\int\frac{T_{s-\rho-1}(x)dx}{\prod_{k=\rho+1}^{k=s}(x-a_k)\sqrt{X_r}}$$

The function $\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}$ is found as just explained, while the functions $\prod_{k=1}^{k=\rho}(x-a_k)$ and $\prod_{k=\rho+1}^{k=s}(x-a_k)$ are found by determining the greatest common measure of the functions $\prod_{k=1}^{k=s}(x-a_k)$ and X_r.

The results of this note applied to the case of $r=2$ give very useful formulas, namely

$$(7)\qquad \int\frac{P_m(x)dx}{\prod_{k=1}^{k=s}(x-a_k)^{n_k}\sqrt{X_2}} = \frac{Q_{n-s-1}(x)\sqrt{X_2}}{\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}}$$

$$+\lambda\int\frac{dx}{\sqrt{X_2}}+\sum_{k=1}^{k=s}\lambda_k\int\frac{dx}{(x-a_k)\sqrt{X_2}}$$

if $X_2(a_k)\neq 0$ $(k\equiv 1, 2, \cdots s)$, n being the greatest of the numbers m and $\sum_{k=1}^{k=s} n_k$; and

$$(8)\qquad \int\frac{P_m(x)dx\,.}{\prod_{k=1}^{k=s}(x-a_k)\sqrt{X_2}} = \frac{Q_{n-s+\rho-1}(x)\sqrt{X_2}}{(x-a_1)\;(x-a_\rho)\prod_{k=1}^{k=s}(x-a_k)^{n_k-1}}$$

$$+\lambda\int\frac{dx}{\sqrt{X_2}}+\sum_{k=\rho+1}^{k=s}\mu_k\int\frac{dx}{(x-a_k)\sqrt{X_2}}$$

if a_1, a_ρ are roots of $X_2=0$.

Formula (7) holds also in the case where a_k is a root of $X_2=0$ provided $n_k=1$.

It will be noticed that the method of reduction here used does not require the degree of the numerator of $R(x)$ to be less than that of the denominator.

JOHNS HOPKINS UNIVERSITY,
October 6, 1897.