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AN EXPLICIT RESULT FOR $|L(1+it,\chi)|$

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Abstract: We give an explicit upper bound for non-principal Dirichlet L-functions on the line $s = 1 + it$. This result can be applied to improve the error in the zero-counting formulae for these functions.

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1. Introduction

Let χ be a non-principal Dirichlet character to the modulus $q \geqslant 1$ and let $L(s,\chi)$ denote its associated L-function (see Davenport's text [3] for a good introduction to this area). Our objective here is to establish some explicit bounds for $|L(1+it,\chi)|$; the main result is given in the following theorem. It should be noted, however, that the theorems of the present paper are, in fact, valid for all $\Re(s) \geqslant 1$.

Theorem 1.1. *Let $q \geqslant 1$ be a non-principal modulus with associated Dirichlet character χ. Then*

$$|L(1+it,\chi)| < \frac{\varphi(q)}{q}\left(\log t - \frac{1}{2}\right) + \log q + \gamma$$

for all $t \geqslant 71$, where $\varphi(q)$ is Euler's totient function and γ is Euler's constant.

We prove this using a method due to Backlund [2]. Of course, we would not expect the above theorem to hold for values of t which are close to zero. To this end, we prove the following theorem using the method of partial summation.

Theorem 1.2. *Let $q \geqslant 1$ be a non-principal modulus with associated Dirichlet character χ. Then*

$$|L(1+it,\chi)| < \log(t + 14/5) + \log q + 1$$

for all $t > 0$.

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It is then a straightforward exercise to combine Theorems 1.1 and 1.2 to get a good result which holds for all $t > 0$. One could simply take the minimum of the bounds given in the above two theorems.

These are not particularly deep results, though the author could not find any explicit bounds in the literature. It should be noted that bounds of this type can be used to explicitly estimate $N(T,\chi)$, the number of zeroes of $L(s,\chi)$ in the critical strip with height at most T. One could use Theorems 1.1 and 1.2 to improve the recent work of Trudgian [8] in this way. Bounds for $N(T,\chi)$ can then be used for explicit estimates involving the prime number theorem for arithmetic progressions; see the work of McCurley [5] for a demonstration of this.

In particular, for such work, one requires a bound on $L(s,\chi)$ in the critical strip $0 \leqslant \Re(s) \leqslant 1$. The equation before (6.2) in Rademacher [7] states that

$$|L(s,\chi)| \leqslant \left(\frac{2\pi}{q}\right)^{\sigma-1/2} |1+s|^{1/2-\sigma} |L(1-s,\overline{\chi})|.$$

It is a straightforward task to substitute $s = it$ into the above inequality with a bound for $|L(1+it,\chi)|$ to furnish a bound for $|L(it,\chi)|$. One can then apply Theorem 2 of the same paper to establish a uniform bound for $|L(s,\chi)|$ on the entire critical strip.

We will prove Theorems 1.1 and 1.2 in the following section. We only need to consider $q \geqslant 2$, noting that the case $q = 1$ corresponds to the Riemann zeta-function. Explicit bounds in this case have been given by Trudgian [9] and Ford [4].

To construct such a bound for a Dirichlet L-function on the line $s = 1+it$, we first consider the Hurwitz zeta-function

$$\zeta(s,c) = \sum_{n=0}^{\infty} \frac{1}{(n+c)^s}$$

for $c \in (0,1]$ and $\Re(s) > 1$. We then have that

$$\begin{aligned} L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} &= \sum_{1 \leqslant a \leqslant q} \sum_{n \equiv a \bmod q} \frac{\chi(n)}{n^s} \\ &= \sum_{1 \leqslant a \leqslant q} \chi(a) \sum_{n \equiv a \bmod q} \frac{1}{n^s} \\ &= \sum_{1 \leqslant a \leqslant q} \frac{\chi(a)}{q^s} \zeta\left(s, \frac{a}{q}\right) \end{aligned} \tag{1}$$

where the manipulations are justified by the absolute convergence of the Dirichlet series for $\mathrm{Re}(s) > 1$. The connection between $L(s,\chi)$ and $\zeta(s,c)$ is given hearty discussion in Chapter 12 of Apostol's [1] introductory text, where he uses the above equation to define $L(s,\chi)$ for *all* $s \in \mathbb{C}$ (where χ is, importantly, non-principal). It so follows from (1) that

$$|L(s,\chi)| \leqslant \frac{1}{q} \sum_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \left| \zeta\left(s, \frac{a}{q}\right) \right|. \tag{2}$$

Therefore, it remains to develop bounds for the Hurwitz zeta-function. The following result is furnished.

Theorem 1.3. *Let $c \in (0,1]$. Then*

$$|\zeta(1+it,c)| < \log t + \frac{1}{c}$$

for all $t \geqslant 50$.

One could substitute the result of Theorem 1.3 into (2) to establish a bound for $|L(1+it,\chi)|$. It turns out, however, that one gets a stronger bound via a symmetry argument suggested by the anonymous referee. As such, we bound the sum

$$\zeta(1+it,a/q)+\zeta(1+it,(q-a)/q)$$

for each $a \leqslant q/2$ such that $(a,q)=1$, and then sum over the required values of a.

2. The proof

2.1. Backlund's method

Our technique for Theorem 1.1 is that used by Backlund [2] in bounding $|\zeta(1+it)|$. We start with the summation formula of Euler and Maclaurin given in Murty's [6] book of problems in analytic number theory.

Lemma 2.1. *Let k be a nonnegative integer and $f(x)$ be $(k+1)$ times differentiable on the interval $[a,b]$. Then*

$$\sum_{a<n\leqslant b} f(n) = \int_a^b f(t)dt + \sum_{r=0}^{k} \frac{(-1)^{r+1}}{(r+1)!}\Big\{f^{(r)}(b)-f^{(r)}(a)\Big\}B_{r+1} + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(x)f^{(k+1)}(x)dx,$$

where $B_j(x)$ is the jth periodic Bernoulli polynomial and $B_j=B_j(0)$.

We consider that for $\Re(s)>1, t>0$ and some integer $N>0$ we have

$$\sum_{n>N}\frac{1}{(n+\frac{a}{q})^s} = \zeta\Big(s,\frac{a}{q}\Big) - \sum_{0\leqslant n\leqslant N}\frac{1}{(n+\frac{a}{q})^s}.$$

We use Lemma 2.1 with $f(n)=(n+a/q)^{-s}$, $k=1$, $a=N$ and $b\to\infty$ to estimate the left side of the above formula, and so we have for the right side:

$$\zeta\Big(s,\frac{a}{q}\Big) - \sum_{0\leqslant n\leqslant N-1}\frac{1}{(n+\frac{a}{q})^s} = \frac{(N+\frac{a}{q})^{1-s}}{s-1} - \frac{1}{2(N+\frac{a}{q})^s} + \frac{s}{12(N+\frac{a}{q})^{s+1}} - \frac{s(s+1)}{2}\int_N^\infty \frac{\{x\}^2-\{x\}+1/6}{(x+\frac{a}{q})^{s+2}}dx.$$

Note that as the right side of the above equation remains valid for $\Re(s) > -1$, it must remain valid for $s = 1 + it$ where $t > 0$. Thus, setting $N = [(t/m) - (a/q)]$, where m is to be chosen later, we obtain the estimate

$$|\zeta(1+it, a/q)| - \sum_{0 \leqslant n \leqslant N-1} \frac{1}{n + a/q} \leqslant \frac{1}{t} + \frac{m}{2(t-m)} + \frac{m^2(1+t)(4+t)}{24(t-m)^2}. \tag{3}$$

From here, we will first prove Theorem 1.3. The result is not required for the proof of the other theorems, but it follows easily from (3) and may be interesting nonetheless. We note that we could have replaced a/q with some real number $c \in (0, 1]$ in the above working. Thus, we have in place of (3) the inequality

$$|\zeta(1+it, c)| - \sum_{0 \leqslant n \leqslant N-1} \frac{1}{n + c} \leqslant \frac{1}{t} + \frac{m}{2(t-m)} + \frac{m^2(1+t)(4+t)}{24(t-m)^2}.$$

The sum can be estimated using the classic bound

$$\sum_{1 \leqslant n \leqslant t} \frac{1}{n} \leqslant \log t + \gamma + \frac{1}{t} \tag{4}$$

where γ is the Euler–Mascheroni constant. We get

$$\begin{aligned} \sum_{0 \leqslant n \leqslant N-1} \frac{1}{n+c} &\leqslant \frac{1}{c} + \sum_{1 \leqslant n \leqslant N} \frac{1}{n} - \frac{1}{N} \\ &\leqslant \log N + \gamma + \frac{1}{c} \\ &\leqslant \log t - \log m + \gamma + \frac{1}{c}. \end{aligned}$$

Putting everything together we have

$$|\zeta(1+it, c)| - \log t - \frac{1}{c} \leqslant -\log m + \gamma + \frac{1}{t} + \frac{m}{2(t-m)} + \frac{m^2(1+t)(4+t)}{24(t-m)^2}.$$

If we now choose $m = 3$ it is easy to verify that the right side of the above inequality is negative for all $t \geqslant 50$. This completes the proof of Theorem 1.3.

We now proceed to prove Theorem 1.1. We consider (3) jointly for a/q and $(q-a)/q$; summing the resulting inequalities yields

$$\begin{aligned} |\zeta(1+it, a/q)| + |\zeta(1+it, (q-a)/q)| \leqslant \sum_{0 \leqslant n \leqslant N-1} \left(\frac{1}{n + a/q} + \frac{1}{n + (q-a)/q} \right) + \\ + \frac{2}{t} + \frac{m}{t-m} + \frac{m^2(1+t)(4+t)}{12(t-m)^2}. \end{aligned} \tag{5}$$

We can pull the term corresponding to $n=0$ from the sum and bound the sum over the remaining range *viz.*

$$\begin{aligned}\sum_{1\leqslant n\leqslant N-1}\left(\frac{1}{n+a/q}+\frac{1}{n+(q-a)/q}\right) &= \sum_{1\leqslant n\leqslant N-1}\frac{2n+1}{(n+a/q)(n+(q-a)/q)}\\ &< \sum_{1\leqslant n\leqslant N-1}\frac{2n+1}{n^2+n}\\ &= 2\sum_{2\leqslant n\leqslant N}\frac{1}{n}+\sum_{2\leqslant n\leqslant N}\frac{1}{n(n-1)}.\end{aligned}$$

We estimate the first sum using (4), whereas the second sum converges to 1 from below as $N\to\infty$. This, when coupled with (5), gives us the inequality

$$\begin{aligned}&|\zeta(1+it,a/q)|+|\zeta(1+it,(q-a)/q)|-2\log t-\frac{q}{a}-\frac{q-a}{a}+1\\ &\qquad< -2\log m+\frac{2m}{t-2m}+2\gamma+\frac{2}{t}+\frac{m}{t-m}+\frac{m^2(1+t)(4+t)}{12(t-m)^2}.\end{aligned}$$

As before, it is a straightforward task to choose $m=3$ and verify that the right side of the above inequality is negative for all $t\geqslant 71$. Thus, for this range of t we have

$$|\zeta(1+it,a/q)|+|\zeta(1+it,(q-a)/q)|<2\log t+\frac{q}{a}+\frac{q-a}{a}-1.$$

We use this explicit bound in conjunction with (2) to get that

$$\begin{aligned}|L(1+it,\chi)| &\leqslant \frac{1}{q}\sum_{\substack{1\leqslant a<q/2\\(a,q)=1}}\left(|\zeta(1+it,a/q)|+|\zeta(1+it,(q-a)/q)|\right)\\ &\leqslant \frac{\varphi(q)}{q}\left(\log t-\frac{1}{2}\right)+\sum_{\substack{1\leqslant a<q\\(a,q)=1}}\frac{1}{a}.\end{aligned}$$

It remains to use (4) to estimate the sum as follows.

$$\sum_{\substack{1\leqslant a\leqslant q\\(a,q)=1}}\frac{1}{a}\leqslant -\frac{1}{q}+\sum_{1\leqslant a\leqslant q}\frac{1}{a}\leqslant \log q+\gamma.$$

This proves Theorem 1.1. The reader should note that there is a more careful way in which one could estimate the above sum (see, for example, Exercise 1.5.1 of Murty [6]). This would introduce a factor of $\varphi(q)/q$ to the front of the logarithm term along with additional error terms. We omit this for the sake of brevity.

2.2. By partial summation

As before, let χ be a non-principal character of modulus q. Using Abel's method of partial summation we have

$$\sum_{n=1}^{N} \frac{\chi(n)}{n^{1+it}} = \Big(\sum_{n=1}^{N} \chi(n)\Big) N^{-1-it} + (1+it) \int_1^N \Big(\sum_{n \leqslant t} \chi(n)\Big) \frac{dt}{t^{2+it}}.$$

For ease of notation, we write

$$A(t) = \sum_{n \leqslant t} \chi(n)$$

to get

$$\sum_{n=1}^{N} \frac{\chi(n)}{n^s} = A(N)N^{-s} + s \int_1^N \frac{A(t)}{t^{s+1}} dt. \tag{6}$$

Note that we can manipulate $L(s, \chi)$ as follows

$$\begin{aligned} L(s,\chi) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} \Big(A(n) - A(n-1)\Big) \frac{1}{n^s} \\ &= \sum_{n=1}^{\infty} A(n) \Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big) \end{aligned}$$

to get

$$L(s,\chi) = s \int_1^{\infty} \frac{A(t)}{t^{s+1}} dt. \tag{7}$$

Subtracting (6) from (7) we have

$$L(s,\chi) - \sum_{n=1}^{N} \frac{\chi(n)}{n^s} = s \int_N^{\infty} \frac{A(t)}{t^{s+1}} dt - A(N)N^{-s}.$$

Clearly, $|A(t)| < q/2$, as $A(N)$ is equal to zero whenever N is a multiple of q. The absolute convergence of the right side for $\Re(s) > 0$ allows us to set $s = 1 + it$ to get

$$|L(1+it,\chi)| \leqslant \sum_{n=1}^{N} \frac{1}{n} + \frac{q(1+t)}{2} \int_N^{\infty} \frac{1}{t^2} dt + \frac{A(N)}{N}.$$

If we set $N = [q(t+b)/m]$ (where m and b are to be chosen later) and use (4) we have

$$|L(1+it,\chi)| - \log(q(t+b)) - 1 \leqslant -\log m + (\gamma - 1) + \frac{m(2+q+qt)}{2q(t+b) - 2m}.$$

If we choose $m = 2$ and $b = 14/5$, it is straightforward to verify that the right hand side is negative for all $q \geqslant 2$ and $t > 0$. This completes our proof.

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