

Hamiltonian structure of gauge-invariant variational problems

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Abstract

Let $C \rightarrow M$ be the bundle of connections of a principal bundle on M . The solutions to Hamilton–Cartan equations for a gauge-invariant Lagrangian density Λ on C satisfying a weak condition of regularity, are shown to admit an affine fibre-bundle structure over the set of solutions to Euler–Lagrange equations for Λ . This structure is also studied for the Jacobi fields and for the moduli space of extremals.

1 Introduction

Let $p: E \rightarrow M$ be a fibred manifold and let Λ be a Lagrangian density on J^1E . The solutions to the Hamilton–Cartan (H–C, in short) equations

(e.g., see [7, 8], or [14]) for Λ , are the sections $\bar{s}: M \rightarrow J^1E$ of the canonical projection $p_1: J^1E \rightarrow M$ fulfilling equation (3.1) below. If Λ is regular, i.e., its Hessian metric is non-singular (cf. Definition 5.4 below), then every solution \bar{s} to the H–C equations is holonomic, i.e., $\bar{s} = j^1s$, and s is a critical section for Λ , that is, a solution to the Euler–Lagrange (E–L, in short) equations. Nevertheless, a Lagrangian defined on the bundle of connections $p: C \rightarrow M$ of a principal bundle $\pi: P \rightarrow M$, which is invariant under the gauge group of P , is never regular. Therefore, Hamilton–Cartan equations and E–L are not equivalent for all the gauge-invariant variational problems on connections.

In the present work we show that the solutions to H–C equations for a gauge-invariant Lagrangian density Λ on the bundle C satisfying a weak condition of regularity (most of interesting gauge-invariant Lagrangians in the field theory satisfies this condition) admits an affine fibre-bundle structure over the set of critical sections of the variational problem defined by the density under consideration. Moreover, the structure of this fibration is completely determined; see Theorem 5.6 below. This proves, in particular, that the E–L equations of a gauge-invariant Lagrangian are essentially of first order.

Such a structure is meaningful even from the point of view of the observables of the field theory, because for every extremal section s the curvature is constant along the fibre over j^1s .

By passing to the quotient such a fibre bundle modulo the gauge group, we obtain a — not necessary trivial — vector bundle associated to the principal bundle of the moduli of extremals of Λ ; see Proposition 5.11 below. Finally, the aforementioned affine structure is also studied for the Jacobi fields; i.e., for the vector fields in the kernel of the linearization of the E–L and H–C operators at any extremal (cf. Theorem 5.12 below).

2 Notations and preliminaries

2.1 Jet bundles

The bundle of r -jets of a fibred manifold $p: E \rightarrow M$ is denoted by $p_r: J^r(p) = J^rE \rightarrow M$ with projections $p_{rk}: J^rE \rightarrow J^kE$, $r \geq k$, the r -jet extension of a section $s: M \rightarrow E$ of p being denoted by $j^r s$. Every p -fibred coordinate system (x^i, y^α) , $1 \leq i \leq n = \dim M$, $1 \leq \alpha \leq m = \dim E - n$, defined on an open subset $U \subseteq E$, induces a coordinate system (x^i, y_I^α) , $0 \leq |I| \leq r$, on $(p_{r0})^{-1}(U) = J^rU$; namely, $y_I^\alpha(j_x^r s) = (\partial^{|I|}(y^\alpha \circ s)/\partial x^I)(x)$, with $y_0^\alpha = y^\alpha$.

Every fibred map $\Phi: E \rightarrow E'$ whose induced map $\phi: M \rightarrow M'$ on the base manifold is a diffeomorphism, induces a map $\Phi^{(r)}: J^r E \rightarrow J^r E'$ by setting, $\Phi^{(r)}(j_x^r s) = j_{\phi(x)}^r(\Phi \circ s \circ \phi^{-1})$, $\forall j_x^r s \in J^r E$.

An automorphism of p is a pair of diffeomorphisms $\Phi: E \rightarrow E$, $\phi: M \rightarrow M$ such that $p \circ \Phi = \phi \circ p$. The set of all automorphisms of p is a group denoted by $\text{Aut}E$.

For every — not necessarily p -projectable — vector field $Y \in \mathfrak{X}(E)$ a unique vector field $Y^{(1)} \in \mathfrak{X}(J^1 E)$ exists (called the one-jet prolongation of Y) such that, 1st) $Y^{(1)}$ is p_{10} -projectable onto Y , and 2nd) $L_{Y^{(1)}}\mathcal{C} \subseteq \mathcal{C}$, where \mathcal{C} is the contact system; i.e., the differential system generated by the contact one-forms $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$ in $\Omega^1(J^1 E)$.

If Y is a p -vertical vector field (the only case that we consider below), then the formulas of one-jet prolongation are as follows (cf. [14, Section 2.4]):

$$\begin{aligned} Y^{(1)} &= v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha}, & v_i^\alpha &= \frac{\partial v^\alpha}{\partial x^i} + \frac{\partial v^\alpha}{\partial y^\beta} y_i^\beta, \\ Y &= v^\alpha \frac{\partial}{\partial y^\alpha}, & v^\alpha &\in C^\infty(E). \end{aligned} \quad (2.1)$$

Lemma 2.1 ([14]). *For every p -vertical vector field $Y \in \mathfrak{X}^v(E)$ the value of the vector field $Y^{(1)}$ at a point $j_x^1 s \in J^1 E$ depends only on $j_x^1(Y \circ s)$.*

2.2 Bundles of connections

An automorphism of a principal G -bundle $\pi: P \rightarrow M$ is an equivariant diffeomorphism $\Phi: P \rightarrow P$; i.e., $\Phi(u \cdot g) = \Phi(u) \cdot g$, $\forall u \in P$, $\forall g \in G$. We denote by $\text{Aut}P$ the group of all automorphisms of P under composition. Every $\Phi \in \text{Aut}P$ determines a unique diffeomorphism $\phi: M \rightarrow M$, such that $\pi \circ \Phi = \phi \circ \pi$. If ϕ is the identity map, then Φ is said to be a gauge transformation (cf. [1, 3.2.1]). We denote by $\text{Gau}P \subset \text{Aut}P$ the subgroup of all gauge transformations.

A vector field $X \in \mathfrak{X}(P)$ is said to be G -invariant if $R_g \cdot X = X$, $\forall g \in G$. If Φ_t is the flow of a vector field $X \in \mathfrak{X}(P)$, then X is G -invariant if and only if $\Phi_t \in \text{Aut}P$, $\forall t \in \mathbb{R}$. Because of this we denote the Lie subalgebra of G -invariant vector fields on P by $\text{aut}P \subset \mathfrak{X}(P)$. Each G -invariant vector field on P is π -projectable. Similarly, a π -vertical vector field $X \in \mathfrak{X}(P)$ is G -invariant if and only if $\Phi_t \in \text{Gau}P$, $\forall t \in \mathbb{R}$. We denote by $\text{gau}P \subset \text{aut}P$

the ideal of all π -vertical G -invariant vector fields on P , which is called the gauge algebra of P .

The group G acts on TP by setting $X \cdot g = (R_g)_*(X)$, $\forall X \in TP$, $\forall g \in G$. The global sections of the quotient vector bundle $T(P)/G$ can naturally be identified to $\text{aut}P$; i.e., $\text{aut}P \cong \Gamma(M, T(P)/G)$. Similarly, $\text{gau}P \cong \Gamma(M, \text{ad}P)$, where $\text{ad}P$ denotes the adjoint bundle: The bundle associated to P by the adjoint representation of G on its Lie algebra \mathfrak{g} , denoted by $\pi_{\mathfrak{g}}: \text{ad}P \rightarrow M$; that is, $\text{ad}P = (P \times \mathfrak{g})/G$, where the action of G on $P \times \mathfrak{g}$ is defined by

$$(u, B) \cdot g = (u \cdot g, \text{Ad}_{g^{-1}}(B)), \quad \forall u \in P, \forall B \in \mathfrak{g}, \forall g \in G.$$

The G -orbit in $\text{ad}P$ of a pair $(u, B) \in P \times \mathfrak{g}$ is denoted by $(u, B)_{\text{ad}}$.

An exact sequence of vector bundles over M (the so-called Atiyah sequence) holds, $0 \rightarrow \text{ad}P \rightarrow T(P)/G \xrightarrow{\pi_*} TM \rightarrow 0$. The fibres $(\text{ad}P)_x$ are endowed with a Lie-algebra structure determined by $[(u, B)_{\text{ad}}, (u, B')_{\text{ad}}] = (u, -[B, B']_{\text{ad}})$, for all $u \in \pi^{-1}(x)$, $B, B' \in \mathfrak{g}$, where $[\cdot, \cdot]$ denotes the bracket in \mathfrak{g} . The sign of the bracket above is needed in order to ensure that the natural identification $\text{gau}P \cong \Gamma(M, \text{ad}P)$ is a Lie-algebra isomorphism, when $\text{gau}P$ is considered as a Lie subalgebra of $\mathfrak{X}(P)$.

Let $X^{h\Gamma} \in \mathfrak{X}(P)$ be the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ with respect to a connection Γ on $\pi: P \rightarrow M$. The vector field $X^{h\Gamma}$ is G -invariant and projects onto X (cf. [10, II. Proposition 1.2]). Hence we have a splitting of the Atiyah sequence, $s_{\Gamma}: TM \rightarrow T(P)/G$, $s_{\Gamma}(X) = X^{h\Gamma}$. Conversely, any splitting $\sigma: TM \rightarrow T(P)/G$ of the Atiyah sequence (i.e., s is a vector bundle homomorphism such that $\pi_* \circ s = \text{id}_{TM}$) comes from a unique connection on P so that there is a natural bijection between connections on P and splittings of the Atiyah sequence. We thus define the bundle of connections $p: C = C(P) \rightarrow M$ as the sub-bundle of $\text{Hom}(TM, T(P)/G)$ determined by all \mathbb{R} -linear mappings $\lambda: T_x M \rightarrow (T(P)/G)_x$ such that $\pi_* \circ \lambda = \text{id}_{T_x M}$. Connections on P can be identified to the global sections of $p: C \rightarrow M$. We also denote by $s_{\Gamma}: M \rightarrow C$ the section of the bundle of connections induced by Γ .

An element $\lambda: T_x M \rightarrow (T(P)/G)_x$ of the bundle C over a point $x \in M$ is nothing but a ‘connection at a point x ’; i.e., λ induces a complementary subspace H_u of the vertical subspace $V_u(P) \subset T_u(P)$ for every $u \in \pi^{-1}(x)$. Any other connection at x can be written as $\lambda' = h + \lambda$, where $h: T_x M \rightarrow (\text{ad}P)_x$ is a linear map. Hence C is an affine bundle modelled over the vector bundle $\text{Hom}(TM, \text{ad}P) \cong T^*M \otimes \text{ad}P$.

Let $(U; x^i)$ be a coordinate open domain in M such that $\pi^{-1}(U) \cong U \times G$. For every $B \in \mathfrak{g}$ we define a flow of gauge transformations over U by setting $\varphi_t^B(x, g) = (x, \exp(tB) \cdot g)$, $x \in U$. Let \tilde{B} be the corresponding infinitesimal generator. If (B_1, \dots, B_m) is a basis of \mathfrak{g} , then $\tilde{B}_1, \dots, \tilde{B}_m$ is a basis of $\Gamma(U, \text{ad}P)$. Let $p: C \rightarrow M$ be the bundle of connections of P . The horizontal lift with respect to Γ of the basic vector field $\partial/\partial x^i$ is given as follows:

$$s_\Gamma \left(\frac{\partial}{\partial x^i} \right) = \left(\frac{\partial}{\partial x^i} \right)^{h_\Gamma} = \frac{\partial}{\partial x^i} - (A_i^\alpha \circ s_\Gamma) \tilde{B}_\alpha.$$

The functions (x^i, A_j^α) , $i, j = 1, \dots, n = \dim M$, $1 \leq \alpha \leq m = \dim G$, induce a coordinate system on $p^{-1}(U) = C(\pi^{-1}U)$ (cf. [2]).

Each automorphism $\Phi \in \text{Aut}P$ acts on connections of P by pulling back connection forms; that is, $\Gamma' = \Phi(\Gamma)$ where $\omega_{\Gamma'} = (\Phi^{-1})^* \omega_\Gamma$ (cf. [10, II. Proposition 6.2-(b)]). If $\Psi \in \text{Aut}P$ is another automorphism, then $(\Psi \circ \Phi)(\Gamma) = \Psi(\Phi(\Gamma))$. For each $\Phi \in \text{Aut}P$ there exists a unique diffeomorphism $\Phi_C: C \rightarrow C$ such that $p \circ \Phi_C = \Phi \circ p$, where $\Phi: M \rightarrow M$ is the diffeomorphism induced by Φ on the base manifold. We thus obtain a group homomorphism $\text{Aut}P \rightarrow \text{Diff}C$. For every connection Γ on P we have $\Phi_C \circ s_\Gamma = s_{\Phi(\Gamma)}$.

If Φ_t is the flow of a G -invariant vector field $X \in \text{aut}P$, then $(\Phi_t)_C$ is a one-parameter group in $\text{Diff}C$ with infinitesimal generator denoted by X_C , and the map $\text{aut}P \rightarrow \mathfrak{X}(C)$, $X \mapsto X_C$ is a Lie-algebra homomorphism.

2.3 Affine-bundle structures

Let $V(p) = \{X \in TE : p_*X = 0\}$ be the vertical subbundle of a fibred manifold $p: E \rightarrow M$.

- (a) Let $p: E \rightarrow M$ be an affine bundle modelled over the vector bundle $p_W: W \rightarrow M$. The directional derivative determines an isomorphism of vector bundles over E , $p^*W \cong V(p)$, $(e, w) \mapsto X_{e,w}$, $p(e) = p_W(w) = x$, where $X_{e,w}$ is the tangent vector at $t=0$ to the curve $t \mapsto tw + e$, which takes values in the fibre $p^{-1}(x)$. In coordinates, $X_{e,w} = w^\alpha(w)(\partial/\partial e^\alpha)_e$.
- (b) Moreover, if $p: E \rightarrow M$ is an arbitrary surjective submersion, then the projection $p_{10}: J^1E \rightarrow E$ is endowed with an affine-bundle structure modelled over $p^*T^*M \otimes V(p)$. In fact, every jet $j_x^1s \in (p_{10})^{-1}(e)$, with $s(x) = e$, can be identified to the section $s_{*,e}: T_xM \rightarrow T_eE$ of $p_{*,e}: T_eE \rightarrow T_xM$. Hence, if $j_x^1s' \in (p_{10})^{-1}(e)$ is another jet, then $p_{*,e} \circ$

$(s'_{*,e} - s_{*,e}) = 0$ and accordingly $s'_{*,e} - s_{*,e}$ takes values into $V_e(p)$. Therefore, $s'_{*,e} - s_{*,e}$ determines an element in $\text{Hom}(T_x M \otimes V_e(p)) = T_x^* M \otimes V_e(p)$. From (a) it follows an isomorphism, $p_{10}^*(p^* T^* M \otimes V(p)) = T^* M \otimes_{J^1 C} V(p) \cong V(p_{10})$.

2.4 The Hessian metric

If $p: E \rightarrow M$ be an affine bundle modelled over the vector bundle $p_W: W \rightarrow M$, then, according to the item (a) in Section 2.3, every $w \in (p_W)^{-1}(x)$ induces a vector field along the fibre $X_w \in \mathfrak{X}(p^{-1}(x))$, $X_w(e) = X_{e,w}$, $\forall e \in p^{-1}(x)$. For every $f \in C^\infty(E)$ and every $e \in E$, with $x = p(e)$, a bilinear form

$$\text{Hess}_e(f): V_e(p) \times V_e(p) \rightarrow \mathbb{R}$$

can be defined as follows: $\text{Hess}_e(f)(w_1, w_2) = X_{w_2}(e)((d_{E/M} f)X_{w_1})$, where the canonical isomorphism $W_x \cong V_e(p)$, defined in the item (a) in Section 2.3, has been used and $d_{E/M}$ denotes the fibred derivative, e.g., see [8]. As this form is proved to be symmetric, $e \mapsto \text{Hess}_e(f)$ defines a section of the vector bundle $S^2 V^*(p) \cong p^* S^2 W^*$.

3 H–C equations

Let $p: E \rightarrow M$ be a fibred manifold, $\dim M = n$, $\dim E = m + n$, where M is assumed to be connected and oriented by a volume form \mathbf{v} . Below, Latin indices run from 1 to n , and Greek indices run from 1 to m . The solutions to the Hamilton–Cartan equations for a density $\Lambda = L\mathbf{v}$, $L \in C^\infty(J^1 E)$ on p , are the sections $\bar{s}: M \rightarrow J^1 E$ of the canonical projection $p_1: J^1 E \rightarrow M$ such that,

$$\bar{s}^*(i_X d\Theta_\Lambda) = 0, \quad \forall X \in \mathfrak{X}^v(J^1 E), \quad (3.1)$$

where

- (i): $\Theta_\Lambda = (-1)^{i-1} (\partial L / \partial y_i^\alpha) \theta^\alpha \wedge \mathbf{v}_i + L\mathbf{v}$ is the Poincaré–Cartan form attached to Λ (cf. [8], [14]);
- (ii): $\mathfrak{X}^v(J^1 E)$ denotes the Lie algebra of p_1 -vertical vector fields;
- (iii): $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$ are the standard contact forms on the one-jet bundle; and

(iv): $(x^i, y^\alpha, y_i^\alpha)$ is the coordinate system on J^1E induced by a fibred coordinate system (x^i, y^α) for the submersion p , adapted to the given volume form; i.e., $\mathbf{v} = dx^1 \wedge \cdots \wedge dx^n$ and $\mathbf{v}_i = (-1)^{i-1} i_{\partial/x^i} \mathbf{v}$.

Lemma 3.1. *A section $\bar{s}: M \rightarrow J^1E$ of $p_1: J^1E \rightarrow M$ is a solution to the H-C equations (3.1) if and only if the following equations hold:*

$$\begin{aligned} (s_j^\beta - \bar{s}_j^\beta) \left(\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\beta} \circ \bar{s} \right) &= 0, \quad 1 \leq i \leq n, 1 \leq \alpha \leq m, \\ -\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial y_j^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial y^\alpha} \circ \bar{s} + (s_j^\beta - \bar{s}_j^\beta) \left(\frac{\partial^2 L}{\partial y^\alpha \partial y_j^\beta} \circ \bar{s} \right) &= 0, \end{aligned}$$

for $1 \leq \alpha \leq m$, where $s^\alpha = y^\alpha \circ \bar{s}$, $s_i^\alpha = \partial s^\alpha / \partial x^i$, and $\bar{s}_i^\alpha = y_i^\alpha \circ \bar{s}$.

Proof. As a simple computation shows, we have

$$d\Theta_\Lambda = \theta^\beta \wedge \left((-1)^j d \left(\frac{\partial L}{\partial y_j^\beta} \right) \wedge \mathbf{v}_j + \frac{\partial L}{\partial y^\beta} \mathbf{v} \right). \quad (3.2)$$

Hence

$$\begin{aligned} \bar{s}^* \left(i_{\partial/\partial y_i^\alpha} d\Theta_\Lambda \right) &= (-1)^{j-1} \left(\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\beta} \circ \bar{s} \right) \bar{s}^* \theta^\beta \wedge \mathbf{v}_j, \\ \bar{s}^* \left(i_{\partial/\partial y^\alpha} d\Theta_\Lambda \right) &= \left(-\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial y_j^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial y^\alpha} \circ \bar{s} \right) \mathbf{v} \\ &\quad + (-1)^{j-1} \left(\frac{\partial^2 L}{\partial y^\alpha \partial y_j^\beta} \circ \bar{s} \right) \bar{s}^* \theta^\beta \wedge \mathbf{v}_j, \end{aligned}$$

and the formulas in the statement follow. \square

If $\bar{s} = j^1 s$ is a holonomic section, then \bar{s} is a solution to the H-C equations if and only if s is a solution to the E-L equations. If L is regular, then the converse holds true: Every solution to the H-C equations, is of the form $\bar{s} = j^1 s$, s being a solution to the E-L equations. Hence, for regular variational problems, H-C equations are equivalent to E-L equations; but this is no longer true for non-regular densities, as is the case for the Yang-Mills Lagrangian.

4 Jacobi fields

4.1 Jacobi fields introduced

Let $p: E \rightarrow M$ be a fibred manifold and let $\Omega^1(E/M) = \Gamma(M, V^*(p))$. Let

$$\mathcal{E}^\Lambda: \Gamma(p) \rightarrow \Omega^1(E/M) \otimes_{C^\infty(M)} \Omega^n(M)$$

be the E–L operator of $\Lambda = L\mathbf{v}$, $L \in C^\infty(J^1E)$, which is the second-order differential operator locally given on a fibred coordinate system $(U; x^i, y^\alpha)$ for the submersion p and for every section s of $p|_U$ by,

$$\mathcal{E}^\Lambda(s) = (\mathcal{E}_\alpha^\Lambda \circ j^2s) d_{E/M}y^\alpha \otimes \mathbf{v},$$

where the functions $\mathcal{E}_\alpha^\Lambda \in C^\infty(J^2E)$ are defined as follows:

$$\mathcal{E}_\alpha^\Lambda(j^2s) = \frac{\partial L}{\partial y^\alpha}(j^1s) - \frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial y_j^\alpha} \circ j^1s \right) (x).$$

The linearization of \mathcal{E}^Λ at $s \in \Gamma(p)$ is the operator

$$\mathcal{L}_s\mathcal{E}^\Lambda: T_s\Gamma(p) = \Gamma(M, s^*V(p)) \rightarrow \Omega^1(E/M) \otimes_{C^\infty(M)} \Omega^n(M)$$

defined as follows.

If $S: (-\varepsilon, \varepsilon) \times U \rightarrow E$ is a one-parameter family of sections, i.e., $p \circ S_t = \text{id}_U$, $|t| < \varepsilon$, then a vector field $X \in \Gamma(U, s^*V(p))$ along $s = S_0$ — called the ‘initial velocity’ of S — is defined to be the tangent vector $X(x) \in V_{s(x)}(p)$ at $t = 0$ to the curve $t \mapsto S_t(x)$, which takes values in the fibre $p^{-1}(x)$ for every point $x \in U$. Expanding $y^\alpha \circ S_t$ up to second order, we obtain $y^\alpha \circ S_t = y^\alpha \circ s + tv^\alpha + t^2f^\alpha$, for certain functions $v^\alpha \in C^\infty(U)$, $f^\alpha \in C^\infty((-\varepsilon, \varepsilon) \times U)$. Hence,

$$X(x) = v^\alpha(x)(\partial/\partial y^\alpha)_{s(x)}, \quad \forall x \in U.$$

Therefore, every vector field $X \in \Gamma(U, s^*V(p))$ is the initial velocity of a one-parameter family of sections S , and we define,

$$\mathcal{L}_s\mathcal{E}^\Lambda(X) = \left. \frac{\partial}{\partial t} \right|_{t=0} \mathcal{E}^\Lambda(S_t).$$

The definition makes sense as it does not depend on the particular one-parameter family of sections chosen. In fact,

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_{t=0} (\mathcal{E}_\alpha^\Lambda \circ j^2 S_t) &= \left(\frac{\partial \mathcal{E}_\alpha^\Lambda}{\partial y^\beta} \circ j^2 s \right) v^\beta + \left(\frac{\partial \mathcal{E}_\alpha^\Lambda}{\partial y_j^\beta} \circ j^2 s \right) \frac{\partial v^\beta}{\partial x^j} \\ &\quad + \left(\frac{\partial \mathcal{E}_\alpha^\Lambda}{\partial y_{jk}^\beta} \circ j^2 s \right) \frac{\partial^2 v^\beta}{\partial x^j \partial x^k}. \end{aligned}$$

This expression also shows that $\mathcal{L}_s \mathcal{E}^\Lambda(X)$ depends linearly on X .

Definition 4.1. A vector field $X \in \Gamma(M, s^*V(p))$ defined along an extremal s of a Lagrangian density Λ on $p: E \rightarrow M$ is said to be a *Jacobi field* if $X \in \ker \mathcal{L}_s \mathcal{E}^\Lambda$.

Proposition 4.2. A vector field $X \in \Gamma(M, s^*V(p))$ defined along an extremal s of a Lagrangian density Λ on $p: E \rightarrow M$ is a Jacobi field if and only if the following equation holds:

$$(j^1 s)^* (i_Y L_{\tilde{X}(1)} d\Theta_\Lambda) = 0, \quad \forall Y \in \mathfrak{X}^v(J^1 E), \quad (4.1)$$

where $\tilde{X} \in \mathfrak{X}^v(E)$ is an arbitrary p -vertical extension of X .

Proof. Equation (4.1) does not depend on the vertical extension chosen. In fact,

$$(j^1 s)^* (i_Y L_{\tilde{X}(1)} d\Theta_\Lambda) = (j^1 s)^* (L_{\tilde{X}(1)}(i_Y d\Theta_\Lambda)) + (j^1 s)^* (i_{[Y, \tilde{X}(1)]} d\Theta_\Lambda), \quad (4.2)$$

and the second term on the right-hand side vanishes, as s is an extremal. Hence

$$(j^1 s)^* (i_Y L_{\tilde{X}(1)} d\Theta_\Lambda) = d((j^1 s)^* (i_{\tilde{X}(1)} i_Y d\Theta_\Lambda)) + (j^1 s)^* (i_{\tilde{X}(1)} d(i_Y \Theta_\Lambda)),$$

and we conclude by simply applying Lemma 2.1.

Moreover, from the formula (3.2) the following identity is obtained:

$$d\Theta_\Lambda = \mathcal{E}_\alpha^\Lambda \theta^\alpha \wedge \mathbf{v} + (-1)^i \frac{\partial^2 L}{\partial y_i^\alpha \partial y^\beta} \theta^\alpha \wedge \theta^\beta \wedge \mathbf{v}_i + (-1)^i \frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\beta} \theta^\alpha \wedge \theta_j^\beta \wedge \mathbf{v}_i,$$

where $\theta_j^\beta = dy_j^\beta - y_{(j^k)}^\beta dx^k$. If $\tilde{\Phi}_t$ is the flow of \tilde{X} , then $j^1 S_t = J^1(\tilde{\Phi}_t) \circ j^1 s$ and from the previous formula we have

$$\begin{aligned} (j^1 s)^* \left\{ J^1(\tilde{\Phi}_t)^*(i_Y d\Theta_\Lambda) \right\} &= (j^1 S_t)^*(i_Y d\Theta_\Lambda) \\ &= (\mathcal{E}_\alpha^\Lambda \circ j^2 S_t) \theta^\alpha(Y) \mathbf{v}. \end{aligned}$$

Taking derivatives with respect to t at $t = 0$ in this formula, we have

$$(j^1 s)^* (L_{\tilde{X}(1)}(i_Y d\Theta_\Lambda)) = \mathcal{L}_s \mathcal{E}^\Lambda(X) \mathbf{v},$$

and we can conclude by simply applying formula (4.2) recalling that s is an extremal. \square

Let \mathcal{S}_Λ (resp. $\bar{\mathcal{S}}_\Lambda$) denote the set of solutions to the E–L equations (resp. H–C equations) attached to a Lagrangian density $\Lambda = L\mathbf{v}$, $L \in C^\infty(J^1 E)$.

Remark 4.3. If $S_t \in \mathcal{S}_\Lambda$ is a one-parameter family of extremals, then its initial velocity X is readily seen to be a Jacobi field along the extremal $s = S_0$; in this case, X is said to be ‘integrable’ (e.g., see [11, Definition 1.2], [12, Section 2.6]). Although important examples of non-integrable Jacobi fields exist, usually Jacobi fields along s are considered as the tangent space $T_s \mathcal{S}_\Lambda$ at an extremal $s \in \mathcal{S}_\Lambda$ to the ‘manifold’ of solutions to the E–L equations for Λ . By the same token, we give the following

Definition 4.4. The tangent space $T_{\bar{s}} \bar{\mathcal{S}}_\Lambda$ at a solution $\bar{s} \in \bar{\mathcal{S}}_\Lambda$ to the ‘manifold’ of solutions to the H–C equations for Λ is defined to be the space of vector fields $\tilde{X} \in \Gamma(M, \bar{s}^* V(p_1))$ that satisfy the equation

$$\bar{s}^*(i_Y L_{\tilde{X}} d\Theta_\Lambda) = 0, \quad \forall Y \in \mathfrak{X}^v(J^1 E),$$

where $\tilde{X} \in \mathfrak{X}^v(J^1 E)$ is a p_1 -vertical extension of \bar{X} .

4.2 The embedding $T_s \mathcal{S}_\Lambda \hookrightarrow T_{j^1 s} \bar{\mathcal{S}}_\Lambda$

Proposition 4.5. *For every $s \in \mathcal{S}_\Lambda$, there is an embedding*

$$\begin{aligned} T_s \mathcal{S}_\Lambda &\hookrightarrow T_{j^1 s} \bar{\mathcal{S}}_\Lambda, \\ X &\mapsto \tilde{X}^{(1)} \circ j^1 s, \end{aligned}$$

where \tilde{X} is any p -vertical extension of $X \in \Gamma(M, s^* V(p))$ to E . If Λ is regular, then $T_s \mathcal{S}_\Lambda \cong T_{j^1 s} \bar{\mathcal{S}}_\Lambda$, $\forall s \in \mathcal{S}_\Lambda$.

Proof. As a straightforward — but rather long — computation shows, a vector field $\bar{X} \in \Gamma(M, \bar{s}^*V(p_1))$ with local expression

$$\bar{X} = v^\alpha \frac{\partial}{\partial y^\alpha} \Big|_{\bar{s}} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha} \Big|_{\bar{s}}, \quad v^\alpha, v_i^\alpha \in C^\infty(M),$$

belongs to $T_{\bar{s}}\bar{\mathcal{S}}_\Lambda$ if and only if the following two equations hold:

$$\begin{aligned} 0 = & \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\sigma} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial y^\alpha \partial y_i^\sigma} \circ \bar{s} + \left(\frac{\partial s^\beta}{\partial x^i} - \bar{s}_i^\beta \right) \left(\frac{\partial^3 L}{\partial y^\alpha \partial y^\sigma \partial y_i^\beta} \circ \bar{s} \right) \right. \\ & - \frac{\partial s^\gamma}{\partial x^j} \left(\frac{\partial^3 L}{\partial y^\alpha \partial y^\gamma \partial y_j^\sigma} \circ \bar{s} \right) - \frac{\partial \bar{s}_h^\gamma}{\partial x^j} \left(\frac{\partial^3 L}{\partial y^\alpha \partial y_h^\gamma \partial y_j^\sigma} \circ \bar{s} \right) \Big) v^\alpha \\ & + \left(\frac{\partial^2 L}{\partial y^\sigma \partial y_i^\alpha} \circ \bar{s} - \frac{\partial^2 L}{\partial y^\alpha \partial y_i^\sigma} \circ \bar{s} \right) \frac{\partial v^\alpha}{\partial x^i} - \left(\frac{\partial^2 L}{\partial y_j^\alpha \partial y_i^\sigma} \circ \bar{s} \right) \frac{\partial v_j^\alpha}{\partial x^i} \\ & + \left(\left(\frac{\partial s^\beta}{\partial x^j} - \bar{s}_j^\beta \right) \left(\frac{\partial^3 L}{\partial y^\sigma \partial y_i^\alpha \partial y_j^\beta} \circ \bar{s} \right) - \frac{\partial^3 L}{\partial x^j \partial y_i^\alpha \partial y_j^\sigma} \circ \bar{s} \right. \\ & \left. - \frac{\partial s^\beta}{\partial x^j} \left(\frac{\partial^3 L}{\partial y^\beta \partial y_i^\alpha \partial y_j^\sigma} \circ \bar{s} \right) - \frac{\partial \bar{s}_h^\beta}{\partial x^j} \left(\frac{\partial^3 L}{\partial y_h^\beta \partial y_i^\alpha \partial y_j^\sigma} \circ \bar{s} \right) \right) v_i^\alpha, \end{aligned} \quad (4.3)$$

$$\begin{aligned} 0 = & \left(\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\sigma} \circ \bar{s} \right) \left(\frac{\partial v^\alpha}{\partial x^i} - v_i^\alpha \right) + \left(\frac{\partial s^\alpha}{\partial x^i} - \bar{s}_i^\alpha \right) \left(\frac{\partial^3 L}{\partial y^\beta \partial y_i^\alpha \partial y_j^\sigma} \circ \bar{s} \right) v^\beta \\ & + \left(\frac{\partial s^\alpha}{\partial x^i} - \bar{s}_i^\alpha \right) \left(\frac{\partial^3 L}{\partial y_i^\alpha \partial y_k^\beta \partial y_j^\sigma} \circ \bar{s} \right) v_k^\beta, \end{aligned} \quad (4.4)$$

where $s = p_{10} \circ \bar{s}$, $s^\alpha = y^\alpha \circ \bar{s} = y^\alpha \circ s$, and $\bar{s}_i^\alpha = y_i^\alpha \circ \bar{s}$. Along a holonomic section $\bar{s} = j^1 s$, equations (4.3) and (4.4) become, respectively,

$$\begin{aligned} 0 = & \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\sigma} \circ j^1 s - \frac{\partial^3 L}{\partial x^i \partial y^\alpha \partial y_i^\sigma} \circ j^1 s - \frac{\partial s^\gamma}{\partial x^j} \left(\frac{\partial^3 L}{\partial y^\alpha \partial y^\gamma \partial y_j^\sigma} \circ j^1 s \right) \right. \\ & \left. - \frac{\partial^2 s^\gamma}{\partial x^h \partial x^j} \left(\frac{\partial^3 L}{\partial y^\alpha \partial y_h^\gamma \partial y_j^\sigma} \circ j^1 s \right) \right) v^\alpha \\ & + \left(\frac{\partial^2 L}{\partial y^\sigma \partial y_i^\alpha} \circ j^1 s - \frac{\partial^2 L}{\partial y^\alpha \partial y_i^\sigma} \circ j^1 s \right) \frac{\partial v^\alpha}{\partial x^i} - \left(\frac{\partial^2 L}{\partial y_j^\alpha \partial y_i^\sigma} \circ j^1 s \right) \frac{\partial v_j^\alpha}{\partial x^i} \end{aligned}$$

$$\begin{aligned}
& - \left(\frac{\partial^3 L}{\partial x^j \partial y_i^\alpha \partial y_j^\sigma} \circ j^1 s + \frac{\partial s^\beta}{\partial x^j} \left(\frac{\partial^3 L}{\partial y^\beta \partial y_i^\alpha \partial y_j^\sigma} \circ j^1 s \right) \right. \\
& \left. + \frac{\partial^2 s^\beta}{\partial x^h \partial x^j} \left(\frac{\partial^3 L}{\partial y_i^\alpha \partial y_h^\beta \partial y_j^\sigma} \circ j^1 s \right) \right) v_i^\alpha, \tag{4.5}
\end{aligned}$$

$$0 = \left(\frac{\partial^2 L}{\partial y_i^\alpha \partial y_j^\sigma} \circ j^1 s \right) \left(\frac{\partial v^\alpha}{\partial x^i} - v_i^\alpha \right). \tag{4.6}$$

In addition, if L is regular, then equation (4.6) is equivalent to saying that $v_i^\alpha = \partial v^\alpha / \partial x^i$, and from formula (2.1) and Lemma 2.1 we conclude that \bar{X} is the one-jet prolongation of a Jacobi field, i.e., $\bar{X} = X^{(1)}$. \square

Remark 4.6. From formula (4.5) we deduce that a vector field

$$\begin{aligned}
X & \in \Gamma(M, s^*V(p)), \\
X & = v^\alpha \frac{\partial}{\partial y^\alpha} \Big|_s, \quad v^\alpha \in C^\infty(M),
\end{aligned}$$

belongs to $T_s\mathcal{S}_\Lambda$ if and only if the following equations hold:

$$\begin{aligned}
0 & = \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\sigma} \circ j^1 s - \frac{\partial^3 L}{\partial x^i \partial y^\alpha \partial y_i^\sigma} \circ j^1 s - \frac{\partial s^\gamma}{\partial x^j} \left(\frac{\partial^3 L}{\partial y^\alpha \partial y^\gamma \partial y_j^\sigma} \circ j^1 s \right) \right. \\
& - \left. \frac{\partial^2 s^\gamma}{\partial x^h \partial x^j} \left(\frac{\partial^3 L}{\partial y^\alpha \partial y_h^\gamma \partial y_j^\sigma} \circ j^1 s \right) \right) v^\alpha \\
& + \left(\frac{\partial^2 L}{\partial y^\sigma \partial y_i^\alpha} \circ j^1 s - \frac{\partial^2 L}{\partial y^\alpha \partial y_i^\sigma} \circ j^1 s - \frac{\partial^3 L}{\partial x^j \partial y_i^\alpha \partial y_j^\sigma} \circ j^1 s \right. \\
& - \left. \frac{\partial s^\beta}{\partial x^j} \left(\frac{\partial^3 L}{\partial y^\beta \partial y_i^\alpha \partial y_j^\sigma} \circ j^1 s \right) - \frac{\partial^2 s^\beta}{\partial x^h \partial x^j} \left(\frac{\partial^3 L}{\partial y_i^\alpha \partial y_h^\beta \partial y_j^\sigma} \circ j^1 s \right) \right) \frac{\partial v^\alpha}{\partial x^i} \\
& - \left(\frac{\partial^2 L}{\partial y_j^\sigma \partial y_i^\alpha} \circ j^1 s \right) \frac{\partial^2 v^\alpha}{\partial x^i \partial x^j}.
\end{aligned}$$

5 H–C equations and gauge invariance

5.1 Gauge-invariant Lagrangians

Definition 5.1. A smooth function $L: J^1C \rightarrow \mathbb{R}$ is said to be *gauge invariant* if $L \circ \Phi_C^{(1)} = L$ for every $\Phi \in \text{Gau}P$.

This equation obviously implies $X_C^{(1)}L = 0$ for every $X \in \text{gau}P$. The converse also holds if the group G is connected. As every $\Phi \in \text{Gau}P$ induces the identity map on M , the function L is gauge invariant if and only if the gauge group is a group of symmetries for the Lagrangian density $\Lambda = Lv$, where v is an arbitrary volume form on the base manifold.

Let

$$\begin{aligned} \Omega: J^1C &\rightarrow \bigwedge^2 T^*M \otimes \text{ad}P, \\ \Omega(j_x^1\sigma_\Gamma) &= (\Omega_\Gamma)_x, \end{aligned} \quad (5.1)$$

be the curvature map. Here, the curvature form Ω_Γ of the connection Γ corresponding to a section s_Γ of p , is seen to be a two form on M with values in the adjoint bundle $\text{ad}P$. On the vector bundle $\bigwedge^2 T^*M \otimes \text{ad}P$ we consider the coordinate systems $(x^i; R_{jk}^\alpha)$, $j < k$, induced by a coordinate system $(U; x^i)$ on M , and a basis (B_α) of \mathfrak{g} , as follows:

$$\eta_2 = \sum_{j < k} \left(R_{jk}^\alpha(\eta_2) dx^j \wedge dx^k \otimes \tilde{B}_\alpha \right)_x, \quad \forall \eta_2 \in \bigwedge^2 T_x^*M \otimes (\text{ad}P)_x.$$

The geometric formulation of Utiyama's Theorem (e.g., see [1]) states that a Lagrangian $L: J^1C \rightarrow \mathbb{R}$ is gauge invariant if and only L factors through Ω as $L = \tilde{L} \circ \Omega$, where

$$\tilde{L}: \bigwedge^2 T^*M \otimes \text{ad}P \rightarrow \mathbb{R} \quad (5.2)$$

is a differentiable function which is invariant under the adjoint representation of G on the curvature bundle. As the curvature map (5.1) is surjective, the function \tilde{L} is unique.

5.2 Projecting $\bar{\mathcal{S}}_\Lambda$ onto \mathcal{S}_Λ

From Lemma 3.1 in Section 3, we readily obtain

Proposition 5.2. *The H-C equations of a Lagrangian L on the bundle of connections $p: C \rightarrow M$ of a principal bundle $\pi: P \rightarrow M$, read as follows:*

$$\begin{aligned} &\left(s_{h,j}^\beta - \bar{s}_{h,j}^\beta \right) \left(\frac{\partial^2 L}{\partial A_{i,k}^\alpha \partial A_{h,j}^\beta} \circ \bar{s} \right) = 0, \quad \forall \alpha, i, k, \\ -\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial A_{i,j}^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial A_i^\alpha} \circ \bar{s} + \left(s_{h,j}^\beta - \bar{s}_{h,j}^\beta \right) \left(\frac{\partial^2 L}{\partial A_i^\alpha \partial A_{h,j}^\beta} \circ \bar{s} \right) &= 0, \quad \forall \alpha, i, \end{aligned}$$

where $\bar{s}: M \rightarrow J^1C$ is a section of $p_1: J^1C \rightarrow M$, and we have set

$$s_i^\alpha = A_i^\alpha \circ p_{10} \circ \bar{s}, \quad s_{i,j}^\alpha = \frac{\partial s_i^\alpha}{\partial x^j}, \quad \bar{s}_{i,j}^\alpha = A_{i,j}^\alpha \circ \bar{s}.$$

Lemma 5.3. *The Hessian metric of a gauge-invariant Lagrangian L on the bundle of connections $p: C \rightarrow M$ of a principal bundle $\pi: P \rightarrow M$, is singular; i.e.,*

$$\det \left(\frac{\partial^2 L}{\partial A_{i,j}^\alpha \partial A_{k,l}^\beta} \right)_{\beta,k,l}^{\alpha,i,j} = 0.$$

Proof. As $L = \tilde{L} \circ \Omega$, we have $\partial L / \partial A_{i,i}^\alpha = 0$, taking the curvature equations into account, i.e.,

$$R_{ij}^\alpha \circ \Omega = A_{i,j}^\alpha - A_{j,i}^\alpha - c_{\beta\gamma}^\alpha A_i^\beta A_j^\gamma.$$

Hence, $\partial^2 L / \partial A_{i,i}^\alpha \partial A_{k,l}^\beta = 0$, for all indices β, k, l . □

Definition 5.4. A Lagrangian $L \in C^\infty(J^1C)$ is said to be *regular* if the Hessian metric $\text{Hess}(L)$ is non-singular.

A gauge-invariant Lagrangian $L \in C^\infty(J^1C)$ is said to be *weakly regular* if the Hessian metric $\text{Hess}(\tilde{L})$ of the function in (5.2) associated to L according to Utiyama's theorem, is non-singular.

In terms of the coordinate system $(x^i; R_{jk}^\alpha)$, $j < k$, on $\bigwedge^2 T^*M \otimes \text{ad}P$ introduced in Section 5.1, this means

$$\det \left(\frac{\partial^2 \tilde{L}}{\partial R_{ij}^\alpha \partial R_{kl}^\beta} \right)_{\beta,k < l}^{\alpha,i < j} \neq 0. \quad (5.3)$$

Remark 5.5. The inequation (5.3) imposes a generic condition on \tilde{L} . In fact, most of gauge-invariant Lagrangians in the field theory satisfy the weak regularity condition (5.3); for example, the general Yang–Mills Lagrangian on the bundle of connections of a principal bundle $P \rightarrow M$ with semisimple Lie group G over a pseudo-Riemannian manifold (M, g) (even when constructed by using a non-degenerate adjoint-invariant pairing on the Lie algebra other than the Cartan–Killing pairing, see [3, 5]) is weakly regular. More generally, any quadratic function as in (5.2), which is simultaneously invariant under the adjoint representation and under the action of the gauge group of the principal bundle of g -orthonormal linear frames, gives rise to a

weakly regular Lagrangian, see [4]. Similarly, Born-Infeld Lagrangians (e.g., see [15,16]) are also weakly regular. We remark on the fact that some special Lagrangians are not weakly regular; for example, if we let the matter field vanish in the Seiberg–Witten Lagrangian (e.g., see [9]) then we obtain a Lagrangian on the bundle of connections, which is not weakly regular (basically, because it involves only the self-dual part of the curvature). Finally, we should also remark that non-gauge-invariant Lagrangians (in the sense of Section 5.1) may produce gauge-invariant actions, as the Chern–Simons Lagrangian. All of them are not considered below.

Given an arbitrary fibred manifold $p: E \rightarrow M$, we recall that the projection $p_{10}: J^1E \rightarrow E$ is endowed with an affine-bundle structure modelled over $T^*M \otimes_E V(p) = p^*T^*M \otimes V(p)$; see the item (b) in Section 2.3. In the particular case of the bundle of connections, which is itself an affine bundle modelled over $T^*M \otimes \text{ad}P$, we conclude that $p_{10}: J^1C \rightarrow C$ is an affine bundle modelled over

$$T^*M \otimes_C V(p) = p^*(\otimes^2 T^*M \otimes \text{ad}P). \quad (5.4)$$

Hence, sections of $T^*M \otimes_C V(p)$ can be considered as $\text{ad}P$ -valued covariant tensors of degree 2 on M with coefficients in C .

Theorem 5.6. *Let $\bar{\mathcal{S}}_\Lambda$ (resp. \mathcal{S}_Λ) denote the set of solutions to H-C (resp. E-L) equations of a weakly regular gauge-invariant Lagrangian L on the bundle of connections $p: C \rightarrow M$ of a principal bundle $\pi: P \rightarrow M$. If $\bar{s}: M \rightarrow J^1C$ belongs to $\bar{\mathcal{S}}_\Lambda$, then the section $s = p_{10} \circ \bar{s}$ belongs to \mathcal{S}_Λ . Hence, a natural projection exists $\varrho: \bar{\mathcal{S}}_\Lambda \rightarrow \mathcal{S}_\Lambda$, $\varrho(\bar{s}) = p_{10} \circ \bar{s}$, which is an affine bundle modelled as follows:*

$$\varrho^{-1}(s) = \{j^1s + t : t \in \Gamma(S^2T^*M \otimes \text{ad}P)\}, \quad \forall s \in \mathcal{S}_\Lambda.$$

Proof. We begin with the first H–C equation in Proposition 5.2. As $L = \tilde{L} \circ \Omega$, we obtain

$$\frac{\partial L}{\partial A_{i,i}^\alpha} = 0, \quad \frac{\partial L}{\partial A_{i,k}^\alpha} = \frac{\partial \tilde{L}}{\partial R_{ik}^\alpha} \circ \Omega, \quad (5.5)$$

where we have set $R_{ik}^\alpha = -R_{ki}^\alpha$ for $i > k$. Hence,

$$\left(s_{h,j}^\beta - \bar{s}_{h,j}^\beta \right) \left(\frac{\partial^2 \tilde{L}}{\partial R_{ik}^\alpha \partial R_{hj}^\beta} \circ (\Omega \circ \bar{s}) \right) = 0.$$

If we assume the weak regularity condition (5.3) holds, then the previous equation yields

$$\left(s_{h,j}^\beta - \bar{s}_{h,j}^\beta\right) - \left(s_{j,h}^\beta - \bar{s}_{j,h}^\beta\right) = 0, \quad \forall \beta, h, j. \quad (5.6)$$

If we write $\bar{s} = j^1s + t$, for a two-tensor t , the condition above means that t is symmetric; that is, t is a section of $S^2T^*M \otimes \text{ad}P \rightarrow M$.

Next, we study the second equation in Proposition 5.2. Taking equations (5.5) into account, we have

$$-\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial A_{i,j}^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial A_i^\alpha} \circ \bar{s} + \left(s_{h,j}^\beta - \bar{s}_{h,j}^\beta \right) \left(\frac{\partial}{\partial A_i^\alpha} \left(\frac{\partial \tilde{L}}{\partial R_{hj}^\beta} \circ \Omega \right) \circ \bar{s} \right) = 0.$$

The last term vanishes identically as $s_{h,j}^\beta - \bar{s}_{h,j}^\beta$ is symmetric by virtue of (5.6) and we have $\partial \tilde{L} / \partial R_{hj}^\beta = -\partial \tilde{L} / \partial R_{jh}^\beta$. Then, the second equation reduces to

$$-\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial A_{i,j}^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial A_i^\alpha} \circ \bar{s} = 0, \quad (5.7)$$

which is precisely the E–L equation, but evaluated at \bar{s} instead of j^1s . Nevertheless, the following formula is readily checked:

$$\frac{\partial L}{\partial A_i^\alpha} = 2 \left(c_{\alpha\gamma}^\beta A_j^\gamma \frac{\partial \tilde{L}}{\partial R_{ij}^\beta} \right) \circ \Omega. \quad (5.8)$$

Moreover, from (5.6) we deduce

$$\begin{aligned} \Omega \circ \bar{s} &= \left(\bar{s}_{i,j}^\alpha - \bar{s}_{j,i}^\alpha - c_{\beta\gamma}^\alpha s_i^\beta s_j^\gamma \right) dx^i \wedge dx^j \otimes \tilde{B}_\alpha \\ &= \left(s_{i,j}^\alpha - s_{j,i}^\alpha - c_{\beta\gamma}^\alpha s_i^\beta s_j^\gamma \right) dx^i \wedge dx^j \otimes \tilde{B}_\alpha \\ &= \Omega \circ j^1s. \end{aligned} \quad (5.9)$$

Therefore, from formulas (5.5) and (5.8), we conclude that equation (5.7) coincides with the E–L equation for s . \square

Remark 5.7. As mentioned in the Introduction, formula (5.9) shows that the curvature remains constant along the fibre of ϱ over any $j^1s \in \mathcal{S}_\Lambda$.

Corollary 5.8. *The H–C equations of a weakly regular gauge-invariant Lagrangian L on the bundle of connections $p: C \rightarrow M$ of a principal bundle $\pi: P \rightarrow M$, are equivalent to the following system:*

$$\begin{aligned} (s_{h,j}^\alpha - \bar{s}_{h,j}^\alpha) - (s_{j,h}^\alpha - \bar{s}_{j,h}^\alpha) &= 0, \quad \forall \alpha, h, j. \\ -\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial A_{i,j}^\alpha} \circ j^1 s \right) + \frac{\partial L}{\partial A_i^\alpha} \circ j^1 s &= 0, \quad \forall \alpha, i, \end{aligned}$$

where $\bar{s}: M \rightarrow J^1 C$ is a section of $p_1: J^1 C \rightarrow M$, and s_i^α , $s_{i,j}^\alpha$, and $\bar{s}_{i,j}^\alpha$ are as in Proposition 5.2.

Remark 5.9. For every section $\bar{s}: M \rightarrow J^1 C$ of $p_1: J^1 C \rightarrow M$, let $s: M \rightarrow C$ be the section of $p: C \rightarrow M$ defined by $s = p_{10} \circ \bar{s}$. As the points $\bar{s}(x)$, $j_x^1 s \in J^1 C$ lie over the same fibre of $p_{10}: J^1 C \rightarrow C$ and this map admits an affine-bundle structure modelled over the vector bundle (5.4), a map

$$\delta_C: J^1(p_1) \rightarrow \otimes^2 T^*M \otimes \text{ad}P$$

exists such that, $\delta_C(j_x^1 \bar{s}) = \bar{s}(x) - j_x^1 s$. If $\text{alt}: \otimes^2 T^*M \otimes \text{ad}P \rightarrow \wedge^2 T^*M \otimes \text{ad}P$ denotes the anti-symmetrization operator, then the first group of H–C equations in Corollary 5.8 means that $j^1 \bar{s}$ takes values into the sub-bundle $\ker(\text{alt} \circ \delta_C)$.

Proposition 5.10. *If Λ is a gauge-invariant Lagrangian density on the bundle of connections of a principal bundle $\pi: P \rightarrow M$ and $\bar{s} \in \bar{\mathcal{S}}_\Lambda$ (resp. $s \in \mathcal{S}_\Lambda$), then $\Phi_C^{(1)} \circ \bar{s} \in \bar{\mathcal{S}}_\Lambda$ (resp. $\Phi_C \circ s \in \mathcal{S}_\Lambda$) for every $\Phi \in \text{Gau}P$. Accordingly, the gauge group of P acts (on the left) on $\bar{\mathcal{S}}_\Lambda$ (resp. \mathcal{S}_Λ) by setting $\Phi \cdot \bar{s} = \Phi_C^{(1)} \circ \bar{s}$ (resp. $\Phi \cdot s = \Phi_C \circ s$), $\forall \bar{s} \in \bar{\mathcal{S}}_\Lambda$ (resp. $\forall s \in \mathcal{S}_\Lambda$), $\forall \Phi \in \text{Gau}P$.*

Proof. We prove that the section $\Phi_C^{(1)} \circ \bar{s}$ of p_1 is a solution to H–C equation (3.1). For every $Z \in \mathfrak{X}^v(J^1 E)$, we set

$$Y = (\Phi_C^{(1)})^{-1} \cdot Z \in \mathfrak{X}^v(J^1 E).$$

As Λ is gauge invariant, we have $(\Phi_C^{(1)})^* \Lambda = \Lambda$, from the functorial character of the Poincaré–Cartan form (see [7]) we obtain

$$\begin{aligned} (\Phi_C^{(1)} \circ \bar{s})^* (i_Z d\Theta_\Lambda) &= \bar{s}^* \left(\Phi_C^{(1)} \right)^* (i_Z d\Theta_\Lambda) \\ &= \bar{s}^* i_Y d \left(\left(\Phi_C^{(1)} \right)^* \Theta_\Lambda \right) \\ &= \bar{s}^* i_Y d\Theta_{(\Phi_C^{(1)})^* \Lambda} \\ &= \bar{s}^* i_Y d\Theta_\Lambda \\ &= 0. \end{aligned}$$

The proof for the solutions to E–L equations, is similar and therefore it is omitted. \square

Theorem 5.6 shows that the set of solutions of H–C trivially fibers over the set of solutions of E–L. On the other hand, for gauge-invariant problems, the moduli space of solutions under the action of the (restricted) gauge group plays a relevant role. We now study the relationship between the moduli of H–C and E–L showing that the first fibres over the second, but not necessarily in a trivial way. First, note that the gauge group $\text{Gau}P$ acts on the adjoint bundle $\text{ad}P$ by setting $\Phi_{\text{ad}}((u, B)_{\text{ad}}) = (\Phi(u), B)_{\text{ad}}$, $\forall \Phi \in \text{Gau}P$, $\forall (u, B) \in P \times \mathfrak{g}$, and this action obviously induces another action on $S^2T^*M \otimes \text{ad}P$ as follows:

$$\begin{aligned} \Phi_{\text{ad}P}(w_1 \odot w_2 \otimes v) &= w_1 \odot w_2 \otimes \Phi_{\text{ad}P}(v), \\ \forall w_1, w_2 \in T_x^*M, \forall v \in (\text{ad}P)_x, \end{aligned} \tag{5.10}$$

where the symbol \odot denotes symmetric product.

Proposition 5.11. *Given a point $x_0 \in M$, let $\text{Gau}_{x_0}P$ be the subgroup of gauge transformations $\Phi \in \text{Gau}P$ such that, $\Phi(u) = u$, $\forall u \in \pi^{-1}(x_0)$. Then*

- (i) *For every gauge-invariant Lagrangian density Λ on the bundle of connections of $\pi: P \rightarrow M$, the quotient map $\kappa_P: \mathcal{S}_\Lambda \rightarrow \mathcal{S}_\Lambda/\text{Gau}_{x_0}P$ is a set-theoretical principal $\text{Gau}_{x_0}P$ -bundle.*
- (ii) *In addition, if Λ is weakly regular, the projection $\varrho: \bar{\mathcal{S}}_\Lambda \rightarrow \mathcal{S}_\Lambda$ defined in Theorem 5.6, induces a mapping*

$$\begin{aligned} \varrho_{\text{Gau}_{x_0}P}: \bar{\mathcal{S}}_\Lambda/\text{Gau}_{x_0}P &\rightarrow \mathcal{S}_\Lambda/\text{Gau}_{x_0}P, \\ \varrho_{\text{Gau}_{x_0}P}(\bar{s} \bmod \text{Gau}_{x_0}P) &= \varrho(\bar{s}) \bmod \text{Gau}_{x_0}P, \end{aligned}$$

*which is the vector bundle associated to the principal bundle κ_P by the action on $S^2\Omega^1(M) \otimes \text{gau}P$ induced on the sections of $S^2T^*M \otimes \text{ad}P$ by the action of $\text{Gau}P$ defined in the formula (5.10) above.*

Proof. As is known, $\text{Gau}_{x_0}P$ acts freely on the space of connections, i.e., on the sections of $p: C \rightarrow M$ and, in particular, on $\bar{\mathcal{S}}_\Lambda$ and on \mathcal{S}_Λ (e.g., see [6, Theorem 2.2.4], [13, III.C]). Nevertheless, the quotients $\bar{\mathcal{S}}_\Lambda/\text{Gau}_{x_0}P$ and $\mathcal{S}_\Lambda/\text{Gau}_{x_0}P$ may be singular, e.g., see [6, p. 134]. Because of this, we consider such structure from the set-theoretical point of view only.

Furthermore, the mapping $\varrho_{\text{Gau}_{x_0}P}$ is well defined as

$$\begin{aligned}\varrho\left(\Phi_C^{(1)} \circ \bar{s}\right) &= \left(p_{10} \circ \Phi_C^{(1)}\right) \circ \bar{s} \\ &= \left(\Phi_C \circ p_{10}\right) \circ \bar{s} \\ &= \Phi_C \circ \varrho(\bar{s}),\end{aligned}$$

and from Theorem 5.6 it follows that every $\bar{s} \in \bar{\mathcal{S}}_\Lambda$ can be uniquely written as $\bar{s} = j^1s + t$, where $s = \varrho(\bar{s})$ and $t \in S^2\Omega^1(M) \otimes \text{gau}P$. Hence, \bar{s} can be identified to the pair (s, t) , i.e., $\bar{\mathcal{S}}_\Lambda \cong \mathcal{S}_\Lambda \times S^2\Omega^1(M) \otimes \text{gau}P$. Recalling that $\Phi_C: C \rightarrow C$ is an affine-bundle morphism whose associated vector bundle is $\text{id}_{T^*M} \otimes \Phi_{\text{ad}}: T^*M \otimes \text{ad}P \rightarrow T^*M \otimes \text{ad}P$, we have

$$\begin{aligned}\Phi_C^{(1)} \circ \bar{s} &= \Phi_C^{(1)} \circ (j^1s + t) \\ &= J^1(\Phi_C \circ s) + \Phi_{\text{ad}} \cdot t,\end{aligned}$$

thus concluding the proof. \square

Theorem 5.12. *With the same notations as in Section 4 and the same assumptions as in Theorem 5.6, if $\bar{X} \in T_{\bar{s}}\bar{\mathcal{S}}_\Lambda$, then $(p_{10})_* \circ \bar{X} \in T_s\mathcal{S}_\Lambda$, where $s = p_{10} \circ \bar{s}$. Hence, the natural map $\varrho: \bar{\mathcal{S}}_\Lambda \rightarrow \mathcal{S}_\Lambda$ induces a linear map $\varrho_*: T_{\bar{s}}\bar{\mathcal{S}}_\Lambda \rightarrow T_s\mathcal{S}_\Lambda$, $\varrho_*(\bar{X}) = (p_{10})_* \circ \bar{X}$. Moreover, $\ker \varrho_* \cong \Gamma(S^2T^*M \otimes \text{ad}P)$.*

Proof. We first begin with the second Jacobi equation (4.4) for a gauge-invariant Lagrangian $L: J^1C \rightarrow \mathbb{R}$ and a Jacobi vector field

$$\bar{X} = v_i^\alpha \frac{\partial}{\partial A_i^\alpha} + v_{i,j}^\alpha \frac{\partial}{\partial A_{i,j}^\alpha}, \quad v_i^\alpha, v_{i,j}^\alpha \in C^\infty(M),$$

along a solution $\bar{s} \in \bar{\mathcal{S}}_\Lambda$. We have

$$\begin{aligned}0 &= \left(\frac{\partial^2 L}{\partial A_{r,i}^\alpha \partial A_{h,j}^\sigma} \circ \bar{s}\right) \left(\frac{\partial v_r^\alpha}{\partial x^i} - v_{r,i}^\alpha\right) \\ &\quad + \left(\frac{\partial s_r^\alpha}{\partial x^i} - \bar{s}_{r,i}^\alpha\right) \left(\frac{\partial^3 L}{\partial A_l^\beta \partial A_{r,i}^\alpha \partial A_{h,j}^\sigma} \circ \bar{s}\right) v_l^\beta \\ &\quad + \left(\frac{\partial s_r^\alpha}{\partial x^i} - \bar{s}_{r,i}^\alpha\right) \left(\frac{\partial^3 L}{\partial A_{r,i}^\alpha \partial A_{l,k}^\beta \partial A_{h,j}^\sigma} \circ \bar{s}\right) v_{l,k}^\beta,\end{aligned}\tag{5.11}$$

for any σ, h, j . From Theorem 5.6, $j^1s - \bar{s}$ is a symmetric tensor. Moreover, taking the formula (5.5) into account, the last two summands of (5.11)

vanish. We thus obtain

$$0 = \left(\frac{\partial^2 L}{\partial A_{r,i}^\alpha \partial A_{s,j}^\sigma} \circ \bar{s} \right) \left(\frac{\partial v_r^\alpha}{\partial x^i} - v_{r,i}^\alpha \right),$$

which, assuming the weak regularity of L , implies

$$v_{r,i}^\alpha = \frac{\partial v_r^\alpha}{\partial x^i} + t_{r,i}^\alpha, \quad (5.12)$$

where $t_{r,i}^\alpha$ are the components of a symmetric tensor $t \in \Gamma(S^2 T^* M \otimes \text{ad} P)$. From formula (2.1) and Lemma 2.1 we conclude $\bar{X} = X^{(1)} + t$, where $X^{(1)}$ is the one-jet prolongation of the vector field along s given by $X = v_i^\alpha \partial / \partial A_i^\alpha$.

Next, we consider the first Jacobi equation (4.3), from which we obtain

$$\begin{aligned} 0 = & \left(\frac{\partial^2 L}{\partial A_r^\alpha \partial A_q^\sigma} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial A_r^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} + (s_{t,i}^\beta - \bar{s}_{t,i}^\beta) \left(\frac{\partial^3 L}{\partial A_r^\alpha \partial A_q^\sigma \partial A_{t,i}^\beta} \circ \bar{s} \right) \right. \\ & - s_{t,j}^\gamma \left(\frac{\partial^3 L}{\partial A_r^\alpha \partial A_t^\gamma \partial A_{q,j}^\sigma} \circ \bar{s} \right) - \frac{\partial \bar{s}_{t,h}^\gamma}{\partial x^j} \left(\frac{\partial^3 L}{\partial A_r^\alpha \partial A_{t,h}^\gamma \partial A_{q,j}^\sigma} \circ \bar{s} \right) \Big) v_r^\alpha \\ & + \left(\frac{\partial^2 L}{\partial A_q^\sigma \partial A_{r,i}^\alpha} \circ \bar{s} - \frac{\partial^2 L}{\partial A_r^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} \right) \frac{\partial v_r^\alpha}{\partial x^i} - \left(\frac{\partial^2 L}{\partial A_{r,j}^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} \right) \frac{\partial v_{r,j}^\alpha}{\partial x^i} \\ & + \left((s_{t,j}^\beta - \bar{s}_{t,j}^\beta) \left(\frac{\partial^3 L}{\partial A_q^\sigma \partial A_{r,i}^\alpha \partial A_{t,j}^\beta} \circ \bar{s} \right) - \frac{\partial^3 L}{\partial x^j \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} \right. \\ & \left. - s_{t,j}^\beta \left(\frac{\partial^3 L}{\partial A_t^\beta \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} \right) - \frac{\partial \bar{s}_{t,h}^\beta}{\partial x^j} \left(\frac{\partial^3 L}{\partial A_{t,h}^\beta \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right) v_{r,i}^\alpha. \end{aligned} \quad (5.13)$$

Again taking the symmetry of the differences $s_{t,i}^\beta - \bar{s}_{t,i}^\beta$ (and their derivatives) into account, the previous equation reduces to the following:

$$\begin{aligned} 0 = & \left(\frac{\partial^2 L}{\partial A_r^\alpha \partial A_q^\sigma} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial A_r^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} - s_{t,j}^\gamma \left(\frac{\partial^3 L}{\partial A_r^\alpha \partial A_t^\gamma \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right. \\ & \left. - \frac{\partial^2 s_t^\gamma}{\partial x^h \partial x^j} \left(\frac{\partial^3 L}{\partial A_r^\alpha \partial A_{t,h}^\gamma \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right) v_r^\alpha \\ & + \left(\frac{\partial^2 L}{\partial A_q^\sigma \partial A_{r,i}^\alpha} \circ \bar{s} - \frac{\partial^2 L}{\partial A_r^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} \right) \frac{\partial v_r^\alpha}{\partial x^i} - \left(\frac{\partial^2 L}{\partial A_{r,j}^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} \right) \frac{\partial v_{r,j}^\alpha}{\partial x^i} \end{aligned}$$

$$\begin{aligned}
 & - \left(\frac{\partial^3 L}{\partial x^j \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} + s_{t,j}^\beta \left(\frac{\partial^3 L}{\partial A_t^\beta \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right. \\
 & \left. + \frac{\partial^2 s_t^\beta}{\partial x^h \partial x^j} \left(\frac{\partial^3 L}{\partial A_{t,h}^\beta \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right) v_{r,i}^\alpha.
 \end{aligned} \tag{5.14}$$

By taking derivatives in (5.12), we obtain

$$\frac{\partial v_{r,j}^\alpha}{\partial x^i} = \frac{\partial v_r^\alpha}{\partial x^j \partial x^i} + \frac{\partial t_{r,j}^\alpha}{\partial x^i},$$

and substituting these expressions into equation (5.14), again by virtue of the skew-symmetry of $\partial L / \partial A_{i,j}^\alpha$, we have

$$\begin{aligned}
 0 &= \left(\frac{\partial^2 L}{\partial A_r^\alpha \partial A_q^\sigma} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial A_r^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} - s_{t,j}^\gamma \left(\frac{\partial^3 L}{\partial A_r^\alpha \partial A_t^\gamma \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right. \\
 & - \left. \frac{\partial^2 s_t^\gamma}{\partial x^j \partial x^h} \left(\frac{\partial^3 L}{\partial A_r^\alpha \partial A_{t,h}^\gamma \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right) v_r^\alpha \\
 & + \left(\frac{\partial^2 L}{\partial A_q^\sigma \partial A_{r,i}^\alpha} \circ \bar{s} - \frac{\partial^2 L}{\partial A_r^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} \right) \frac{\partial v_r^\alpha}{\partial x^i} - \left(\frac{\partial^2 L}{\partial A_{r,j}^\alpha \partial A_{q,i}^\sigma} \circ \bar{s} \right) \frac{\partial v_r^\alpha}{\partial x^i \partial x^j} \\
 & - \left(\frac{\partial^3 L}{\partial x^j \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} + s_{t,j}^\beta \left(\frac{\partial^3 L}{\partial A_t^\beta \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right. \\
 & \left. + \frac{\partial^2 s_t^\beta}{\partial x^h \partial x^j} \left(\frac{\partial^3 L}{\partial A_{t,h}^\beta \partial A_{r,i}^\alpha \partial A_{q,j}^\sigma} \circ \bar{s} \right) \right) \frac{\partial v_r^\alpha}{\partial x^i}.
 \end{aligned}$$

Finally, taking the equations $\Omega \circ \bar{s} = \Omega \circ j^1 s$, $L = \bar{L} \circ \Omega$, into account, we conclude that \bar{s} can be replaced by $j^1 s$ into the previous equation and we can end the proof by simply applying Remark 4.6. \square

Corollary 5.13. *A Jacobi field $\bar{X} \in T_{\bar{s}} \bar{\mathcal{S}}_\Lambda$ is integrable if and only if the Jacobi field $\varrho_*(\bar{X}) = X \in T_s \mathcal{S}_\Lambda$, $s = p_{10} \circ \bar{s}$, is integrable.*

Proof. If $\bar{X} \in T_{\bar{s}} \bar{\mathcal{S}}_\Lambda$ is an integrable Jacobi field, then $\bar{X} = d/d\varepsilon|_{\varepsilon=0} \bar{s}_\varepsilon$ where $\bar{s}_\varepsilon = j^1 s_\varepsilon + t_\varepsilon$, with $\pi_{10} \circ \bar{s} = s_\varepsilon \in \mathcal{S}_\Lambda$, t_ε being a symmetric tensor. Then,

$$\varrho_*(\bar{X}) = (\pi_{10})_* \bar{X} = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \pi_{10} \circ \bar{s}_\varepsilon = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} s_\varepsilon.$$

Hence $X = \varrho_*(\bar{X})$ is integrable. Conversely, assume $X = d/d\varepsilon|_{\varepsilon=0}s_\varepsilon$, $s_\varepsilon \in \mathcal{S}_\Lambda$, and $\bar{X} \in (\varrho_*)^{-1}X$. We know that $\bar{X} = X^{(1)} + t$, where t is a symmetric tensor. Then, $\bar{X} = d/d\varepsilon|_{\varepsilon=0}\bar{s}_\varepsilon$ with $\bar{s}_\varepsilon = j^1s_\varepsilon + \varepsilon t \in \bar{\mathcal{S}}_\Lambda$, and \bar{X} is integrable. \square

6 H–C self-dual and anti-self-dual connections

Let (M, g) be a pseudo-Riemannian n -dimensional oriented connected manifold of signature (n^+, n^-) , $n = n^+ + n^-$, and let $\mathbf{v}_g = \sqrt{|\det(g_{ij})|}dx^1 \wedge \cdots \wedge dx^n$, $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$, be its pseudo-Riemannian volume form. The canonical duality isomorphism attached to g is denoted by $T_x M \rightarrow T_x^* M$, $X \mapsto X^\flat$, with inverse map $T_x^* M \rightarrow T_x M$, $w \mapsto w^\sharp$. Let $g^{(r)}$ be the metric on $\bigwedge^r T^* M$ given by $g^{(r)}(w^1 \wedge \cdots \wedge w^r, \bar{w}^1 \wedge \cdots \wedge \bar{w}^r) = \det(g((w^i)^\sharp, (\bar{w}^j)^\sharp))_{i,j=1}^r$.

Let $V \rightarrow M$ be a vector bundle. The Hodge star can be extended to V -valued forms as follows: $\star(\omega_r \otimes v) = (\star\omega_r) \otimes v$, $\forall \omega_r \in \bigwedge^r T_x^* M$, $\forall v \in V_x$.

Let $\pi: P \rightarrow M$ be a principal G -bundle and let $\pi_{\text{ad}P}: \text{ad}P \rightarrow M$ be the adjoint bundle; i.e., the bundle associated with P under the adjoint representation of G on its Lie algebra \mathfrak{g} . For every $B \in \mathfrak{g}$ and every $u \in P$, let $(u, B)_G$ be the coset of $(u, B) \in P \times \mathfrak{g}$ modulo G . A symmetric bilinear form $\langle \cdot, \cdot \rangle \in S^2 \mathfrak{g}^*$ is said to be invariant under the adjoint representation if the following equation holds: $\langle \text{Ad}_g B, \text{Ad}_g C \rangle = \langle B, C \rangle$, $\forall g \in G$, $\forall B, C \in \mathfrak{g}$. By taking derivatives on this equation we obtain $\langle [A, B], C \rangle + \langle B, [A, C] \rangle = 0$, $\forall A, B, C \in \mathfrak{g}$. If the group G is connected, then both equations above are equivalent.

Every symmetric bilinear form $\langle \cdot, \cdot \rangle \in S^2 \mathfrak{g}^*$ invariant under the adjoint representation induces a fibred metric $\langle\langle \cdot, \cdot \rangle\rangle: \text{ad}P \oplus \text{ad}P \rightarrow \mathbb{R}$ by setting

$$\langle\langle (u, B)_G, (u, C)_G \rangle\rangle = \langle B, C \rangle, \quad \forall u \in P, \forall B, C \in \mathfrak{g}. \quad (6.1)$$

We further assume that the pairing (6.1) is non-degenerate.

Every pseudo-Riemannian metric g on M and every fibred $\langle\langle \cdot, \cdot \rangle\rangle$ on $\text{ad}P$ induce a fibred metric on the vector bundle of $\text{ad}P$ -valued differential r -forms on M as follows: $((\alpha_r \otimes a, \beta_r \otimes b)) = g^{(r)}(\alpha_r, \beta_r) \langle\langle a, b \rangle\rangle$, $\forall \alpha_r, \beta_r \in \bigwedge^r T_x^* M$, and $\forall a, b \in (\text{ad}P)_x$. Moreover, the pairing (6.1) defines an exterior product (see [1]),

$$\begin{aligned} \wedge: \left(\bigwedge^\bullet T^* M \otimes \text{ad}P \right) \oplus \left(\bigwedge^\bullet T^* M \otimes \text{ad}P \right) &\rightarrow \bigwedge^\bullet T^* M, \\ (\alpha_q \otimes a) \wedge (\beta_r \otimes b) &= (\alpha_q \wedge \beta_r) \langle\langle a, b \rangle\rangle. \end{aligned}$$

Let $p: C \rightarrow M$ be the bundle of connections of P . According to the previous definitions, a pseudo-Riemannian metric g on M and an adjoint-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ allow one to define a quadratic Lagrangian density $\Lambda = L\mathbf{v}_g$ on J^1C by setting,

$$\begin{aligned}\Lambda(j_x^1s) &= ((\Omega^s(x), \Omega^s(x))) \mathbf{v}_g(x) \\ &= \Omega^s(x) \hat{\wedge} \star \Omega^s(x),\end{aligned}\tag{6.2}$$

where s is a local section of p defining a principal connection whose curvature form is denoted by Ω^s . In [3] it is proved that the E-L equations of the Lagrangians above are seen to be independent of the pairing (6.1) and they coincide with the classical Yang–Mills equations: $\nabla^s \star \Omega^s = 0$.

Theorem 6.1. *In addition to the hypotheses above, assume $\dim M = 4$ and g is a Riemannian metric. Let \mathcal{S}_Λ^+ (resp. \mathcal{S}_Λ^-) be the set of self-dual (resp. anti-self-dual) connections with respect to the Lagrangian density (6.2). A section \bar{s} of $p_1: J^1C \rightarrow M$ belongs to the fibre $\varrho^{-1}(s)$, with $s \in \mathcal{S}_\Lambda^+$ (resp. $s \in \mathcal{S}_\Lambda^-$) if and only if the following equations hold:*

$$\text{alt}(\bar{s} - j^1s) = 0,\tag{6.3}$$

$$\star(\Omega \circ \bar{s}) = (\Omega \circ \bar{s}) \quad (\text{resp. } \star(\Omega \circ \bar{s}) = -(\Omega \circ \bar{s})),\tag{6.4}$$

where $\text{alt}: \otimes^2 T^*M \otimes \text{ad}P \rightarrow \wedge^2 T^*M \otimes \text{ad}P$ is the alternating operator.

Remark 6.2. Equation (6.3) is a first-order differential equation, whereas (6.4) is a purely algebraic equation. The condition $\text{alt}(\bar{s} - j^1s) = 0$ is not specific of the Yang–Mills Lagrangian but general for any weakly regular gauge-invariant Lagrangian. In fact, it defines the subset of $\Gamma(p_1)$ given by

$$\{j^1s + t : s \in \Gamma(p), t \in \Gamma(S^2T^*M \otimes \text{ad}P)\}.\tag{6.5}$$

It is thus interesting to note that the group of equations really defined by self-dual (resp. anti-self-dual) connections, is not longer a differential equation but an algebraic constrain on the subset (6.5).

Proof of Theorem 6.1. As L is weakly regular, if $\bar{s} \in \bar{\mathcal{S}}_\Lambda$, then $\text{alt}(\bar{s} - j^1s) = 0$ by virtue of Remark 5.9. Moreover, as

$$-\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial A_{i,j}^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial A_i^\alpha} \circ \bar{s} = -\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial A_{i,j}^\alpha} \circ j^1s \right) + \frac{\partial L}{\partial A_i^\alpha} \circ j^1s,$$

as $\Omega \circ \bar{s} = \Omega \circ j^1 s$, we also obtain $\nabla^s \star \Omega^s = 0$, where $\Omega^s = \Omega \circ j^1 s$ and $s = p_{10} \circ \bar{s}$. Hence, the definition of a self-dual connection (i.e., $\star(\Omega \circ j^1 s) = (\Omega \circ j^1 s)$) can be written as $\star(\Omega \circ \bar{s}) = (\Omega \circ \bar{s})$ for every $\bar{s} \in \varrho^{-1}(\mathcal{S}_\Lambda^+)$. Similarly for \mathcal{S}_Λ^- . \square

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References

- [1] D. Bleecker, *Gauge theory and variational principles*, Global Analysis Pure and Applied Series A, 1, Addison-Wesley Publishing Co., Reading, MA, 1981.
- [2] M. Castrillón López and J. Muñoz Masqué, *The geometry of the bundle of connections*, Math. Z. **236** (2001), 797–811.
- [3] M. Castrillón López and J. Muñoz Masqué, *Independence of Yang–Mills equations with respect to the invariant pairing in the Lie algebra*, Int. J. Theor. Phys. **46**(4) (2007), 1020–1026.
- [4] M. Castrillón López and J. Muñoz Masqué, *Gauge-invariant characterization of Yang–Mills–Higgs equations*, Ann. Henri Poincaré **8** (2007), 203–217.
- [5] M. Castrillón López, J. Muñoz Masqué and T. Ratiu, *Gauge invariance and variational trivial problems on the bundle of connections*, Differ. Geom. Appl. **19**(2) (2003), 127–145.
- [6] P. Cotta-Ramusino and C. Reina, *The action of the group of bundle-automorphisms on the space of connections and the geometry of gauge theories*, J. Geom. Phys. **1**(3) (1984), 121–155.
- [7] P.L. García Pérez, *The Poincaré–Cartan invariant in the calculus of variations*, Symposia Mathematica, Volume XIV (Convegno di Geometria Simplettica e Fisica Matematica, INDAM, Rome, 1973), London, Academic Press, 1974, 219–246.
- [8] H. Goldschmidt and S. Sternberg, *The Hamilton–Cartan formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble) **23** (1973), 203–267.
- [9] J. Jost, X. Peng and G. Wang, *Variational aspects of the Seiberg–Witten functional*, Calc. Var. Partial Differ. Equ. **4**(3) (1996), 205–218.

- [10] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, New York, John Wiley & Sons, Inc. (Interscience Division), Vol I, 1963.
- [11] L. Lemaire and J.C. Wood, *Jacobi fields along harmonic 2-spheres in $\mathbb{C}P^2$ are integrable*, J. Lond. Math. Soc. (2) **66**(2) (2002), 468–486.
- [12] F. Lin and Ch. Wang, *The analysis of harmonics maps and their heat lows*, Singapore, World Scientific Publishing Co. Pte. Ltd., 2008.
- [13] P.K. Mitter and C.M. Viallet, *On the bundle of connections and the gauge orbit manifold in Yang–Mills theory*, Comm. Math. Phys. **79**(4) (1981), 457–472.
- [14] J. Muñoz Masqué and L.M. Pozo Coronado, *Parameter invariance in field theory and the Hamiltonian formalism*, Fortschr. Phys. **48** (2000), 361–405.
- [15] Y. Okawa, *Derivative corrections to Dirac-Born-Infeld Lagrangian and non-commutative gauge theory*, Nucl. Phys. B **566**(1–2) (2000), 348–362.
- [16] T. Suzuki, *Born-Infeld action in $(n + 2)$ -dimension, the field equation and a soliton solution*, Lett. Math. Phys. **47**(2) (1999), 159–171.

