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Semidualizing Module and Gorenstein Homological Dimensions

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Abstract

Let C be a semidualizing module over any commutative ring R. We investigate the semidualizing module C with finite injective dimension. In particular, we obtain some equivalent characterizations of C under the trivial extension of R by C. Moreover, we get that the supremum of the C-Gorenstein projective dimensions of all R-modules and the supremum of the C-Gorenstein injective dimensions of all R-modules are equal. Hence the C-Gorenstein global dimension of the ring R is definable. At last, we consider the weak C-Gorenstein global dimension.

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Keywords: semidualizing module; trivial extension; C-Gorenstein global dimension.

1 Introduction

Throughout the note, R is always a commutative ring and C is a semidualizing R-module. The notion of semidualizing module was studied more than 27 years ago under other names by, e.g., Foxby [4] (PG-module of rank 1), Golod [5] (suitable module) and Vasconcelos [15] (spherical module), which can be viewed as a generalization of dualizing module and free module of rank one. Relative algebra with respect to the semidualizing module C has caught many authors' attention. In this field, projective (injective, flat) modules are generalized to C-projective (injective, flat) modules. Recently, H. Holm, P. Jørgensen, S. Sather-Wagstaff, and D. White extended the Gorenstein projective (injective, flat) modules to C-Gorenstein projective (injective, flat) modules. Note that if the semidualizing module C is the regular module R, then C-Gorenstein projective (injective or flat) modules are just Gorenstein projective (injective or flat) and the classical homological algebra is generalized to the Gorenstein homological algebra with respect to the semidualizing module C, for this topic, we refer the readers to see [8, 12, 16].

In classical homological algebra, we use the projective (injective, flat) modules to resolve an R-module, and we get the definitions of homological dimensions. And the homological dimensions can be used to characterize some rings, which provided for us a new method to study the classical ring theory. As the counterpart, many authors studied the

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Gorenstein homological dimensions to characterize the Gorenstein rings. Recall that a ring R is called n-Gorenstein for a nonnegative integer n, if and only if R is both left and right noetherian and the self-injective dimension of R on both left and right is no more than n. Enochs and Jenda [3] introduced the Gorenstein ring and showed the homological properties of modules over such rings, see [3, Chapter 9]. As the generalization of free modules, the semidualizing modules can replace the regular module R in many cases. Hence it is natural for us to consider the homological property of a ring when the semidualizing module C has finite injective dimension.

Throughout this paper, we use $pd_R(M)$, $id_R(M)$, *and* $fd_R(M)$ to denote, respectively, the classical projective, injective, and flat dimension of R-module M; we use $Gpd_R(M)$, $Gid_R(M)$, and $Gfd_R(M)$ to denote, respectively, the Gorenstein projective, injective, and flat dimension of M; furthermore, we use $C\text{-}Gpd_R(M)$, $C\text{-}Gid_R(M)$ and $C\text{-}Gfd_R(M)$ to denote, respectively, the C-Gorenstein projective, injective, and flat dimension of M.

We show the following theorem over any commutative ring R, see Theorems 3.4 and 3.8.

Theorem For any nonnegative integer n, if both $sup\{C\text{-}Gid_R(M) \mid M \in ModR\}$ and $sup\{C\text{-}Gpd_R(M) \mid M \in ModR\}$ are finite, then the following are equivalent.

(1) $id_R(C) \leq n$, i.e., C is dualizing;

(2) $\sup\{C\text{-}Gpd_R(M) \mid M \in ModR\} \leq n$;

(3) $\sup\{C\text{-}Gid_R(M) \mid M \in ModR\} \leq n$.

Moreover, we show that the C-Gorenstein global dimension of R, denoted by G_C-$gldim(R)$, which is defined following Corollary 3.6, can be computed by a simple formula.

Corollary Let C be a semidualizing R-module. If $G_C\text{-}gldim(R) < \infty$, then $G_C\text{-}gldim(R) = sup\{C\text{-}Gpd_R(R/I) \mid I \text{ is an ideal of } R\}$.

At the end, we consider the weak C-Gorenstein global dimension, $\sup\{C\text{-}Gfd_R(M) \mid M \in ModR\}$, and denote it by $wG_C\text{-}gldim(R)$. Obviously, it is a generalization of weak Gorenstein global dimension of R. We compare the C-Gorenstein global dimension with the weak C-Gorenstein global dimension of ring R, by our main theorem, we get the following result.

Theorem Let C be a semidualizing R-module. Then $wG_C\text{-}gldim(R) \leq G_C\text{-}gldim(R)$ and when R is Noetherian, they are equal.

It is worthy to note that as an R-module, R is semidualizing. So if we set C to be R, then we recover the counterpart results in homological algebra and Gorenstein homological algebra. But the proofs of results in this paper are not trivial generalizations of the existing proofs.

2 Preliminaries

In this section, we recall a number of definitions, notions and results which will be used throughout the paper. For unexplained concepts and notations, we refer the reader to [8, 13, 16].

Definition 2.1. *[16, 1.8]* An R-module C is called *semidualizing* if

(1) C admits a degreewise finitely generated projective resolution;

(2) the natural homothety map $R \longrightarrow \mathrm{Hom}_R(C,C)$ is an isomorphism;

(3) $\mathrm{Ext}_R^{\geq 1}(C,C) = 0$.

Let C be a semidualizing R-module. We denote the class of C-flat R-modules by $\mathcal{F}_C(R)$, the C-projective R-modules by $\mathcal{P}_C(R)$ and the C-injective R-modules by $\mathcal{I}_C(R)$, respectively. By [10, Definition 5.1], we have that

(1) $\mathcal{F}_C(R) = \{C \otimes_R F \mid F\ is\ flat\}$;

(2) $\mathcal{P}_C(R) = \{C \otimes_R P \mid P\ is\ projective\}$;

(3) $\mathcal{I}_C(R) = \{\mathrm{Hom}_R(C,I) \mid I\ is\ injective\}$.

As the generalization of Gorenstein injective (projective, flat) modules, Holm and Jørgensen defined the C-Gorenstein injective (projective, flat) modules over commutative Noetherian ring R, [8, Definition 2.7], in which the definition of C-Gorenstein projective modules has been extended to the non-Noetherian ring by White[16], where she called G_C-projective modules, we refer the reader to see [8, 16].

Remark 2.2. By [8, Example 2.8], we know that projective modules are C-Gorenstein projective, injective modules are C-Gorenstein injective and flat modules are C-Gorenstein flat. Hence every R-module M admits C-Gorenstein (projective, injective and flat) resolution. It is easy to see from the proof of [8, Example 2.8] that the condition of R being Noetherian is not needed.

By [8, Definition 9], for any R-module M, we have the C-Gorenstein projective (injective, flat) dimension, which was denoted by $C\text{-}Gpd_R(M)$, ($C\text{-}Gid_R(M)$, $C\text{-}Gfd_R(M)$).

At last, we recall the definition of trivial extension:

Definition 2.3. Let R be a ring and C a semidualizing module. The direct sum $R \oplus C$ can be equipped with the product:

$$(r,c) \cdot (r',c') = (rr', rc' + r'c).$$

This turns $R \oplus C$ into a ring which is called the *trivial extension* of R by C and denoted by $R \ltimes C$.

There are canonical ring homomorphisms, $R \rightleftarrows R \ltimes C$, which enable us to view R-modules as $R \ltimes C$-modules, and vice versa. Hence as R-module $R \ltimes C \cong R \oplus C$.

For any R-module M, Holm and Jørgensen demonstrated the relation between the C-Gorenstein homological dimensions over ring R and the Gorenstein homological dimensions over ring $R \ltimes C$, see [8, Theorem 2.16]. Note that the conclusion in [8, Theorem 2.16] also holds true for non-Noetherain rings, as we have the following lemma:

Lemma 2.4. *Let R be any commutative ring and I an injective R-module. For any R-module M, we have $Ext^i_{R\ltimes C}(\mathrm{Hom}_R(R \ltimes C, I), M) \cong Ext^i_R(\mathrm{Hom}_R(C,I), M)$ for all $i \geq 0$.*

Proof. By Definition 2.1, C has a degreewise finitely generated projective resolution. By Definition 2.3, there exist an R-module isomorphism $R\ltimes C\cong R\oplus C$, so as R-module $R\ltimes C$ admits a degreewise finitely generated projective resolution. So by [3, Theorem 3.2.11], we have the following isomorphisms,

$$\mathrm{Hom}_R(R\ltimes C,I)\cong \mathrm{Hom}_R(\mathrm{Hom}_R(R\ltimes C,C),I)\cong (R\ltimes C)\otimes_R \mathrm{Hom}_R(C,I).$$

As an R-module, $\mathrm{Hom}_R(C,I)$ has the following projective resolution

$$\mathbb{P}=\cdots\to P_1\to P_0\to \mathrm{Hom}_R(C,I)\to 0.$$

Since $\mathrm{Hom}_R(C,I)\in\mathcal{A}_C(R)$, $Tor^R_{i\geq 1}(R\ltimes C,\ \mathrm{Hom}_R(C,I))=0$. Thus applying the functor $(R\ltimes C)\otimes_R -$ to $\mathbb{P}$, we get another exact sequence

$$\cdots\to (R\ltimes C)\otimes_R P_1\to (R\ltimes C)\otimes_R P_0\to (R\ltimes C)\otimes_R \mathrm{Hom}_R(C,I)\to 0.$$

By [8, Lemmas 1.5], we know that $(R\ltimes C)\otimes_R P_j$ is a projective $R\ltimes C$-module for any $j\geq 0$, thus the above exact sequence is the projective resolution of the $R\ltimes C$-module $(R\ltimes C)\otimes_R \mathrm{Hom}_R(C,I)$. Hence we have that

$$\begin{aligned}
&Ext^i_{R\ltimes C}(\mathrm{Hom}_R(R\ltimes C,I),M)\\
&\cong Ext^i_{R\ltimes C}((R\ltimes C)\otimes_R \mathrm{Hom}_R(C,I),M)\\
&= H_i\mathrm{Hom}_{R\ltimes C}((R\ltimes C)\otimes_R \mathbb{P},M)\\
&\cong H_i\mathrm{Hom}_R(\mathbb{P},M)\\
&= Ext^i_R(\mathrm{Hom}_R(C,I),M).
\end{aligned}$$

Remark 2.5. By [8, Lemma 1.4], each injective $R\ltimes C$-module is a direct summand in a module $\mathrm{Hom}_R(R\ltimes C,I)$ for an injective R-module I. Hence, when R is a commutative ring, $C\text{-}Gid_R(M)=Gid_{R\ltimes C}(M)$ for any R-module M by [6, Theorem 2.22]. Similarly, we can prove $C\text{-}Gpd_R(M)=Gpd_{R\ltimes C}(M)$ and $C\text{-}Gfd_R(M)=Gfd_{R\ltimes C}(M)$ over any commutative ring R.

3 Main results

We give our main results in this section. Firstly, we give a characterization of semidualizing module C.

For any ring R, we denote $Ggldim(R)$ by the Gorenstein global dimension of R. By [1, Theorem 1.1],

$$Ggldim(R)=sup\{Gpd_R(M)\mid M\in ModR\}=sup\{Gid_R(M)\mid M\in ModR\}.$$

Proposition 3.1. *Let C be a semidualizing R-module and n a non-negative integer. If $id_R(C)\leq n$, then*

(1) $\mathcal{I}_C\text{-}id_R(Q)=id_R(C\otimes_R Q)\leq n$ for every projective R-module Q;

(2) $id_{R\ltimes C}(P)\leq n$ for every projective $R\ltimes C$-module P.

Proof. (1). Let Q be a projective R-module. As $C\cong C\otimes R$, the C-projective R-module $C\otimes Q$ is the summand of the any direct sum of the $C\otimes R$. Since $id_R(C)\leq n$, we have $id_R(C\otimes Q)\leq n$. On the other hand, we denote the C-injective dimension of Q by $\mathcal{I}_C\text{-}id_R(Q)$, by[14, Theorem 2.11], we have that $\mathcal{I}_C\text{-}id_R(Q)=id_R(C\otimes_R Q)$. So $\mathcal{I}_C\text{-}id_R(Q)\leq n$.

(2). By [8, Lemma 1.5], we only need to show $id_{R\ltimes C}((R\ltimes C)\otimes_R Q)\leq n$. In fact, there are R-module isomorphisms

$$(R\ltimes C)\otimes_R Q\cong (R\oplus C)\otimes_R Q\cong Q\oplus (C\otimes_R Q).$$

By [8, Example 2.8], C-injective R-modules are C-Gorenstein injective R-modules. So

$$C-Gid_R(Q) \leq \mathcal{I}_C - id_R(Q) \leq n \qquad and \qquad C-Gid_R(C \otimes_R Q) \leq id_R(C \otimes_R Q) \leq n.$$

Thus, $C\text{-}Gid_R((R \ltimes C) \otimes_R Q) = C\text{-}Gid_R(Q \oplus (C \otimes_R Q) \leq n$. By Remark 1.5, we get that $Gid_{R\ltimes C}((R \ltimes C) \otimes_R Q) \leq n$. On the other hand, $(R \ltimes C) \otimes_R Q$ is projective as an $R \ltimes C$-module, so by [7, Theorem 2.2], we have that

$$id_{R\ltimes C}((R \ltimes C) \otimes_R Q) = Gid_{R\ltimes C}((R \ltimes C) \otimes_R Q).$$

Therefore, $id_{R\ltimes C}((R \ltimes C) \otimes_R Q) \leq n$.

If we set $C = R$, the following theorem is exactly [3, Proposition 9.1.7].

Theorem 3.2. *Let R be any commutative ring such that $Ggldim(R \ltimes C) < \infty$. The following are equivalent for a non-negative integer n,*

(1) $id_R(C) \leq n$;

(2) $id_{R\ltimes C}(P) \leq n$ for every projective $R \ltimes C$-module P;

(3) $pd_{R\ltimes C}(E) \leq n$ for every injective $R \ltimes C$-module E.

Proof. (1) $\Rightarrow$ (2). It follows by Proposition 3.1(2).

(2) $\Rightarrow$ (3). For any $R \ltimes C$-module M and all $i > n$, we have $Ext^i_{R\ltimes C}(M,P) = 0$ by (2). Since $Ggldim(R \ltimes C) < \infty$, $sup\{Gpd_{R\ltimes C}(M) \mid M \in Mod(R \ltimes C)\} < \infty$ by [1, Theorem 1.1]. So $Gpd_{R\ltimes C}(M) \leq n$ by [6, Theorem 2.20] and $Gid_{R\ltimes C}(M) \leq n$ by [1, Theorem 1.1]. Hence $pd_{R\ltimes C}(E) \leq n$ for every injective $R \ltimes C$-module E by [6, Theorem 2.22].

(3) $\Rightarrow$ (1). By (3), $Ext^i_{R\ltimes C}(E,N) = 0$ for all $R \ltimes C$-module N and all $i > n$. Since $Ggldim(R \ltimes C) < \infty$, $Gid_{R\ltimes C}(N) \leq n$ by [6, Theorem 2.22]. So $Gpd_{R\ltimes C}(N) \leq n$. For any R-module M, then $C\text{-}Gpd_R(M) = Gpd_{R\ltimes C}(M) \leq n$ by Remark 2.5. Hence $id_R(T) \leq n$ for any C-projective R-module T by [16, Proposition 2.12]. As $C \cong C \otimes R$, we get $id_R(C) \leq n$.

Now, we consider the C-Gorenstein global dimension of R.

Proposition 3.3. *Let C be a semidualizing R-module and n a nonnegative integer.*

(1) If $C\text{-}Gpd_R(M) \leq n$ for every R-module M, then $pd_{R\ltimes C}(E) \leq n$ for every injective $R \ltimes C$-module E.

(2) If $C\text{-}Gid_R(M) \leq n$ for every R-module M, then $id_{R\ltimes C}(P) \leq n$ for every projective $R \ltimes C$-module P.

Proof. We only prove (1) and the proof of (2) is similar.

To show that $pd_{R\ltimes C}(E) \leq n$ for every injective $R \ltimes C$-module E, we only need to show $pd_{R\ltimes C}(\mathrm{Hom}_R(R \ltimes C, I)) \leq n$ for any injective R-module I by [8, Lemma 1.4]. In fact, since R is commutative, $\mathrm{Hom}_R(R \ltimes C, I)$ is an R-module. So we have that $C\text{-}Gpd_R(\mathrm{Hom}_R(R \ltimes C, I)) \leq n$. Thus $Gpd_{R\ltimes C}(\mathrm{Hom}_R(R \ltimes C, I)) \leq n$ by Remark 2.5. Following from [7, Theorem 2.1], we get that $pd_{R\ltimes C}(\mathrm{Hom}_R(R \ltimes C, I)) = Gpd_{R\ltimes C}(\mathrm{Hom}_R(R \ltimes C, I)) \leq n$.

Theorem 3.4. *For any nonnegative integer n, if both $sup\{C\text{-}Gid_R(M) \mid M \in ModR\}$ and $sup\{C\text{-}Gpd_R(M) \mid M \in ModR\}$ are finite, then the following are equivalent:*

(1) $id_R(C) \leq n$, *i.e.,* C *is dualizing;*

(2) $sup\{C\text{-}Gpd_R(M) \mid M \in ModR\} \leq n$*;*

(3) $sup\{C\text{-}Gid_R(M) \mid M \in ModR\} \leq n$*.*

Proof. (1) $\Rightarrow$ (2). Since $id_R(C) \leq n$, $id_{R\ltimes C}(P) \leq n$ for every projective $R\ltimes C$-module P by Proposition 2.1(2). So we have that $Ext^{i>n}_{R\ltimes C}(M,P) = 0$ for any R-module M. But C-$Gpd_R(M) < \infty$, so $Gpd_{R\ltimes C}(M) < \infty$ by Remark 2.5. Thus $Gpd_{R\ltimes C}(M) \leq n$ by [6, Theorem 2.20]. Therefore $C\text{-}Gpd_R(M) \leq n$ by [8, Theorem 2.16], and (2) follows.

(2) $\Rightarrow$ (1). Since $C \cong C \otimes_R R$, C is C-projective. So $Ext^{i>n}_R(M,C) = 0$. Hence $id_R(C) \leq n$ by [16, Proposition 2.12].

(2) $\Rightarrow$ (3). By (2), $C\text{-}Gpd_R(M) \leq n$ for every R-module M. So $pd_{R\ltimes C}(E) \leq n$ for every injective $R\ltimes C$-module E by Proposition 3.3(1). Hence we have that $Ext^{i>n}_{R\ltimes C}(E,M) = 0$. As $C\text{-}Gid_R(M) < \infty$, $Gid_{R\ltimes C}(M) < \infty$. By [6, Theorem 2.22], $Gid_{R\ltimes C}(M) \leq n$. Thus C-$Gid_R(M) \leq n$ also by [8, Theorem 2.16] and (3) follows.

(3) $\Rightarrow$ (2). By (3), $C\text{-}Gid_R(M) \leq n$ for every R-module M. So $id_{R\ltimes C}(P) \leq n$ for every projective $R\ltimes C$-module P by Proposition 3.3(2). Hence $Ext^{i>n}_{R\ltimes C}(M,P) = 0$. As $C\text{-}Gpd_R(M) < \infty$, $Gpd_{R\ltimes C}(M) < \infty$. By [6, Theorem 2.20], $Gpd_{R\ltimes C}(M) \leq n$. Thus $C\text{-}Gpd_R(M) \leq n$ also by [8, Theorem 2.16] and (2) follows.

Remark 3.5. By Theorem 3.4, we know that if both $sup\{C\text{-}Gid_R(M) \mid M \in ModR\}$ and $sup\{C\text{-}Gpd_R(M) \mid M \in ModR\}$ are finite, then $sup\{C\text{-}Gid_R(M) \mid M \in ModR\} = sup\{C\text{-}Gpd_R(M) \mid M \in ModR\}$. When $sup\{C\text{-}Gid_R(M) \mid M \in ModR\}$ is infinite, then there exists an R-module M, such that $C\text{-}Gid_R M = \infty$, then $Gid_{R\ltimes C} M = \infty$ by Remark 2.5. Hence $sup\{Gid_{R\ltimes C}(M) \mid M \in ModR\} = \infty$. But $sup\{Gid_{R\ltimes C}(M) \mid M \in ModR\} = sup\{Gpd_{R\ltimes C}(M) \mid M \in ModR\}$ by [1, Theorem 1.1], so there exist an $R\ltimes C$-module N such that $Gpd_{R\ltimes C}(N) = \infty$, thus C-$Gpd_R N = \infty$ also by Remark 2.5. So $sup\{C\text{-}Gpd_R(M) \mid M \in ModR\} = \infty$ and vice versa.

Therefore we get the following equality.

Corollary 3.6. *Let R be any commutative ring and C a semidualizing R-module. Then*
$$sup\{C\text{-}Gid_R(M) \mid M \text{ is an } R\text{-module}\} = sup\{C\text{-}Gpd_R(M) \mid M \text{ is an } R\text{-module}\}.$$

We call the common value in the above Corollary C-Gorenstein global dimension of R and denote it by $G_C\text{-}gldim(R)$. It is easy to see that C-Gorenstein global dimension extends Gorenstein global dimension.

In classical homological algebra, the global dimension of a ring R, denoted by $gldim(R)$, can be computed via the following formula:
$$gldim(R) = sup\{pd(R/I) \mid I \text{ is an ideal of } R\}.$$
And by Theorem 3.4, the C-Gorenstein global dimension of R can also be computed via a similar formula.

Corollary 3.7. *Let C be a semidualizing R-module. If $G_C\text{-}gldim(R) < \infty$, then $G_C\text{-}gldim(R) = sup\{C\text{-}Gpd_R(R/I) \mid I \text{ is an ideal of } R\}$.*

Proof. It is clear that

$$sup\{C-Gpd_R(R/I) \mid I \text{ } is \text{ } an \text{ } ideal \text{ } of \text{ } R\} \leq G_C-gldim(R).$$

Let $sup\{C\text{-}Gpd_R(R/I) \mid I \text{ } is \text{ } an \text{ } ideal \text{ } of \text{ } R\} = n < \infty$. Since $C \cong C \otimes_R R$, C is C-projective. So by [16, Proposition 2.12], we have that $Ext_R^{n+1}(R/I, C) = 0$ for every R ideal I. Consider the injective resolution of C,

$$0 \to C \to E_0 \to \cdots \to E_{n-1} \to T^{'} \to 0.$$

Applying $\mathrm{Hom}_R(R/I, -)$, we get that $Ext_R^1(R/I, T^{'}) \cong Ext_R^{n+1}(R/I, C) = 0$. By [17, Theorem 9.11], we know that $T^{'}$ is injective. So $id_R(C) \leq n$. By Theorem 3.4, we have that $sup\{C\text{-}Gpd_R(M) \mid M \in ModR\} \leq n$ and thus G_C-gldim(R)$\leq n$ by Corollary 3.6. Hence G_C-$gldim(R) = sup\{C\text{-}Gpd_R(R/I) \mid I \text{ } is \text{ } an \text{ } ideal \text{ } of \text{ } R\}$.

At last, we give the definition of the weak C-Gorenstein dimension of ring R and we denote it by wG_C-$gldim(R)$, i.e., wG_C-$gldim(R)$=sup$\{C\text{-}Gfd_R(M) \mid M \in ModR\}$. Obviously, it is the generalization of weak Gorenstein global dimension of R. By Remark 2.2, flat modules are C-Gorenstein flat, hence wG_C-$gldim(R) \leq wgldim(R)$, where $wgldim(R)$ denotes the weak global dimension of R. Moreover, we show the connection between the C-Gorenstein global dimension and the weak C-Gorenstein dimension of ring R.

Theorem 3.8. *Let C be a semidualizing R-module. Then wG_C-$gldim(R) \leq G_C$-$gldim(R)$ and when R is Noetherian, they are equal.*

Proof. If $G_C - gldim(R) = \infty$, it is obviously that $wG_C - gldim(R) \leq G_C - gldim(R)$. If G_C-$gldim(R) = n < \infty$, then $id_R(C) \leq n$ by Theorem 3.4 and Corollary 3.6. Thus $fd_R(Hom(C,E)) \leq n$ and $Tor_R^{i>n}(Hom(C,E), M) = 0$ for every R-module M. On the other hand, by [1, Corollary 1.2(2)] and Remark 2.5, $wG_C - gldim(R) < \infty$. Hence C-$Gfd_R M \leq n$ for every R-module M by [6, Theorem 3.14]. Therefore wG_C-$gldim(R) \leq n$ and wG_C-$gldim(R) \leq G_C$-$gldim(R)$.

When R is Noetherian, we will show that G_C-$gldim(R) \leq wG_C$-$gldim(R)$. In fact, suppose that wG_C-$gldim(R) = n$ for some nonnegative integer n, then for every finitely generated R-module M, we get that C-$Gfd_R(M) \leq n$. Consider the projective resolution of M: $0 \to G_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$, with P_i projective for $0 \leq i \leq n-1$. By the definition of C-Gorenstein flat dimension, we know that G_n is C-Gorenstein flat. On the other hand, R is Noetherian and M is finitely generated, so G_n is finitely presented C-Gorenstein flat. Thus as $R \ltimes C$-module, G_n is finitely presented Gorenstein flat by [8, Theorem 2.16]. Moreover, R is Noetherian implies that $R \ltimes C$ is Noetherian by [11, Page 87]. By [2, Proposition 1.3], we conclude that G_n is a Gorenstein projective $R \ltimes C$-module. So G_n is a C-Gorenstein projective R-module also by [8, Theorem 2.16]. Thus C-$Gpd_R(M) \leq n$. Particularly, we have that C-$Gpd_R(R/I) \leq n$ for any R ideal I. So by Corollary 3.7, we have that $G_C - gldim(R) \leq n$. Hence $G_C - gldim(R) \leq wG_C - gldim(R)$ and so

$$G_C - gldim(R) = wG_C - gldim(R).$$

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References

[1] D. Bennis and N. Mahdou, Global Gorenstein dimensions, *Proc. Amer. Math. Soc.*, **138** (2010), 461-465

[2] D. Bennis and N. Mahdou, Strongly Gorenstein projective, injective, and flat modules, *J. pure and applied algebra*, **210** (2007), 437-445

[3] E. E. Enochs, O. M. G. Jenda, *Relative homological algebra*, Exp. Math 30, de Gruyter 2000.

[4] H.-B. Foxby, Gorenstein modules and related modules, *Math. Scand*, **31** (1972), 267-284

[5] E. S. Golod, G-dimension and generalized perfect ideals, *Trudy Mat. Inst. Steklov*, **165** (1984), 62-66.

[6] H. Holm, Gorenstein homological dimensions, *J.Pure Appl. Algebra*, **189** (2004), 167-193.

[7] H. Holm, Rings with finite Gorenstein injective dimension, *Proc. Amer. Math. Soc.*, **132** (2004), 1279-1283.

[8] H. Holm and P. JØrgensen, Semi-dualizing modules and related Gorenstein homologica dimensions, *J. Pure Appl. Algebra*, **205(2)** (2006), 423-445.

[9] H. Holm and P. JØrgensen, Cohen-Macaulay injective,projective, and flat dimension. (2004), available from http://www.arxiv.org/abs/math. AC/0405523.

[10] H. Holm and D. White, Foxby equivalence over associative rings, *J. Math. Kyoto Univ.*, **47** (2008), 781-808.

[11] R. M. Fossum, P. A. Griffith , I. Reiten, *Trivial extensions of abelian categories*, Berlin, Springerverlag 1975.

[12] S. Sather-Wagstaff, T.Sharif and D. White, Tate cohomology with respect to semidualizing modules, *J. Algebra* **324** (2010), 2336-2368.

[13] S. Sather-Wagstaff, T.Sharif and D. White, Comparison of relative cohomology theories with respect to semidualizing modules, *Math. Z.*, **264(3)** (2010), 571-600.

[14] R. Takahashi and D. White, Homological aspects of semidualizing modules, *Math. Scand*, **106(1)** (2010), 5-22.

[15] W. V. Vasconcelos, *Divisor theory in module categories*, North-Holland Publishing Co., Amsterdam: Noth-Holland Math. Stud, 1974.

[16] D. White, Gorenstein projective dimension with respect to a semidualizing module, *J. Commut. Algebra*, **2(1)** (2010), 111-137.

[17] C. A. Weibel, *An introduction to homological algebra*, Cambridge Stud. Adv. Math., vol. 38, Cambridge University Press, Cambridge 1994.