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ON THE CRITICAL VALUES OF L-FUNCTIONS OF BASE CHANGE FOR HILBERT MODULAR FORMS II

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Abstract: In this paper we generalize some results, obtained by Shimura, Yoshida and the author, on critical values of L-functions of l-adic representations attached to Hilbert modular forms twisted by finite order characters, to the critical values of L-functions of arbitrary base change to totally real number fields of l-adic representations attached to Hilbert modular forms twisted by some finite-dimensional representations.

Keywords: L-functions, Base change, special values, Hilbert modular forms.

1. Introduction

For F a totally real number field of degree n, let J_F be the set of infinite places of F, and let $\Gamma_F := \mathrm{Gal}(\bar{\mathbb{Q}}/F)$. Let f be a normalized Hecke eigenform of $\mathrm{GL}(2)/F$ of weight $k = (k(\tau))_{\tau \in J_F}$, where all $k(\tau)$ have the same parity and $k(\tau) \geqslant 2$. We denote by Π the cuspidal automorphic representation of $\mathrm{GL}(2)/F$ generated by f. In this paper we assume that Π is non-CM. We denote by ρ_Π the l-adic representation of Γ_F attached to Π. Define $k_0 = \max\{k(\tau) | \tau \in J_F\}$ and $k^0 = \min\{k(\tau) | \tau \in J_F\}$. Any integer $m \in \mathbb{Z}$ such that $(k_0 - k^0)/2 < m < (k_0 + k^0)/2$ is called a critical value for f or Π. Let F' be a totally real finite extension of F. Consider a finite-dimensional continuous representation

$$\psi : \Gamma_{F'} \to \mathrm{GL}_N(\mathbb{C}).$$

We assume throughout this paper that $\psi = \phi \otimes \chi$, where χ is a continuous abelian representation of $\Gamma_{F'}$, and ϕ is a continuous representation of $\Gamma_{F'}$ satisfying the following property: the number field $K := \bar{\mathbb{Q}}^{\ker \phi}$ is a $(\mathbb{Z}/2\mathbb{Z})^r$-extension of a totally real field for some non-negative integer r, i.e. there exists a totally real subfield F'' of K such that $\mathrm{Gal}(K/F'') \cong (\mathbb{Z}/2\mathbb{Z})^r$. Let V_ψ be the space corresponding to ψ. We denote by $d^+_{\tau'}(\psi)$ the dimension of the subspace of V_ψ on which the complex

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conjugation corresponding to $\tau' \in J_{F'}$ acts by $+1$, and by $d^-_{\tau'}(\psi)$ the dimension of the subspace of V_ψ on which the complex conjugation corresponding to $\tau' \in J_{F'}$ acts by -1. Throughout this paper we write $a \sim b$ for $a, b \in \mathbb{C}$ if $b \neq 0$ and $a/b \in \bar{\mathbb{Q}}$.

In this paper we prove the following result:

Theorem 1.1. Assume $k(\tau) \geqslant 3$ for all $\tau \in J_F$ and $k(\tau) \bmod 2$ is independent of τ. Let F' be a totally real finite extension of F. Let $\psi = \phi \otimes \chi$ be a finite-dimensional complex-valued continuous representation of $\Gamma_{F'}$. Assume that the continuous representation χ is abelian, and that the continuous representation ϕ satisfies the following property: $K := \bar{\mathbb{Q}}^{ker\phi}$ is a $(\mathbb{Z}/2\mathbb{Z})^r$-extension of a totally real number field for some non-negative integer r. Then

$$L(m, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi) \sim \pi^{m[F':\mathbb{Q}]\dim\psi} \prod_{\tau' \in J_{F'}} c_{\tau'|F}^{(-1)^{(m+1)}}(\Pi)^{d^-_{\tau'}(\psi)} c_{\tau'|F}^{(-1)^m}(\Pi)^{d^+_{\tau'}(\psi)}$$

for any integer m satisfying

$$(k_0+1)/2 \leqslant m < (k_0+k^0)/2,$$

where $c^-_{\tau'|F}(\Pi)$ and $c^+_{\tau'|F}(\Pi)$ appear in Propositions 2.2 and 2.3 below.

Theorem 1.1 is a generalization of Theorem 4.3 of [S], the main theorem of [Y], Theorem 4 of [Y], Theorems 1.1, 1.2 and 1.3 of [V1] (i.e. Propositions 2.1, 2.2 and 2.3 below; when ψ is abelian, Theorem 1.1 could be deduced easily from Propositions 2.1, 2.2 and 2.3 below), and of [V4]. It is conjectured that the result obtained in Theorem 1.1 should be true for arbitrary finite-dimensional complex-valued continuous representations ψ of $\Gamma_{F'}$.

2. Known results

Consider F a totally real number field and let J_F be the set of infinite places of F. If Π is a cuspidal automorphic representation (discrete series at infinity) of weight $k = (k(\tau))_{\tau \in J_F}$ of $\mathrm{GL}(2)/F$, where all $k(\tau)$ have the same parity and all $k(\tau) \geqslant 2$, then there exists ([T]) a λ-adic representation

$$\rho_\Pi := \rho_{\Pi,\lambda} : \Gamma_F \to \mathrm{GL}_2(O_\lambda) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l),$$

which satisfies $L(s, \rho_{\Pi,\lambda}) = L(s, \Pi) = L(s, f)$ (the equality up to finitely many Euler factors) and is unramified outside the primes dividing $\mathbf{n}l$ (by fixing a specific isomorphism $i : \bar{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$ one can regard ρ_Π as a complex-valued representation). Here O is the coefficients ring of Π and λ is a prime ideal of O above some prime number l, $\mathbf{n}$ is the level of Π and f is the normalized Hecke eigenform of $\mathrm{GL}(2)/F$ of weight k corresponding to Π. We denote by $F_\infty^\times$ the archimedean part of the idele group $F_{\mathbb{A}}^\times$ of F.

We know (this is Theorem 1.1 of [V1], which is a generalization of Theorem 4.3 of [S]):

Proposition 2.1. Assume $k(\tau) \geqslant 3$ for all $\tau \in J_F$ and $k(\tau)$ mod 2 is independent of τ. Let F' be a totally real extension of F. Then for every $\epsilon = (\epsilon(\tau)) \in (\mathbb{Z}/2\mathbb{Z})^{J_{F'}}$, there exists a constant $u(\epsilon, \Pi) \in \mathbb{C}^\times/\bar{\mathbb{Q}}^\times$ with the following property. If ψ is a finite order Hecke character of F' such that

$$\psi_\infty(x) = \prod_{\tau \in J_{F'}} sgn(x_\tau)^{\epsilon(\tau)+m}, \qquad x = (x_\tau) \in F_\infty'^\times,$$

then

$$L(m, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi) \sim \pi^{m[F':\mathbb{Q}]} u(\epsilon, \Pi)$$

for any integer m satisfying

$$(k_0 + 1)/2 \leqslant m < (k_0 + k^0)/2.$$

We know (this is Theorem 1.2 of [V1], which is a generalization of the main theorem of [Y]):

Proposition 2.2. Assume that $k(\tau) \geqslant 3$ for all $\tau \in J_F$ and $k(\tau)$ mod 2 is independent of τ. Let F' be a totally real extension of F. Then, for every $\tau \in J_{F'}$, there exist constants $c_\tau^\pm(\Pi) \in \mathbb{C}^\times$ which are determined uniquely mod $\bar{\mathbb{Q}}^\times$ such that

$$u(\epsilon, \Pi) \sim \prod_{\tau \in J_{F'}} c_\tau^{\epsilon(\tau)}(\Pi),$$

where $\epsilon = (\epsilon(\tau)) \in (\mathbb{Z}/2\mathbb{Z})^{J_{F'}}$, and $u(\epsilon, \Pi)$ was defined in Proposition 1.1. The constants $c_\tau^\pm(\Pi) \in \mathbb{C}^\times$ are uniquely determined mod $\bar{\mathbb{Q}}^\times$.

Here we understand that $c_\tau^0(\Pi) = c_\tau^+(\Pi)$, $c_\tau^1(\Pi) = c_\tau^-(\Pi)$ by identifying $\mathbb{Z}/2\mathbb{Z}$ with $\{0, 1\}$.

We know (this is Theorem 1.3 of [V1], which is a generalization of Theorem 4 of [Y]):

Proposition 2.3. Assume that $k(\tau) \geqslant 3$ for all $\tau \in J_F$ and $k(\tau)$ mod 2 is independent of τ. Let F' be a totally real extension of F. Then we have

$$c_\tau^\pm(\Pi) = c_{\tau|F}^\pm(\Pi) \qquad \text{for every } \tau \in J_{F'}.$$

3. The proof of Theorem 1.1

We fix a non-CM cuspidal automorphic representation Π of $\mathrm{GL}(2)/F$ as in Theorem 1.1, and let F'/F be a totally real finite extension. Let $\psi = \phi \otimes \chi$ be a finite-dimensional representation of $\Gamma_{F'}$ as in Theorem 1.1. We denote by F'' the maximal totally real subfield of $K := \bar{\mathbb{Q}}^{\ker\phi}$. Obviously K is a $(\mathbb{Z}/2\mathbb{Z})^r$-extension of F'' for some r, and χ is a direct sum of one-dimensional representations.

From the beginning of §15 of [CR] we know that there exist some subfields $E_i \subseteq K$ such that $\mathrm{Gal}(K/E_i)$ are solvable (actually we don't use this solvability in this paper), and some integers n_i, such that

$$[K:F']\phi = \sum_{i=1}^{u} n_i \mathrm{Ind}_{\mathrm{Gal}(K/E_i)}^{\mathrm{Gal}(K/F')} 1_{E_i},$$

where $1_{E_i} : \mathrm{Gal}(K/E_i) \to \mathbb{C}^\times$ is the trivial representation. In particular we have $[K:F'] \dim \phi = \sum_{i=1}^{u} n_i [E_i : F']$. Then

$$\begin{aligned} L\left(s, \rho_\Pi|_{\Gamma_{F'}} \otimes \phi\right)^{[K:F']} &= \prod_{i=1}^{u} L\left(s, \rho_\Pi|_{\Gamma_{F'}} \otimes \mathrm{Ind}_{\Gamma_{E_i}}^{\Gamma_{F'}} 1_{E_i}\right)^{n_i} \\ &= \prod_{i=1}^{u} L\left(s, \mathrm{Ind}_{\Gamma_{E_i}}^{\Gamma_{F'}} (\rho_\Pi|_{\Gamma_{E_i}})\right)^{n_i} = \prod_{i=1}^{u} L\left(s, \rho_\Pi|_{\Gamma_{E_i}}\right)^{n_i}. \end{aligned}$$

Let $F_i := E_i \cap F''$. One can see easily that E_i is a $(\mathbb{Z}/2\mathbb{Z})^{r_i}$-extension of F_i for some r_i. Hence

$$\begin{aligned} L\left(s, \rho_\Pi|_{\Gamma_{F'}} \otimes \phi\right)^{[K:F']} &= \prod_{i=1}^{u} L\left(s, \rho_\Pi|_{\Gamma_{E_i}}\right)^{n_i} \\ &= \prod_{i=1}^{u} \prod_{\phi_i :\, \mathrm{Gal}(E_i/F_i) \to \mathbb{C}^\times} L\left(s, \rho_\Pi|_{\Gamma_{F_i}} \otimes \phi_i\right)^{n_i}. \end{aligned}$$

Also one has

$$\begin{aligned} L\left(s, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi\right)^{[K:F']} &= L\left(s, \rho_\Pi|_{\Gamma_{F'}} \otimes \phi \otimes \chi\right)^{[K:F']} \\ &= \prod_{i=1}^{u} \prod_{\phi_i :\, \mathrm{Gal}(E_i/F_i) \to \mathbb{C}^\times} L\left(s, \rho_\Pi|_{\Gamma_{F_i}} \otimes \phi_i \otimes \chi|_{\Gamma_{F_i}}\right)^{n_i}. \end{aligned}$$

Using the potential modularity of the representation $\rho_\Pi|_{\Gamma_{F_i}}$ (see Theorem A of [BGGT], Theorem 2.1 of [V2] or Theorem 1.1 of [V3]), one can prove easily the meromorphic continuation to the entire complex plane of the functions $L(s, \rho_\Pi|_{\Gamma_{F_i}} \otimes \phi_i \otimes \chi|_{\Gamma_{F_i}})$ (for details see for example the proof of Theorem 1.1 of [V1]), and hence one gets the meromorphic continuation to the entire complex plane of the function $L(s, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi)^{[K:F']}$. Moreover, from the proof of Theorem 1.1 of [V1] we know that the function $L(s, \rho_\Pi|_{\Gamma_{F_i}} \otimes \phi_i \otimes \chi|_{\Gamma_{F_i}})^{[K:F']}$ has no poles or zeros at $s = m$ for each integer m satisfying $(k_0+1)/2 \leqslant m < (k_0+k^0)/2$. Thus for each integer m satisfying $(k_0+1)/2 \leqslant m < (k_0+k^0)/2$, we get the identity

$$L\left(m, \rho_\Pi|_{\Gamma_{F'}} \otimes \psi\right)^{[K:F']} = \prod_{i=1}^{u} \prod_{\phi_i :\, \mathrm{Gal}(E_i/F_i) \to \mathbb{C}^\times} L\left(m, \rho_\Pi|_{\Gamma_{F_i}} \otimes \phi_i \otimes \chi|_{\Gamma_{F_i}}\right)^{n_i}.$$

We have:

$$\begin{aligned}
[K:F']\phi &= \sum_{i=1}^{u} n_i \mathrm{Ind}_{\mathrm{Gal}(K/E_i)}^{\mathrm{Gal}(K/F')} 1_{E_i} = \sum_{i=1}^{u} n_i \mathrm{Ind}_{\mathrm{Gal}(K/F_i)}^{\mathrm{Gal}(K/F')} \mathrm{Ind}_{\mathrm{Gal}(K/E_i)}^{\mathrm{Gal}(K/F_i)} 1_{E_i} \\
&= \sum_{i=1}^{u} n_i \mathrm{Ind}_{\mathrm{Gal}(K/F_i)}^{\mathrm{Gal}(K/F')} \left(\sum_{\phi_i : \mathrm{Gal}(E_i/F_i) \to \mathbb{C}^\times} \phi_i \right) \\
&= \sum_{i=1}^{u} n_i \sum_{\phi_i :\, \mathrm{Gal}(E_i/F_i) \to \mathbb{C}^\times} \mathrm{Ind}_{\mathrm{Gal}(K/F_i)}^{\mathrm{Gal}(K/F')} \phi_i .
\end{aligned}$$

From this identity by tensoring by χ one can deduce easily that

$$d_{\tau'}^{-}(\psi) = \sum_{i=1}^{u} n_i \sum_{\phi_i :\, \mathrm{Gal}(E_i/F_i) \to \mathbb{C}^\times} \sum_{\{\tau_i \in J_{F_i} | \tau_i|_{F'} = \tau'\}} d_{\tau_i}^{-} \left(\phi_i \otimes \chi|_{\Gamma_{F_i}} \right)$$

and

$$d_{\tau'}^{+}(\psi) = \sum_{i=1}^{u} n_i \sum_{\phi_i :\, \mathrm{Gal}(E_i/F_i) \to \mathbb{C}^\times} \sum_{\{\tau_i \in J_{F_i} | \tau_i|_{F'} = \tau'\}} d_{\tau_i}^{+} \left(\phi_i \otimes \chi|_{\Gamma_{F_i}} \right),$$

for any $\tau' \in J_{F'}$.

Recall that $[K:F'] \dim \phi = \sum_{i=1}^{u} n_i [E_i : F']$, and thus $[K:\mathbb{Q}] \dim \psi = \sum_{i=1}^{u} n_i [E_i : \mathbb{Q}] \dim \chi$. Now from Propositions 2.1, 2.2 and 2.3 one gets easily that

$$\begin{aligned}
&L\left(m, \rho_\Pi|_{\Gamma_{F_i}} \otimes \phi_i \otimes \chi|_{\Gamma_{F_i}}\right) \\
&\quad \sim \pi^{m[F_i:\mathbb{Q}] \dim \chi} \prod_{\tau_i \in J_{F_i}} c_{\tau_i|F'}^{-1^{(m+1)}}(\Pi)^{d_{\tau_i}^{-}(\phi_i \otimes \chi|_{\Gamma_{F_i}})} c_{\tau_i|F'}^{(-1)^m}(\Pi)^{d_{\tau_i}^{+}(\phi_i \otimes \chi|_{\Gamma_{F_i}})},
\end{aligned}$$

and hence

$$
\begin{aligned}
L\left(m,\rho_\Pi|_{\Gamma_{F'}}\otimes\psi\right)^{[K:F']} &= \prod_{i=1}^{u}\prod_{\phi_i:\mathrm{Gal}(E_i/F_i)\to\mathbb{C}^\times} L(m,\rho_\Pi|_{\Gamma_{F_i}}\otimes\phi_i\otimes\chi|_{\Gamma_{F_i}})^{n_i}\\
&\sim \prod_{i=1}^{u}\prod_{\phi_i:\,\mathrm{Gal}(E_i/F_i)\to\mathbb{C}^\times}\Bigg(\pi^{mn_i[F_i:\mathbb{Q}]\dim\chi}\prod_{\tau_i\in J_{F_i}} c_{\tau_i|F'}^{-1^{(m+1)}}\\
&\quad\times(\Pi)^{n_i d^-_{\tau_i}(\phi_i\otimes\chi|_{\Gamma_{F_i}})} c_{\tau_i|F'}^{(-1)^m}(\Pi)^{n_i d^+_{\tau_i}(\phi_i\otimes\chi|_{\Gamma_{F_i}})}\Bigg)\\
&= \pi^{m[F':\mathbb{Q}]\dim\psi}\\
&\quad\times\prod_{\tau'\in J_{F'}} c_{\tau'}^{-1^{(m+1)}}(\Pi)^{\sum_{i=1}^{u} n_i\sum_{\phi_i:\mathrm{Gal}(E_i/F_i)\to\mathbb{C}^\times}\sum_{\{\tau_i\in J_{F_i}|\tau_i|_{F'}=\tau'\}} d^-_{\tau_i}(\phi_i\otimes\chi|_{\Gamma_{F_i}})}\\
&\quad\times\prod_{\tau'\in J_{F'}} c_{\tau'}^{(-1)^m}(\Pi)^{\sum_{i=1}^{u} n_i\sum_{\phi_i:\mathrm{Gal}(E_i/F_i)\to\mathbb{C}^\times}\sum_{\{\tau_i\in J_{F_i}|\tau_i|_{F'}=\tau'\}} d^+_{\tau_i}(\phi_i\otimes\chi|_{\Gamma_{F_i}})}\\
&= \pi^{m[K:\mathbb{Q}]\dim\psi}\prod_{\tau'\in J_{F'}} c_{\tau'}^{-1^{(m+1)}}(\Pi)^{d^-_{\tau'}(\psi)} c_{\tau'}^{(-1)^m}(\Pi)^{d^+_{\tau'}(\psi)}\\
&= \pi^{m[K:\mathbb{Q}]\dim\psi}\prod_{\tau'\in J_{F'}} c_{\tau'|F}^{-1^{(m+1)}}(\Pi)^{d^-_{\tau'}(\psi)} c_{\tau'|F}^{(-1)^m}(\Pi)^{d^+_{\tau'}(\psi)},
\end{aligned}
$$

and thus

$$
L(m,\rho_\Pi|_{\Gamma_{F'}}\otimes\psi)\sim\pi^{m[F':\mathbb{Q}]\dim\psi}\prod_{\tau'\in J_{F'}} c_{\tau'|F}^{-1^{(m+1)}}(\Pi)^{d^-_{\tau'}(\psi)} c_{\tau'|F}^{(-1)^m}(\Pi)^{d^+_{\tau'}(\psi)},
$$

which proves Theorem 1.1.

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