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ATOMIC, MOLECULAR AND WAVELET DECOMPOSITION OF 2-MICROLOCAL BESOV AND TRIEBEL-LIZORKIN SPACES WITH VARIABLE INTEGRABILITY

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Abstract: We introduce 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability and give characterizations by decompositions in atoms, molecules and wavelets. These spaces cover the usual Besov and Triebel-Lizorkin spaces as well as spaces of variable smoothness and integrability. We emphasize that the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ which were defined recently in [12] are included in this approach.

Keywords: 2-microlocal spaces, wavelet decomposition, Besov spaces, Triebel-Lizorkin spaces, variable smoothness, variable integrability.

1. Introduction

We present function spaces of Besov and Triebel-Lizorkin type with variable smoothness and integrability. We describe the variable smoothness of these spaces in terms of 2-microlocal weight sequences, see Definition 2.1. This paper can be seen as a continuation of [22], where a characterization with local means is given for 2-microlocal Besov and Triebel-Lizorkin spaces with variable integrability.

Using the results from [22] we present characterizations of these spaces by decompositions in atoms, molecules and wavelets.

2-microlocal function spaces initially appeared in the book of Peetre [34] and have independently been studied by Bony [7] in connection with pseudodifferential operators. Later on, Jaffard in [19] gave a wavelet characterization of 2-microlocal Hölder-Zygmund spaces and Jaffard & Meyer [20] elaborated them widely. In [27] Levy Véhel & Seuret developed the 2-microlocal formalism and it turned out that 2-microlocal function spaces are an useful tool to measure local regularity of functions.

In Lemma 2.6 we will show that spaces of variable smoothness are included in the scale of 2-microlocal function spaces. Sobolev and Besov spaces of vari-

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able smoothness $\sigma(x)$ have been introduced by Beauzamy in [4]. In [25] and [26] Leopold defined and elaborated Besov spaces $B^{s,a}_{p,q}(\mathbb{R}^n)$ of variable smoothness for a class of hypoelliptic pseudodifferential operators. The symbols $a(x,\xi) = \langle\xi\rangle^{\sigma(x)}$ are special cases in the works of Leopold and correspond to Besov spaces $B^{\sigma(\cdot)}_{p,p}(\mathbb{R}^n)$ with variable smoothness. In [5] Besov gave a characterization by differences for these spaces $B^{s,a}_{p,p}(\mathbb{R}^n)$. Furthermore, in [6] Besov generalized the result and characterized 2-microlocal Besov and Triebel-Lizorkin spaces $B^{\boldsymbol{a}}_{p,q}(\mathbb{R}^n)$ and $F^{\boldsymbol{a}}_{p,q}(\mathbb{R}^n)$ by differences. Here $\boldsymbol{a}(x) = \{a_k(x)\}_{k\in\mathbb{N}_0}$ is a 2-microlocal weight sequence.

The second concept, we rely on, is the concept of variable exponent spaces $L_{p(\cdot)}(\mathbb{R}^n)$. They can be traced back to Orlicz [33] 1931. A good survey of $L_{p(\cdot)}(\mathbb{R}^n)$ spaces and its fundamental properties is given by Kováčik & Rákosník in [23]. From the point of view of harmonic analysis, the breakthrough for variable exponent spaces was achieved by Diening, when he showed in [10] that the Hardy-Littlewood maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$ for p satisfying some regularity condition inside a large ball B_R and constant outside. Inspired by this work, Cruz-Uribe, Fiorenza and Neugebauer in [8] and Nekvinda in [32] elaborated the conditions on p.

The spaces $L_{p(\cdot)}(\mathbb{R}^n)$ possess interesting applications in fluid dynamics, image processing, PDE and variational calculus (see [12] and references therein). So it was natural, that several extensions in various more advanced function spaces as Bessel potential spaces $H^s_{p(\cdot)}$ and Besov and Triebel-Lizorkin spaces $F^s_{p(\cdot),q}(\mathbb{R}^n)$, $B^s_{p(\cdot),q}(\mathbb{R}^n)$ were made ([3], [18] and [43]).

The concept of function spaces with variable smoothness and the concept of variable integrability were firstly mixed up by Diening, Hästö & Roudenko in [12]. They defined Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and proved a discretization by the so called φ-transform and used it to derive trace results. From the trace theorem on $\mathbb{R}^{n-1}$ it became clear, why it is natural to have all parameters variable. Due to

$$\mathrm{tr}F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{s(\cdot)-1/p(\cdot)}_{p(\cdot),p(\cdot)}(\mathbb{R}^{n-1})$$

([12, Theorem 3.13]) we see the necessity of taking s and q variable if p is not constant.

The main aim of this paper is to present a decomposition by wavelets for $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$, $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Since we want to have this characterization with compactly supported Daubechies wavelets as well as with C^∞ Meyer wavelets, we also need characterizations with atoms and molecules as tools. In Section 2 we define the spaces $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and we prove their basic properties. The next section contains the needed characterizations by decompositions in atoms and molecules. Using the results from Section 3 in Section 4 we prove the wavelet characterization for $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$, $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$. Finally, in Section 5 we present all previous results for the most general spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and in Section 6 we collect the important results for the spaces of variable smoothness denoted as $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, $B^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$.

2. Definitions and basic properties

In this section we present the Fourier analytic definition of 2-microlocal Besov and Triebel-Lizorkin spaces, $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$, and we prove the basic properties in analogy to the classical Besov and Triebel-Lizorkin spaces.

As usual $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space, $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbols $\mathbb{Z}$ and $\mathbb{C}$ stand for the sets of integers and complex numbers, respectively.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all complex valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Its topology is generated by the norms $\|\varphi\|_{k,l} = \sup_{x\in\mathbb{R}^n} \langle x\rangle^k \sum_{|\beta|\leqslant l} |D^\beta\varphi(x)|$, where $\langle x\rangle^k = (1+|x|^2)^{k/2}$. By $\mathcal{S}'(\mathbb{R}^n)$ we denote its dual space. We denote the Fourier transform by $\mathcal{F}$ and its inverse by $\mathcal{F}^{-1}$ on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ respectively and we use the symbols $\hat{f}$ and $f^\vee$ for $\mathcal{F}f$ and $\mathcal{F}^{-1}f$.

The constant c stands for unimportant positive constants. So the value of the constant c may change from one occurrence to another. By $a_k \sim b_k$ we mean that there are two constants $c_1, c_2 > 0$ such that $c_1 a_k \leqslant b_k \leqslant c_2 a_k$ for all admissible k.

2.1. Definition of the spaces

Before introducing the spaces, we define admissible weight sequences $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0}$, we explain smooth resolutions of unity and we give the basic knowledge about spaces of variable integrability.

Definition 2.1 (Admissible weight sequence). *Let $\alpha \geqslant 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leqslant \alpha_2$. A sequence of non-negative measurable functions $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0}$ belongs to the class $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ if, and only if,*

(i) *There exists a constant* $\mathrm{C} > 0$ *such that*

$$0 < w_j(x) \leqslant \mathrm{C} w_j(y) \left(1 + 2^j|x-y|\right)^\alpha \qquad \text{for all } j \in \mathbb{N}_0 \text{ and all } x, y \in \mathbb{R}^n.$$

(ii) *For all* $j \in \mathbb{N}_0$ *we have*

$$2^{\alpha_1} w_j(x) \leqslant w_{j+1}(x) \leqslant 2^{\alpha_2} w_j(x) \qquad \text{for all } x \in \mathbb{R}^n.$$

Such a system $\{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ is called admissible weight sequence*.*

A non-negative measurable function ϱ is called an *admissible weight function* if there exist constants $\alpha_\varrho \geqslant 0$ and $\mathrm{C}_\varrho > 0$ such that

$$0 < \varrho(x) \leqslant \mathrm{C}_\varrho \varrho(y)(1+|x-y|)^{\alpha_\varrho} \qquad \text{holds for every } x, y \in \mathbb{R}^n. \tag{1}$$

If $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0}$ is an admissible weight sequence, each w_j is an admissible weight function, but in general the constant C_{w_j} depends on $j \in \mathbb{N}_0$. If we use $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ without any restrictions, then $\alpha \geqslant 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ are arbitrary but fixed numbers.

A fundamental example of an admissible weight sequence is given by the 2-microlocal weights. For a fixed nonempty set $U \subset \mathbb{R}^n$ and $s, s' \in \mathbb{R}$ they are given by

$$w_j(x) := 2^{js}\left(1 + 2^j \operatorname{dist}(x, U)\right)^{s'}, \tag{2}$$

where $\operatorname{dist}(x, U) = \inf_{z \in U} |x - z|$ is the distance of $x \in \mathbb{R}^n$ from U.

A special case is $U = \{x_0\}$ for $x_0 \in \mathbb{R}^n$. Then $\operatorname{dist}(U, x) = |x - x_0|$ and we get the well known 2-microlocal weights ([20] and [1]) treated by many authors

$$w_j(x) = 2^{js}(1 + 2^j|x - x_0|)^{s'} \qquad \text{for } j \in \mathbb{N}_0. \tag{3}$$

If U is an open subset of $\mathbb{R}^n$, we get the weight sequence used by Moritoh and Yamada in [30].

To give a Fourier analytic definition of these spaces we need smooth decompositions of unity. We start with an arbitrary function $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) \geqslant 0$ and

$$\varphi_0(x) = \begin{cases} 1, & |x| \leqslant 1 \\ 0, & |x| \geqslant 2. \end{cases} \tag{4}$$

Furthermore, we define $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ and set $\varphi_j(x) = \varphi(2^{-j}x)$ for all $j \in \mathbb{N}$. Then $\{\varphi_j\}_{j \in \mathbb{N}_0}$ is a smooth dyadic resolution of unity and we have

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1 \qquad \text{for all } x \in \mathbb{R}^n.$$

Furthermore, we give a short survey on variable exponent spaces $L_{p(\cdot)}(\mathbb{R}^n)$. A very good resource is [23] for more details. The class of exponents $\mathcal{P}(\mathbb{R}^n)$ consists of all measurable functions $p : \mathbb{R}^n \to (0, \infty]$ which are bounded away from zero. For a set $A \subset \mathbb{R}^n$ we denote $p_A^+ = \operatorname{ess-sup}_{x \in A} p(x)$ and $p_A^- = \operatorname{ess-inf}_{x \in A} p(x)$; we use the abbreviations $p^+ = p_{\mathbb{R}^n}^+$ and $p^- = p_{\mathbb{R}^n}^-$. The Lebesgue space $L_{p(\cdot)}(\mathbb{R}^n)$ of variable integrability consists of all measurable functions f such that for some $\lambda > 0$ the modular $\varrho_{L_{p(\cdot)}(\mathbb{R}^n)}(f/\lambda)$ is finite, where

$$\varrho_{L_{p(\cdot)}(\mathbb{R}^n)}(f) = \int_{\mathbb{R}_0^n} |f(x)|^{p(x)} dx + \operatorname*{ess-sup}_{x \in \mathbb{R}_\infty^n} |f(x)|.$$

Here, $\mathbb{R}_\infty^n$ denotes the subset of $\mathbb{R}^n$, where $p(x) = \infty$ and $\mathbb{R}_0^n = \mathbb{R}^n \setminus \mathbb{R}_\infty^n$. The Luxemburg norm of a function $f \in L_{p(\cdot)}(\mathbb{R}^n)$ is given by

$$\big\| f \big| L_{p(\cdot)}(\mathbb{R}^n) \big\| = \inf\{\lambda > 0 : \varrho_{L_{p(\cdot)}(\mathbb{R}^n)}(f/\lambda) \leqslant 1\}.$$

Now, we define the spaces under consideration.

Definition 2.2. *Let* $\{\varphi_j\}_{j\in\mathbb{N}_0}$ *be a resolution of unity and let* $\boldsymbol{w} = \{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. *Further, let* $0 < q \leqslant \infty$ *and* $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, *then we define*

$$B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\| f \,|\, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|_{\varphi} < \infty \right\},$$

where

$$\left\| f \,|\, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|_{\varphi} = \left(\sum_{j=0}^{\infty} \left\| w_j (\varphi_j \hat{f})^{\vee} \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q}.$$

For $p^+ < \infty$ *we define*

$$F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \left\| f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|_{\varphi} < \infty \right\},$$

where

$$\left\| f \,|\, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|_{\varphi} = \left\| \left(\sum_{j=0}^{\infty} |w_j(x) (\varphi_j \hat{f})^{\vee}(x)|^q \right)^{1/q} \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\|.$$

The norms get modified as usual, if q *equals infinity.*

Remark 2.3.

(1) We use the notation $\|\cdot\|_{\varphi}$ to indicate that formally our definition depends on the start function φ of the resolution of unity. Fortunately, from the local means characterization of these spaces [22] we obtain, that another start function $\tilde{\varphi}$ defines an equivalent norm. Therefore, we suppress the index φ in the notation of the norm for the rest of the paper.

(2) Later on, it is sometimes convenient to use another notation for the norms. We use

$$\left\| f_j \,|\, \ell_q(L_{p(\cdot)}(\mathbb{R}^n)) \right\| = \left(\sum_{j=0}^{\infty} \left\| f_j(\cdot) \,|\, L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q}$$

and

$$\left\| f_j \,|\, L_{p(\cdot)}(\ell_q) \right\| = \left\| \left(\sum_{j=0}^{\infty} |f_j(\cdot)|^q \right)^{1/q} \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\|.$$

2.2. Connection to known spaces

For **p** =**const** and $w_j(x) = 2^{js}$ we get back to the usual Besov and Triebel-Lizorkin spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$.

If p is constant, then the spaces from Definition 2.2 coincide with the spaces in [6] presented by Besov. He used the theory of ultra-distributions to allow exponential growth of the weights. Restricted to the usual distributions, our approach is more general because negative smoothness is allowed ($\alpha_1, \alpha_2 \in \mathbb{R}$ in contrast to $0 \leqslant \alpha_1 \leqslant \alpha_2$ in [6]).

Regarding p constant the entire literature on 2-microlocal spaces $B^{s,s'}_{pq}(\mathbb{R}^n,U)$ with

$$w_j(x) = 2^{js}(1+2^j \operatorname{dist}(x,U))^{s'}$$

([1], [7], [19], [20], [21], [29], [30], [27]) is included in this approach.

Moreover, the spaces of generalized smoothness are contained in this approach ([14], [31]) by taking

$$w_j(x) = 2^{js}\Psi(2^{-j}), \qquad \text{or more general } \ w_j(x) = \sigma_j.$$

Here, $\{\sigma_j\}_{j\in\mathbb{N}_0}$ is an admissible sequence that means there exist $d_0, d_1 > 0$ with $d_0\sigma_j \leqslant \sigma_{j+1} \leqslant d_1\sigma_j$ and Ψ is a slowly varying function.

If $w_j(x) = 2^{js}w_0(x)$ for all $j \in \mathbb{N}$, then we obtain the weighted Besov and Triebel-Lizorkin spaces with constant p ([13, Chapter 4]).

For **variable** $\boldsymbol{p \in \mathcal{P}(\mathbb{R}^n)}$ and $w_j(x) = 2^{js}$ the spaces considered by Xu in [43] are contained in the above scale.

Moreover, also the classical spaces of variable integrability are contained in $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ for $w_j(x) = 2^{js}$. For example we have $F^s_{p(\cdot),2}(\mathbb{R}^n) = H^s_{p(\cdot)}(\mathbb{R}^n)$, where $H^s_{p(\cdot)}(\mathbb{R}^n)$ are the Bessel potential spaces (fractional Sobolev spaces) of variable integrability which were introduced in [3] and in [18]. The integrability p has to belong to $C^{\log}(\mathbb{R}^n)$ (see Definition 2.4) with $1 < p^- \leqslant p^+ < \infty$ and $s \geqslant 0$ ([12, Theorem 4.5]). Especially, we get under these conditions that $F^k_{p(\cdot),2}(\mathbb{R}^n) = W^k_{p(\cdot)}(\mathbb{R}^n)$, where $k \in \mathbb{N}_0$.

If in [12] one chooses the parameter q as a constant function, then the spaces from Definition 2.2 include the Triebel-Lizorkin spaces $F^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ of [12]. Later on, in Definition 5.1 we work with the spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ where also $q(\cdot)$ is a variable function. If $s : \mathbb{R}^n \to \mathbb{R}$ is out of $L_\infty(\mathbb{R}^n) \cap C^{\log}_{loc}(\mathbb{R}^n)$ (see Definition 2.4), then

$$F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n), \qquad \text{where } \ w_j(x) = 2^{js(x)}.$$

Our approach is more general since $s(\cdot)$ can be negative, whereas in [12] $s(x) \geqslant 0$. In Lemma 2.6 we will show that we obtain a 2-microlocal weight sequence by the construction $w_j(x) = 2^{js(x)}$, whenever $s \in L_\infty(\mathbb{R}^n) \cap C^{\log}_{loc}(\mathbb{R}^n)$.

2.3. Basic properties

We introduce the Hardy-Littlewood maximal operator $\mathcal{M}_t$ which is defined for a locally integrable function $f \in L^{loc}_1(\mathbb{R}^n)$ and for $0 < t \leqslant 1$ by

$$\mathcal{M}_t(f)(x) = \left(\sup_{x\in Q} \int_Q |f(y)|^t dy\right)^{1/t},$$

and $\mathcal{M}(f)(x) = \mathcal{M}_1(f)(x)$. The Hardy-Littlewood maximal operator is a key tool in harmonic analysis. It is known that the boundedness of many operators follows from the boundedness of the maximal operator. So it was a breakthrough, when Diening discovered in [10], that the maximal operator is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$ under certain regularity conditions on $p(\cdot)$. Let us define the important classes.

Definition 2.4. *Let $g \in C(\mathbb{R}^n)$. We say that g is* locally log-Hölder continuous*, abbreviated $g \in C^{\log}_{loc}(\mathbb{R}^n)$, if there exists $c_{\log} > 0$ such that*

$$|g(x) - g(y)| \leqslant \frac{c_{\log}}{\log(\mathrm{e}+1/|x-y|)} \tag{5}$$

holds for all $x, y \in \mathbb{R}^n$.

We say, that g is globally log-Hölder continuous*, abbreviated $g \in C^{\log}(\mathbb{R}^n)$, if g is locally* log*-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ such that*

$$|g(x) - g_\infty| \leqslant \frac{c_{\log}}{\log(\mathrm{e}+|x|)} \tag{6}$$

holds for all $x \in \mathbb{R}^n$.

If $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ belongs to these classes, then the maximal operator is bounded.

Proposition 2.5 (Theorem 3.6 in [11]). *Let $p \in \mathcal{P}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ \leqslant \infty$. If $\frac{1}{p(\cdot)} \in C^{\log}(\mathbb{R}^n)$, then $\mathcal{M}$ is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$ i.e., there exists $c > 0$ such that for all $f \in L_{p(\cdot)}(\mathbb{R}^n)$*

$$\left\| \mathcal{M}f \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\| \leqslant C \left\| f \middle| L_{p(\cdot)}(\mathbb{R}^n) \right\|.$$

We write $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, if $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $1/p(\cdot) \in C^{\log}(\mathbb{R}^n)$. If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ then $\mathcal{M}$ is bounded on $L_{p(\cdot)/p_0}(\mathbb{R}^n)$ for every $p_0 < p^-$ or, equivalently, $\mathcal{M}_t$ is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$, where $t = \min(1, p_0)$.

Furthermore, from [9, Corollary 2.1] we obtain that $\mathcal{M}_t$ is bounded on $L_{p(\cdot)}(\ell_q)$ if $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ for every $p_0 < \min(p^-, q)$ and $t = \min(p_0, 1)$.

The next lemma gives the connection between 2-microlocal spaces and spaces of variable smoothness.

Lemma 2.6. *Let $s : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then $s \in C^{\log}_{loc}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ if, and only if, $\{w_j\}_{j\in\mathbb{N}_0}$ defined by*

$$w_j(x) = 2^{js(x)}$$

belongs to $\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ for some $\alpha \geqslant 0$ and $\alpha_1 \leqslant \alpha_2$.

Proof. In [22] it was already shown that from $s \in C^{\log}_{loc}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$ it follows $\{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ for some $\alpha \geqslant 0$ and $\alpha_1, \alpha_2 \in \mathbb{R}$. Only the reverse direction remains to be proved. It is trivial that the second condition in Definition 2.1 on $w_j(x)$ is equivalent to the boundedness of $s(x)$ and we get $\alpha_1 \leqslant s(x) \leqslant \alpha_2$. Let $x, y \in \mathbb{R}^n$ with $|x - y| > 1$, then we have $\log(\mathrm{e}+1/|x-y|) \geqslant 1$ and

$$|s(x) - s(y)| \leqslant 2\|s\|_\infty \leqslant \frac{2\|s\|_\infty}{\log(\mathrm{e}+1/|x-y|)}.$$

So only the case $|x-y| \leqslant 1$ is open. Without loss of generality we assume that $w_j(x) \geqslant w_j(y)$, then we have

$$|s(x)-s(y)| = \frac{1}{j}\log_2\frac{w_j(x)}{w_j(y)} \leqslant \frac{\log_2 \mathrm{C}(1+2^j|x-y|)^\alpha}{j}$$

for all $j \in \mathbb{N}$. We choose $2^{-j+2} \leqslant |x-y| \leqslant 2^{-j+3}$ and obtain

$$\begin{aligned}|s(x)-s(y)| &\leqslant \frac{\log_2 \mathrm{C}(1+2^3)^\alpha}{2+\log_2 1/|x-y|} \leqslant \frac{\log_2 \mathrm{C}(1+2^3)^\alpha}{\log_2(\mathrm{e}+1/|x-y|)}\\ &\leqslant \frac{c_{\log}}{\log(\mathrm{e}+1/|x-y|)}.\end{aligned}$$

■

Remark 2.7.

(1) There exists also a more local class of exponent functions called $C^{\log}_{1-loc}(\mathbb{R}^n)$ defined in [40]. Here condition (5) has to hold only locally for $|x-y| \leqslant 1$. This is somewhat weaker than the class $C^{\log}_{loc}(\mathbb{R}^n)$ condition, but we have $C^{\log}_{1-loc}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n) = C^{\log}_{loc}(\mathbb{R}^n) \cap L_\infty(\mathbb{R}^n)$.

(2) Lemma 2.6 does not state the equivalence of spaces of variable smoothness and 2-microlocal function spaces. Only if the weight sequence $\{w_j\}_{j\in\mathbb{N}_0}$ is constructed like $w_j(x) = 2^{js(x)}$, then they coincide. For example, the heavily used ([27], [29] and [20]) weight sequence $w_j(x) = 2^{js}(1+2^j|x-x_0|)^{s'}$ with $x_0 \in \mathbb{R}^n$ and $s, s' \in \mathbb{R}$ can not be reformulated into a variable smoothness function for $s' \neq 0$.

Let us recall the characterization of $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ by local means, proved in [22, Theorem 2.2]. For $\{\psi_k\}_{k\in\mathbb{N}_0} \in \mathcal{S}(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$ and $a > 0$ we define the Peetre maximal operator by

$$(\psi_k^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|(\psi_k * f)(y)|}{1+|2^k(y-x)|^a}, \qquad \text{where } k \in \mathbb{N}_0 \text{ and } x \in \mathbb{R}^n. \tag{7}$$

Proposition 2.8. *Let $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < q \leqslant \infty$, $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and let $a \in \mathbb{R}$, $R \in \mathbb{N}_0$ with $R > \alpha_2$. Further, let ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$ with*

$$D^\beta \psi_1(0) = 0, \qquad \text{for } 0 \leqslant |\beta| < R, \tag{8}$$

and

$$|\psi_0(x)| > 0 \qquad \text{on} \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\} \tag{9}$$

$$|\psi_1(x)| > 0 \qquad \text{on} \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\} \tag{10}$$

for some $\varepsilon > 0$.

(1) *For $a > \frac{n}{p^-} + \alpha$ and for all $f \in \mathcal{S}'(\mathbb{R}^n)$ we have*

$$\big\| f | B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\big\| \sim \big\| (\Psi_k * f) w_k | \ell_q(L_{p(\cdot)})\big\| \sim \big\| (\Psi_k^* f)_a w_k | \ell_q(L_{p(\cdot)})\big\| .$$

(2) *If $p^+ < \infty$, then for $a > \frac{n}{\min(p^-,q)} + \alpha$*

$$\left\| f \right| F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\big\| \sim \left\| (\Psi_k * f) w_k \right| L_{p(\cdot)}(\ell_q)\big\| \sim \left\| (\Psi_k^* f)_a w_k \right| L_{p(\cdot)}(\ell_q)\big\|$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

If $R = 0$, then we do not need any moment conditions (8) on ψ_1. The local means characterization gives us that the definition of the spaces $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ is independent on the start function φ_0 of the resolution of unity. To this end, we do not need an index φ in the notation of the norm.

3. Decomposition by Atoms and Molecules

In this section we present two decomposition theorems. We characterize the spaces $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ via decompositions by atoms and molecules. First we introduce the basic notation.

3.1. Preliminaries

For $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$ we define the closed cube $Q_{\nu m}$ with center in $2^{-\nu} m$ and with sides parallel to the axes and length $2^{-\nu}$. By $\chi_{\nu m}$ we denote the characteristic function of the cube $Q_{\nu m}$.

Definition 3.1. *Let $\boldsymbol{w} = \{w_k\}_{k \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < q \leqslant \infty$ and let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. Then for all complex valued sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ we define*

$$b^{\boldsymbol{w}}_{p(\cdot),q} = \left\{ \lambda : \left\| \lambda \right| b^{\boldsymbol{w}}_{p(\cdot),q} \right\| < \infty \right\}$$

where

$$\left\| \lambda \right| b^{\boldsymbol{w}}_{p(\cdot),q}\right\| := \left(\sum_{\nu=0}^{\infty} \left\| \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m) \chi_{\nu m}(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q}.$$

Furthermore, for $p^+ < \infty$ we define

$$f^{\boldsymbol{w}}_{p(\cdot),q} = \left\{ \lambda : \left\| \lambda \right| f^{\boldsymbol{w}}_{p(\cdot),q} \right\| < \infty \right\}$$

where

$$\left\| \lambda \right| f^{\boldsymbol{w}}_{p(\cdot),q}\right\| := \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^q w_\nu^q(2^{-\nu} m) \chi_{\nu m}(x) \right)^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|.$$

We define atoms which are the building blocks for atomic decompositions.

Definition 3.2. *Let $K, L \in \mathbb{N}_0$ and let $\gamma > 1$. A K-times continuous differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K,L]$-atom centered at $Q_{\nu m}$, $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if*

$$\operatorname{supp} a \subseteq \gamma Q_{\nu m}, \tag{11}$$

$$|D^\beta a(x)| \leqslant 2^{|\beta|\nu}, \qquad \textit{for } 0 \leqslant |\beta| \leqslant K \tag{12}$$

and if

$$\int_{R^n} x^\beta a(x)dx = 0 \qquad \textit{for } 0 \leqslant |\beta| < L \textit{ and } \nu \geqslant 1. \tag{13}$$

If an atom a is centered at $Q_{\nu m}$, that means if it fulfills (11), then we denote it by $a_{\nu m}$. We recall the definition $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ and point out that for $\nu = 0$ or $L = 0$ there are no moment conditions (13) required.

Definition 3.3. *Let $K, L \in \mathbb{N}_0$ and let $M > 0$. A K-times continuous differentiable function $\mu \in C^K(\mathbb{R}^n)$ is called $[K, L, M]$-molecule concentrated in $Q_{\nu m}$, if for some $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$*

$$|D^\beta \mu(x)| \leqslant 2^{|\beta|\nu}(1 + 2^\nu|x - 2^{-\nu}m|)^{-M}, \qquad \textit{for } 0 \leqslant |\beta| \leqslant K \tag{14}$$

and

$$\int_{R^n} x^\beta \mu(x)dx = 0 \qquad \textit{if } 0 \leqslant |\beta| < L \textit{ and } \nu \geqslant 1. \tag{15}$$

Remark 3.4.

(1) For $L = 0$ or $\nu = 0$ there are no moment conditions (15) required. If a molecule is concentrated in $Q_{\nu m}$, that means it satisfies (14), then it is denoted by $\mu_{\nu m}$.
(2) If $a_{\nu m}$ is a $[K, L]$-atom, then it is a $[K, L, M]$-molecule for every $M > 0$.

3.2. Basic results

For proving the decomposition by atoms and molecules we need four basic lemmas. The next lemmas go back to Frazier & Jawerth [15] and a proof adapted to our setting can be found in [21].

Lemma 3.5 (Lemma 3.3 in [15]). *Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be resolution of unity and let $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ be $[K, L, M]$ molecules. Then*

$$|(\varphi_j^\vee * \mu_{\nu m})(x)| \leqslant c2^{-(\nu-j)(L+n)}(1 + 2^j|x - 2^{-\nu}m|)^{L+n-M}, \qquad \textit{for } j \leqslant \nu$$

and

$$|(\varphi_j^\vee * \mu_{\nu m})(x)| \leqslant c2^{-(j-\nu)K}(1 + 2^j|x - 2^{-\nu}m|)^{-M}, \qquad \textit{for } j \geqslant \nu.$$

We also need a partition of unity of Calderon type; a proof can be found in [15].

Lemma 3.6 (Theorem 6 in [15]). *Let $\{\varphi_j\}_{j\in\mathbb{N}_0} \in \Phi(\mathbb{R}^n)$ be a resolution of unity and let $R \in \mathbb{N}$. Then there exist functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ with:*

$$\operatorname{supp}\theta_0, \operatorname{supp}\theta \subseteq \{x \in \mathbb{R}^n : |x| \leqslant 1\}, \tag{16}$$

$$|\hat{\theta}_0(\xi)| \geqslant c_0 > 0 \qquad \textit{for } \ |\xi| \leqslant 2, \tag{17}$$

$$|\hat{\theta}(\xi)| \geqslant c > 0 \qquad \textit{for } \ \frac{1}{2} \leqslant |\xi| \leqslant 2, \tag{18}$$

$$\int_{\mathbb{R}^n} x^\gamma \theta(x) dx = 0 \qquad \textit{for } \ 0 \leqslant |\gamma| \leqslant R \tag{19}$$

such that

$$\hat{\theta}_0(\xi)\hat{\psi}_0(\xi) + \sum_{j=1}^{\infty} \hat{\theta}(2^{-j}\xi)\hat{\psi}(2^{-j}\xi) = 1, \qquad \textit{for all } \xi \in \mathbb{R}^n, \tag{20}$$

where the functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ are defined via

$$\hat{\psi}_0(\xi) = \frac{\varphi_0(\xi)}{\hat{\theta}_0(\xi)} \qquad \textit{and} \qquad \hat{\psi}(\xi) = \frac{\varphi_1(2\xi)}{\hat{\theta}(\xi)}. \tag{21}$$

The next lemma can be found in [24, Lemma 7.1] and a proof in our notation is postponed to the appendix.

Lemma 3.7. *Let $0 < t \leqslant 1$ and $R > \frac{n}{t}$. For any $j, \nu \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, $x \in Q_{jl}$ and any sequence $\{h_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ of complex numbers, we have*

$$\sum_{m\in\mathbb{Z}^n} |h_{\nu m}|(1 + 2^j|x - 2^{-\nu}m|)^{-R} \leqslant c \max(2^{(\nu-j)\frac{n}{t}}, 1)\mathcal{M}_t\left(\sum_{m\in\mathbb{Z}^n} |h_{\nu m}\chi_{\nu m}|\right)(x).$$

The next lemma is a Hardy-type inequality which is easy to prove.

Lemma 3.8 (Lemma 3 in [43]). *Let $0 < q \leqslant \infty$, $\delta > 0$ and $p \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leqslant p^+ \leqslant \infty$. Let $\{g_k\}_{k\in\mathbb{N}_0}$ be a sequence of non-negative measurable functions on $\mathbb{R}^n$ and denote*

$$G_\nu(x) = \sum_{k=0}^{\infty} 2^{-|\nu-k|\delta} g_k(x), \qquad x \in \mathbb{R}^n, \ \nu \in \mathbb{N}_0.$$

Then there exist constants $C_1, C_2 \geqslant 0$ such that

$$\left\| G_k | \, \ell_q(L_{p(\cdot)}) \right\| \leqslant C_1 \left\| g_k | \, \ell_q(L_{p(\cdot)}) \right\|$$

and

$$\left\| G_k | \, L_{p(\cdot)}(\ell_q) \right\| \leqslant C_2 \left\| g_k | \, L_{p(\cdot)}(\ell_q) \right\|.$$

We introduce the numbers σ_p and $\sigma_{p,q}$ by

$$\sigma_p = n\left(\frac{1}{\min(1,p^-)} - 1\right) \qquad \text{and} \qquad \sigma_{p,q} = n\left(\frac{1}{\min(1,p^-,q^-)} - 1\right). \tag{22}$$

If p or q are constant exponent functions, then we get back the usual definitions of σ_p and $\sigma_{p,q}$, see [39].

The last task in this subsection is to clarify the convergence of the sum

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}, \tag{23}$$

where $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K,L,M]$ molecules and $\{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ belongs to some sequence space from Definition 3.1. At least, we have to show the convergence of (23) in $\mathcal{S}'(\mathbb{R}^n)$. To this end, we need some embedding theorems between the sequence spaces. A first result are the Sobolev embeddings, which were recently proved in [40, Theorem 3.1] and [2, Theorem 6.4] in the variable smoothness setting. If one looks at the proofs in [40] and [2] it is easy to verify the Sobolev embeddings with admissible weight sequences. The condition needed in the proof of the F-case ([40, Lemma 2.9]) is

$$\frac{w_j(x)}{w_j(y)} \leqslant c \qquad \text{for} \;\; |x-y| \leqslant 2^{-j}$$

which is trivially fulfilled by $\{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$. In the B-case the weights have to fulfill a similar condition ([2, Lemma 4.3]) which is also easy to check.

Now, we state the needed Sobolev embeddings with 2-microlocal weight sequences. We also use the notation $f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ which gets defined in Section 5. To avoid confusions read the proposition as if $0 < q \leqslant \infty$ is a fixed parameter.

Proposition 3.9. *Let $\boldsymbol{w^0}, \boldsymbol{w^1} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ with $\frac{w_j^0(\cdot)}{w_j^1(\cdot)} \geqslant 1$ and $p_0(\cdot), p_1(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with*

$$\frac{w_j^0(x)}{w_j^1(x)} = 2^{j\left(\frac{n}{p_0(x)} - \frac{n}{p_1(x)}\right)} \qquad \textit{for all} \;\; x \in \mathbb{R}^n.$$

(1) *Let $0 < p_0(x) \leqslant p_1(x) < \infty$ with $0 < p_0^- \leqslant p_1^- \leqslant p_1^+ < \infty$ and let $q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $q(x) = \infty$ for all $x \in \mathbb{R}^n$ or $0 < q^- \leqslant q(x) < \infty$ for all $x \in \mathbb{R}^n$. Then*

$$f^{\boldsymbol{w^0}}_{p_0(\cdot),q(\cdot)} \hookrightarrow f^{\boldsymbol{w^1}}_{p_1(\cdot),q(\cdot)}.$$

(2) *If $0 < q \leqslant \infty$, then*

$$b^{\boldsymbol{w^0}}_{p_0(\cdot),q} \hookrightarrow b^{\boldsymbol{w^1}}_{p_1(\cdot),q}.$$

Further, we need an embedding between B and F spaces. This was already done in [2, Theorem 6.1]. Since we only need the case where q is constant, it is easy to have also $q = \infty$ included.

Proposition 3.10. *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $0 < q \leqslant \infty$, then*

$$b^{\boldsymbol{w}}_{p(\cdot),\min(p^-,q)} \hookrightarrow f^{\boldsymbol{w}}_{p(\cdot),q} \hookrightarrow b^{\boldsymbol{w}}_{p(\cdot),\max(p^+,q)}.$$

Finally, we can state the lemma on the convergence of (23) in $\mathcal{S}'(\mathbb{R}^n)$. The proof is again postponed to the appendix.

Lemma 3.11. *Let $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ and $0 < q \leqslant \infty$. Furthermore, let $K, L \in \mathbb{N}_0$ and $M > 0$ with*

$$L > \sigma_p - \alpha_1, \qquad K \text{ arbitrary and } M \text{ large enough.}$$

If $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q}$ or $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q}$ and $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K, L, M]$-molecules concentrated in $Q_{\nu m}$, then the sum

$$\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(x) \tag{24}$$

converges in $\mathcal{S}'(\mathbb{R}^n)$.

3.3. Atomic and molecular decompositions

Now we state one direction of the atomic decomposition theorem.

Theorem 3.12. *Let $\{w_\nu\}_{\nu\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $0 < q \leqslant \infty$. Further, let $K, L \in \mathbb{N}_0$ and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $0 < p^- \leqslant p^+ \leqslant \infty$.*

(1) *If*

$$K > \alpha_2 \qquad \text{and} \qquad L > \sigma_p - \alpha_1,$$

then for each $f \in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ there exist $[K, L]$-atoms $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ centered at $Q_{\nu m}$ and $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q}$ such that

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \qquad \text{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{25}$$

holds. Moreover

$$\left\| \lambda | \, b^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|$$

where the constant $c > 0$ is universal for all $f \in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

(2) *If* $p^+ < \infty$ *and*

$$K > \alpha_2 \qquad \text{and} \qquad L > \sigma_{p,q} - \alpha_1,$$

then for each $f \in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ *there exist* $[K,L]$*-atoms* $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *centered at* $Q_{\nu m}$ *and* $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q}$ *such that* (25) *holds. Moreover*

$$\left\| \lambda | \, f^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f | \, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|$$

where the constant $c > 0$ *is universal for all* $f \in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

Proof. The proof follows the ideas in [15] and [14]. Because every $[K,L]$ atom is a $[K,L,M]$ molecule for arbitrary $M > 0$ from Lemma 3.11 we have the convergence of (25) in $\mathcal{S}'(\mathbb{R}^n)$. We use Lemma 3.6 with $R = L - 1$, the functions $\theta_0, \theta \in \mathcal{S}(\mathbb{R}^n)$ with the properties (16)-(20) and the functions $\psi_0, \psi \in \mathcal{S}(\mathbb{R}^n)$ with (21). Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then from Lemma 3.6 we get

$$f = f * \theta_0 * \psi_0 + \sum_{\nu=1}^{\infty} 2^{\nu n} \theta(2^\nu \cdot) * \psi_\nu * f,$$

where $\psi_\nu(\cdot) = 2^{\nu n}\psi(2^\nu \cdot)$. Now, splitting the integration with respect to the cubes $Q_{\nu m}$ we derive

$$\begin{aligned} f(x) = & \sum_{m\in\mathbb{Z}^n} \int_{Q_{0m}} \theta_0(x-y)(\psi_0 * f)(y)dy \\ & + \sum_{\nu=1}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{\nu n} \int_{Q_{\nu m}} \theta(2^\nu(x-y))(\psi_\nu * f)(y)dy. \end{aligned}$$

For each $\nu \in \mathbb{N}$ and all $m \in \mathbb{Z}^n$ we define

$$\lambda_{\nu m} = C_\theta \sup_{y\in Q_{\nu m}} |(\psi_\nu * f)(y)|,$$

where $C_\theta = \max\{\sup_{|x|\leqslant 1} |D^\beta \theta(x)| : |\beta| \leqslant K\}$. If $\lambda_{\nu m} \neq 0$, then we define

$$a_{\nu m}(x) = \frac{1}{\lambda_{\nu m}} 2^{\nu n} \int_{Q_{\nu m}} \theta(2^\nu(x-y))(\psi_\nu * f)(y)dy,$$

otherwise, we set $a_{\nu m}(x) = 0$. The a_{0m} atoms and λ_{0m} are defined similarly by using θ_0 and ψ_0. Clearly, (25) holds and the properties of θ_0, ψ_0, θ and ψ_ν ensure that $a_{\nu m}$ are $[K,L]$-atoms. It remains to prove, that there exists a constant c such that $\left\| \lambda | \, a^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f | \, A^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|$; where a stands for b or f and A for B or F respectively.

For fixed $\nu \in \mathbb{N}_0$ and $a > \frac{n}{\min(p^-,q)} + \alpha$ we obtain

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} w_\nu(x)\lambda_{\nu m}\chi_{\nu m}(x) &\leqslant c \sum_{m\in\mathbb{Z}^n} w_\nu(x) \sup_{y\in Q_{\nu m}} |(\psi_\nu * f)(y)|\chi_{\nu m}(x) \\ &\leqslant c' w_\nu(x) \sup_{|z|\leqslant c2^{-\nu}} \frac{|(\psi_\nu * f)(x-z)|}{1+|2^\nu z|^a}(1+|2^\nu z|^a) \\ &\leqslant c'' w_\nu(x)(\psi_\nu^* f)_a(x), \end{aligned} \tag{26}$$

since $|x-y| \leqslant c2^{-\nu}$ for $x, y \in Q_{\nu m}$ and $\sum_{m\in\mathbb{Z}^n}\chi_{\nu m}(x) = 1$. Here, $(\psi_\nu^* f)_a$ denotes the Peetre maximal operator, defined in (7). In applying the $L_{p(\cdot)}(\mathbb{R}^n)$ and the ℓ_q norm we get

$$\left\| \lambda |\, b^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left(\sum_{\nu=0}^{\infty} \left\| \sum_{m\in\mathbb{Z}^n} |2^{\nu s} w_\nu(x)(\psi_\nu^* f)_a(x)| \,\Big|\, L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q}. \tag{27}$$

If we apply the $L_{p(\cdot)}(\ell_q)$ norm on (26), then we obtain

$$\left\| \lambda |\, f^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |2^{\nu s} w_\nu(x)(\psi_\nu^* f)_a(x)|^q \right)^{1/q} \Big|\, L_{p(\cdot)}(\mathbb{R}^n) \right\|. \tag{28}$$

Since $\psi_0 \in \mathcal{S}(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ are two kernels which fulfill the moment conditions (8) and the Tauberian conditions (9) and (10), we can use Proposition 2.8 with $a > \frac{n}{\min(p^-,q)} + \alpha$ and derive from (27) and (28)

$$\left\| \lambda |\, b^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f |\, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|$$

and

$$\left\| \lambda |\, f^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f |\, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|,$$

which prove both parts of the theorem. ■

To get the reverse direction of the atomic decomposition theorem we use the more general molecules. Afterwards the atomic decomposition theorem follows easily.

Theorem 3.13. *Let* $\{w_\nu\}_{\nu\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ *and* $0 < q \leqslant \infty$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $0 < p^- \leqslant p^+ \leqslant \infty$.

(1) *Let* $K, L \in \mathbb{N}_0$ *with*

$$K > \alpha_2 \qquad \text{and} \qquad L > \sigma_p - \alpha_1$$

and $M > 0$ large enough. If $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$-molecules and $\lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n} \in b^{\boldsymbol{w}}_{p(\cdot),q}$, then

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}, \text{ convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{29}$$

is an element of $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and

$$\left\| f | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\| \leqslant c \left\| \lambda | \, b^{\boldsymbol{w}}_{p(\cdot),q}\right\|. \tag{30}$$

(2) *Let $p^+ < \infty$ and $K, L \in \mathbb{N}_0$ with*

$$K > \alpha_2 \qquad \text{and} \qquad L > \sigma_{p,q} - \alpha_1,$$

and $M > 0$ large enough. If $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$-molecules and $\lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n} \in f^{\boldsymbol{w}}_{p(\cdot),q}$, then (29) is an element of $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and

$$\left\| f | \, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\| \leqslant c \left\| \lambda | \, f^{\boldsymbol{w}}_{p(\cdot),q}\right\|. \tag{31}$$

Proof. The convergence of the sum (29) in $\mathcal{S}'(\mathbb{R}^n)$ follows from Lemma 3.11. As usual we divide the summation (29) in dependence on $j \in \mathbb{N}_0$ into two parts

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m} = \sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} \ldots + \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} \ldots = f_j + f^j.$$

Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity from (4). Now, let us prove the first part of the theorem for the Besov spaces. We have

$$\begin{aligned}
\left\| f | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\| &\leqslant c \left(\left\| f_j | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\| + \left\| f^j | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\| \right) \\
&= c \left(\sum_{j=0}^{\infty} \left\| \sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} (\varphi_j^\vee * \mu_{\nu m})(\cdot) w_j(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} \\
&\quad + c \left(\sum_{j=0}^{\infty} \left\| \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} (\varphi_j^\vee * \mu_{\nu m})(\cdot) w_j(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} \\
&= \sigma_1 + \sigma_2.
\end{aligned}$$

We estimate σ_1. We use Lemma 3.5 with $\nu \leqslant j$ and by the properties of the weight sequence we obtain

$$\begin{aligned}
&\sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} |(\varphi_j^\vee * \mu_{\nu m})(x) w_j(x) \lambda_{\nu m}| \\
&\qquad \leqslant c \sum_{\nu=0}^{j} \sum_{m\in\mathbb{Z}^n} 2^{-(j-\nu)(K-\alpha_2)} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)(1 + 2^j|x - 2^{-\nu}m|)^{\alpha - M}
\end{aligned}$$

and using Lemma 3.7 with $t < \min(1,p^-)$ and $M-\alpha > n/t$ we can estimate this from above by

$$\leqslant c' \sum_{\nu=0}^{j} 2^{-(j-\nu)(K-\alpha_2)} \mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(x).$$

Further, using Lemma 3.8 with $\delta = K-\alpha_2 > 0$ and the fact that $\mathcal{M}_t$ is bounded on $L_{p(\cdot)}(\mathbb{R}^n)$

$$\begin{aligned}
\sigma_1 &\leqslant c \left(\sum_{j=0}^{\infty} \left\| \sum_{\nu=0}^{j} 2^{-(j-\nu)\delta} \mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} \\
&\leqslant c' \left(\sum_{\nu=0}^{\infty} \left\| \mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} \\
&\leqslant c'' \left(\sum_{\nu=0}^{\infty} \left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} \\
&= c'' \left\| \lambda \right| b^{\boldsymbol{w}}_{p(\cdot),q} \right\|. \qquad (32)
\end{aligned}$$

For σ_2 we use Lemma 3.5 with $\nu \geqslant j$ and obtain

$$\begin{aligned}
&\sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} |(\varphi_j^\vee * \mu_{\nu m})(x) w_j(x) \lambda_{\nu m}| \\
&\qquad \leqslant c \sum_{\nu=j+1}^{\infty} \sum_{m\in\mathbb{Z}^n} 2^{-(\nu-j)(L+n+\alpha_1)} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)(1+2^j|x-2^{-\nu}m|)^{\alpha-M+L+n}
\end{aligned}$$

and using Lemma 3.7 with $t < \min(1,p^-)$ and $M-\alpha-L-n > n/t$ we estimate further

$$\leqslant c' \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(L+n+\alpha_1-\frac{n}{t})} \mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(x).$$

Finally, using Lemma 3.8 with $\delta = L+n+\alpha_1-\frac{n}{t} > 0$ we obtain

$$\begin{aligned}
\sigma_2 &\leqslant c \left(\sum_{j=0}^{\infty} \left\| \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)\delta} \mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} \\
&\leqslant c' \left(\sum_{\nu=0}^{\infty} \left\| \mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} \qquad (33) \\
&\leqslant c'' \left(\sum_{\nu=0}^{\infty} \left\| \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^q \right)^{1/q} = c'' \left\| \lambda \right| b^{\boldsymbol{w}}_{p(\cdot),q} \right\|.
\end{aligned}$$

Now, by (32) and (33) we get $\left\| f \right| B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\| \leqslant c \left\| \lambda \right| b^{\boldsymbol{w}}_{p(\cdot),q} \right\|$.

In the F-case we do the same estimates and calculations as above, use the second part of Lemma 3.8 and the boundedness of $\mathcal{M}_t$ on $L_{p(\cdot)}(\ell_q)$ for $t < \min(p^-, q, 1)$ to obtain (31). ■

Remark 3.14. Closer inspection of the proof shows that $M > \sigma_p + L + 2n + 2\alpha$ ($\sigma_{p,q}$ in the F-case) is a sufficient condition for M.

Since every $[K, L]$-atom is a $[K, L, M]$-molecule for every $M > 0$ we get the following corollary.

Corollary 3.15. *Let $\{w_j\}_{j\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and let $0 < q \leqslant \infty$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$.*

(1) *Let $K, L \in \mathbb{N}_0$ with*

$$K > \alpha_2 \qquad \text{and} \qquad L > \sigma_p - \alpha_1.$$

(a) *If $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q}$ and $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K, L]$-atoms centered at $Q_{\nu m}$, then*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \qquad \text{convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{34}$$

belongs to the space $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and there exists a constant $c > 0$ with

$$\left\| f | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\| \leqslant c \left\| \lambda | \, b^{\boldsymbol{w}}_{p(\cdot),q} \right\|.$$

The constant $c > 0$ is universal for all λ and $a_{\nu m}$.

(b) *For every $f \in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$, there exists $\lambda \in b^{\boldsymbol{w}}_{p(\cdot),q}$ and $[K, L]$-atoms centered at $Q_{\nu m}$ such that there exists a representation (34), converging in $\mathcal{S}'(\mathbb{R}^n)$, with*

$$\left\| \lambda | \, b^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|,$$

where the constant $c > 0$ is universal for all $f \in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

(2) *Let $p^+ < \infty$ and $K, L \in \mathbb{N}_0$ with*

$$K > \alpha_2 \qquad \text{and} \qquad L > \sigma_{p,q} - \alpha_1.$$

(a) If $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q}$ and $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K, L]$-atoms centered at $Q_{\nu m}$, then f represented by (34) belongs to the space $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and there exists a constant $c > 0$ with

$$\left\| f | \, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\| \leqslant c \left\| \lambda | \, f^{\boldsymbol{w}}_{p(\cdot),q} \right\|.$$

The constant $c > 0$ is universal for all λ and $a_{\nu m}$.

(b) For every $f \in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$, there exists $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q}$ and $[K, L]$-atoms centered at $Q_{\nu m}$ such that there exists a representation (34), converging in $\mathcal{S}'(\mathbb{R}^n)$, with

$$\left\| \lambda | \, f^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f | \, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|,$$

where the constant $c > 0$ is universal for all $f \in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

4. Wavelet decomposition

In this section we describe the function spaces $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ by a decomposition with wavelets. The proof relies on [21] and [24]. The ingredients are the local means characterization (Proposition 2.8) and the just proved decompositions by molecules and atoms.

4.1. Preliminaries

First, we recall some results from wavelet theory.

Proposition 4.1.

(1) *There are a real scaling function $\psi_F \in \mathcal{S}(\mathbb{R})$ and a real associated wavelet $\psi_M \in \mathcal{S}(\mathbb{R})$ such that their Fourier transforms have compact supports, $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ and*

$$\operatorname{supp} \widehat{\psi_M} \subseteq \left[-\frac{8}{3}\pi, -\frac{2}{3}\pi\right] \cup \left[\frac{2}{3}\pi, \frac{8}{3}\pi\right].$$

(2) *For any $k \in \mathbb{N}$ there exist a real compactly supported scaling function $\psi_F \in C^k(\mathbb{R})$ and a real compactly supported associated wavelet $\psi_M \in C^k(\mathbb{R})$ such that $\widehat{\psi_F}(0) = (2\pi)^{-1/2}$ and*

$$\int_{\mathbb{R}} x^l \psi_M(x) dx = 0 \qquad \text{for all } \ l \in \{0, \dots, k-1\}.$$

In both cases we have, that $\{\psi_{\nu m} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}\}$ is an orthonormal basis in $L_2(\mathbb{R})$, where

$$\psi_{\nu m}(t) := \begin{cases} \psi_F(t-m), & \text{if } \nu = 0,\ m \in \mathbb{Z} \\ 2^{\frac{\nu-1}{2}} \psi_M(2^{\nu-1}t - m), & \text{if } \nu \in \mathbb{N},\ m \in \mathbb{Z} \end{cases}$$

and the functions ψ_M, ψ_F are according to (i) or (ii).

This proposition is taken over from [39, Theorem 1.61]. The wavelets in the first part are called Meyer wavelets. They do not have a compact support but they are fast decaying functions ($\psi_F, \psi_M \in \mathcal{S}(\mathbb{R})$) and ψ_M has infinitely many moment conditions. The wavelets from the second part are called Daubechies wavelets. Here the functions ψ_M, ψ_F do have compact support, but they only have limited smoothness. Both types of wavelets are well described in [41, Chapters 3 and 4].

This orthonormal basis can be generalized to the n-dimensional case by a tensor product procedure. We take over the notation from [39, Section 4.2.1] with $l = 0$. Let ψ_M, ψ_F be the Meyer or Daubechies wavelets described above. Now, we define

$$G^0 = \{F, M\}^n \qquad \text{and} \qquad G^\nu = \{F, M\}^{n*} \quad \text{if } \nu \geqslant 1,$$

where the * indicates, that at least one G_i of $G = (G_1, \dots, G_n) \in \{F, M\}^{n*}$ must be an M. It is clear from the definition, that the cardinal number of $\{F, M\}^{n*}$ is $2^n - 1$. Let for $x \in \mathbb{R}^n$

$$\Psi^{\nu}_{Gm}(x) = 2^{\nu\frac{n}{2}} \prod_{r=1}^{n} \psi_{G_r}(2^{\nu} x_r - m_r), \tag{35}$$

where $G \in G^{\nu}$, $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$. Then $\{\Psi^{\nu}_{Gm} : \nu \in \mathbb{N}_0, G \in G^{\nu}, m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$. Finally, we have to adjust the sequence spaces $b^{\boldsymbol{w}}_{p(\cdot),q}$ and $f^{\boldsymbol{w}}_{p(\cdot),q}$ to our situation.

Definition 4.2. *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < q \leqslant \infty$ and $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leqslant p^+ \leqslant \infty$.*

(1) *Then*

$$\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q} := \Big\{ \lambda = \{\lambda^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^{\nu}, m\in\mathbb{Z}^n} \subset \mathbb{C} : \Big\| \lambda | \, \tilde{b}^{\boldsymbol{w}}_{p(\cdot),q} \Big\| < \infty \Big\}$$

where

$$\Big\| \lambda | \, \tilde{b}^{\boldsymbol{w}}_{p(\cdot),q} \Big\| = \left(\sum_{\nu=0}^{\infty} \sum_{G\in G^{\nu}} \left\| \sum_{m\in\mathbb{Z}^n} w_{\nu}(2^{-\nu}m) |\lambda^{\nu}_{Gm}| \chi_{\nu m}(x) \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|^{q} \right)^{1/q}.$$

(2) *For $p^+ < \infty$ we define*

$$\tilde{f}^{\boldsymbol{w}}_{p(\cdot),q} := \Big\{ \lambda = \{\lambda^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^{\nu}, m\in\mathbb{Z}^n} \subset \mathbb{C} : \Big\| \lambda | \, \tilde{f}^{\boldsymbol{w}}_{p(\cdot),q} \Big\| < \infty \Big\}$$

where

$$\Big\| \lambda | \, \tilde{f}^{\boldsymbol{w}}_{p(\cdot),q} \Big\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{G\in G^{\nu}} \sum_{m\in\mathbb{Z}^n} w^{q}_{\nu}(2^{-\nu}m) |\lambda^{\nu}_{Gm}|^{q} \chi_{\nu m}(x) \right)^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\|.$$

To get the wavelet characterization we use local means with kernels which only have limited smoothness and we use the molecular decomposition described in the previous section. This idea goes back to [38], [24] and [17]. First, we recall the local means with kernel k

$$k(t, f)(x) = t^{-n} \int_{\mathbb{R}^n} k\left(\frac{y - x}{t}\right) f(y) dy.$$

With $t = 2^{-j}$, $x = 2^{-j} l$ where $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$, one gets

$$k(2^{-j}, f)(2^{-j} l) = 2^{jn} \int_{\mathbb{R}^n} k(2^j y - l) f(y) dy = \int_{\mathbb{R}^n} k_{jl}(y) f(y) dy = k_{jl}(f). \tag{36}$$

We have that (36) is a dual pairing if k_{jl} are the Meyer wavelets, because they are $\mathcal{S}(\mathbb{R}^n)$ functions, and for Daubechies wavelets because of their compact support. Now the usual properties on k get shifted to the kernels k_{jl}.

Definition 4.3. *Let $A, B \in \mathbb{N}_0$ and $c > 0$. Further, let $k_{jl}(x) \in C^A(\mathbb{R}^n)$ with $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$ be functions in R^n with*

$$|D^\beta k_{jl}(x)| \leqslant c 2^{j|\beta| + jn}(1 + 2^j|x - 2^{-j}l|)^{-C}, \qquad |\beta| \leqslant A,$$

for all $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, and

$$\int_{\mathbb{R}^n} x^\beta k_{jl}(x) dx = 0, \qquad |\beta| < B,$$

for $j \geqslant 1$ and $l \in \mathbb{Z}^n$.

From the definition it is clear that $\{2^{-jn}k_{jl}\}_{j\in\mathbb{N}_0, l\in\mathbb{Z}^n}$ are $[A, B, C]$ molecules.

4.2. Wavelet isomorphism

We want to use the molecular decomposition obtained in the last section. We assume that $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K, L, M]$ molecules and that the $\{k_{jl}\}_{j\in\mathbb{N}_0, l\in\mathbb{Z}^n}$ are the above given functions from Definition 4.3.

Before coming to the theorem we recall two fundamental lemmas. First, we have to give estimates of the quantity $|\langle \mu_{\nu m}, k_{jl}\rangle|$.

Lemma 4.4 (Appendix B in [16]).

(i) *Let $\nu \geqslant j$, $M > A + n$ and $L \geqslant A$, then*

$$|\langle \mu_{\nu m}, k_{jl}\rangle| \leqslant c 2^{-(\nu-j)(A+n)}(1 + 2^j|2^{-\nu}m - 2^{-j}l|)^{-\min(M-A-n, C)}. \quad (37)$$

(ii) *Let $\nu \leqslant j$, $C > K + n$ and $B \geqslant K$, then*

$$|\langle \mu_{\nu m}, k_{jl}\rangle| \leqslant c 2^{-(j-\nu)K}(1 + 2^\nu|2^{-\nu}m - 2^{-j}l|)^{-\min(M, C-K-n)}. \quad (38)$$

The second lemma is just a reformulation of Lemma 3.7. For the proof, just take Lemma 3.7 and remember that $x \in Q_{jl}$, which means $0 \leqslant |x - 2^{-j}l| \leqslant 2^{-j}$.

Lemma 4.5. *Let $0 < t \leqslant 1$ and $R > \frac{n}{t}$. For any $j, \nu \in \mathbb{N}_0$, any $l \in \mathbb{Z}^n$ and any sequence $\{h_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ of complex numbers, we have with $x \in Q_{jl}$*

$$\sum_{m\in\mathbb{Z}^n} |h_{\nu m}|(1 + 2^j|2^{-j}l - 2^{-\nu}m|)^{-R} \leqslant c \max(2^{(\nu-j)\frac{n}{t}}, 1)\mathcal{M}_t\left(\sum_{m\in\mathbb{Z}^n} |h_{\nu m}\chi_{\nu m}|\right)(x).$$

Now, we are ready to state the first theorem, which gives us one direction of the wavelet decomposition. We define $k(f) = \{k_{jl}(f) : j \in \mathbb{N}_0, l \in \mathbb{Z}^n\}$.

Theorem 4.6. *Let $\boldsymbol{w} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < q \leqslant \infty$ and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. Further, let $\{k_{jl}\}_{j\in\mathbb{N}_0, l\in\mathbb{Z}^n}$ be as in Definition 4.3 with $C > 0$ large enough and $A, B \in \mathbb{N}_0$.*

(1) *If*

$$A > \sigma_p - \alpha_1 \qquad \text{and} \qquad B > \alpha_2,$$

then for some $c > 0$ and all $f \in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$

$$\left\| k(f) | \, b^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|.$$

(2) *Let* $p^+ < \infty$. *If*

$$A > \sigma_{p,q} - \alpha_1 \qquad \text{and} \qquad B > \alpha_2,$$

then for some $c > 0$ *and all* $f \in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$

$$\left\| k(f) | \, f^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| f | \, F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \right\|.$$

Proof. This time we only prove the F-part of the Theorem. The Besov spaces part follows the same line of arguments. We apply the decomposition by atoms to $f \in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and get

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m}, \tag{39}$$

where $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ are $[K, L]$-atoms with $K = B > \alpha_2$ and $L = A > \sigma_{p,q} - \alpha_1$. By Corollary 3.15 it is sufficient to find a $c > 0$ with

$$\left\| k(f) | \, f^{\boldsymbol{w}}_{p(\cdot),q} \right\| \leqslant c \left\| \lambda | \, f^{\boldsymbol{w}}_{p(\cdot),q} \right\|. \tag{40}$$

As usual, we decompose the sum in (39) in

$$f = \sum_{\nu=0}^{j} \ldots + \sum_{\nu=j+1}^{\infty} \ldots = f_j + f^j$$

and derive

$$k_{jl}(f) = \int_{\mathbb{R}^n} k_{jl}(y) f_j(y) dy + \int_{\mathbb{R}^n} k_{jl}(y) f^j(y) dy.$$

We use Lemma 4.4 where $\mu_{\nu m} = a_{\nu m}$; that means that $M = \infty$ in the formulation of the lemma. For $\nu \leqslant j$ we obtain by Lemma 4.4 with $C > K + n$ and Lemma 4.5 with appropriately chosen $t < \min(1, p^-, q)$

$$\begin{aligned}
w_j(x) |k_{jl}(f_j)| &\leqslant c \sum_{\nu=0}^{j} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m} \langle k_{jl}, a_{\nu m} \rangle | w_j(x) \\
&\leqslant c \sum_{\nu=0}^{j} 2^{-(j-\nu)(K-\alpha_2)} \\
&\quad \times \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m)(1 + 2^j |2^{-\nu} m - 2^{-j} l|)^{-C+K+n+\alpha} \\
&\leqslant c' \sum_{\nu=0}^{j} 2^{-(j-\nu)(K-\alpha_2)} \mathcal{M}_t \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m) \chi_{\nu m} \right)(x)
\end{aligned}$$

for $x \in Q_{jl}$ and $C - K - n - \alpha > n/t$. For the norm with $\delta = K - \alpha_2 > 0$ we obtain

$$\begin{aligned}
\left\| k_{jl}(f_j) | f^{\boldsymbol{w}}_{p(\cdot),q} \right\| &= \left\| \left(\sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}^n} |k_{jl}(f_j) w_j(\cdot) \chi_{jl}(\cdot)|^q \right)^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\| \\
&\leqslant c \left\| \left(\sum_{j=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \left[\sum_{\nu=0}^{j} 2^{-(j-\nu)\delta} \right.\right.\right. \\
&\qquad \left.\left.\left. \times \mathcal{M}_t \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m) \chi_{\nu m} \right)(\cdot) \right]^q \chi_{jl}(\cdot) \right)^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\| .
\end{aligned}$$

Now apply Lemma 3.8 and use the vector-valued maximal inequality to estimate further

$$\begin{aligned}
&\leqslant c' \left\| \left(\sum_{\nu=0}^{\infty} \left[\mathcal{M}_t \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m) \chi_{\nu m} \right)(\cdot) \right]^q \right)^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\| \\
&\leqslant c'' \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}|^q w_\nu^q(2^{-\nu} m) \chi_{\nu m}(\cdot) \right)^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\| = c'' \left\| \lambda | f^{\boldsymbol{w}}_{p(\cdot),q} \right\| .
\end{aligned}$$

For $\nu > j$ we use Lemma 4.4 again and obtain

$$\begin{aligned}
|w_j(x) k_{jl}(f^j)| \leqslant c \sum_{\nu=j+1}^{\infty} & 2^{-(\nu-j)(A+n+\alpha_1)} \\
& \times \sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m)(1 + 2^j |2^{-\nu} m - 2^{-j} l|)^{-C+\alpha}.
\end{aligned}$$

By Lemma 4.5 with suitably chosen $t < \min(1, p^-, q)$ and $x \in Q_{jl}$ this can be further estimated to

$$\leqslant c' \sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)(A+n+\alpha_1-n/t)} \mathcal{M}_t \left(\sum_{m \in \mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu} m) \chi_{\nu m} \right)(x).$$

We set $\delta = A - \sigma_{p,q} + \alpha_1 = \delta > 0$ and get in applying the norm and Lemma 3.8

$$
\begin{aligned}
\left\| k_{jl}(f^j) \right| f^{\boldsymbol{w}}_{p(\cdot),q} \Big\| &= \left\| \left(\sum_{j=0}^{\infty} \sum_{l\in\mathbb{Z}^n} |k_{jl}(f^j) w_j(\cdot)\chi_{jl}(\cdot)|^q \right)^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\| \\
&\leqslant c \left\| \left(\sum_{j=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \left[\sum_{\nu=j+1}^{\infty} 2^{-(\nu-j)\delta} \right.\right.\right. \\
&\quad \times \left.\left. \mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(\cdot) \right]^q \chi_{jl}(\cdot) \right)^{1/q} \left| L_{p(\cdot)}(\mathbb{R}^n) \right\| \\
&\leqslant c' \left\| \left[\sum_{\nu=j+1}^{\infty} \left(\mathcal{M}_t \left(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| w_\nu(2^{-\nu}m)\chi_{\nu m} \right)(\cdot) \right)^q \right]^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\| \\
&\leqslant c'' \left\| \left[\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^q w_\nu^q(2^{-\nu}m)\chi_{\nu m}(\cdot) \right]^{1/q} \right| L_{p(\cdot)}(\mathbb{R}^n) \Bigg\| \\
&= c'' \left\| \lambda \right| f^{\boldsymbol{w}}_{p(\cdot),q} \Big\| .
\end{aligned}
$$

Finally, we obtain (40) and Corollary 3.15 ensures

$$
\left\| k_{jl}(f) \right| f^{\boldsymbol{w}}_{p(\cdot),q} \Big\| \leqslant c \left\| f \right| F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n) \Big\| .
$$

■

Remark 4.7. Deeper inspection shows that

$$
C > \max(A, B) + 2\alpha + 2n + \sigma_{p,q}
$$

is a sufficient condition ($C > \max(A, B) + 2\alpha + 2n + \sigma_p$ in the B-case).

We come to the wavelet decomposition theorem. Let us assume that

$$
\psi_M \in C^k(\mathbb{R}^n) \qquad \text{and} \qquad \psi_F \in C^k(\mathbb{R}^n) \tag{41}
$$

are the real compactly supported Daubechies wavelets from Proposition 4.1, with

$$
\int_{\mathbb{R}^n} x^\beta \psi_M(x)dx = 0 \qquad \text{for } |\beta| < k. \tag{42}
$$

By the tensor product procedure (35) we have that $\{\Psi^\nu_{G,m} : \nu \in \mathbb{N}_0, G \in G^\nu$ and $m \in \mathbb{Z}^n\}$ is an orthonormal basis in $L_2(\mathbb{R}^n)$.

Before coming to the theorem we clarify the convergence of

$$
\sum_{\nu=0}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \Psi^\nu_{Gm} \qquad \text{with } \lambda \in b^{\boldsymbol{w}}_{p(\cdot),q}. \tag{43}
$$

We say that a series converges unconditionally, if any rearrangement of the series also converges to the same outcome. We know that $\{2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ are $[k,k,M]$-molecules for every $M>0$ and therefore we have the unconditional convergence of (43) in $\mathcal{S}'(\mathbb{R}^n)$ from Lemma 1 in [21] with $k>\sigma_p-\alpha_1$.

Moreover, the following proof shows the unconditional convergence of (43) in $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ for $0<q<\infty$ and $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ with $p^+<\infty$. If $0<p^+<\infty$ and $0<q\leqslant\infty$, then we have unconditional convergence in $B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$, where $\{\varrho_j\}_{j\in\mathbb{N}_0}$ is an admissible weight sequence with $\sup_{x\in\mathbb{R}^n}\frac{\varrho_j(x)}{w_j(x)}\to 0$.

In the case of no restrictions on $p(\cdot),q$ we also have unconditional convergence in $B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$ where $\{\varrho_j\}_{j\in\mathbb{N}_0}$ is an admissible weight sequence with $\sup_{x\in\mathbb{R}^n}\frac{\varrho_j(x)}{w_j(x)}\to 0$ and $\lim_{|x|\to\infty}\sup_{j\in\mathbb{N}_0}\frac{\varrho_j(x)}{w_j(x)}=0$. For the Triebel-Lizorkin spaces we have the same convergence assertions; the last case is missing due to the restriction $p^+<\infty$.

Theorem 4.8. *Let $\boldsymbol{w}\in\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0<q\leqslant\infty$ and $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$.*

(1) *Let $f\in\mathcal{S}'(\mathbb{R}^n)$ and*

$$k>\max(\sigma_p-\alpha_1,\alpha_2) \tag{44}$$

in (41) and (42). Then $f\in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f=\sum_{\nu=0}^{\infty}\sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n}\lambda^{\nu}_{Gm}2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm}\qquad \text{with}\ \ \lambda\in\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}, \tag{45}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any space $B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$ with $\frac{\varrho_\nu(x)}{w_\nu(x)}\to 0$ for $|x|\to\infty$ and all ν and also $\sup_{x\in\mathbb{R}^n}\frac{\varrho_\nu(x)}{w_\nu(x)}\to 0$ for $\nu\to\infty$. The representation (45) is unique, we have

$$\lambda^{\nu}_{Gm}=\lambda^{\nu}_{Gm}(f)=2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle \tag{46}$$

and

$$I: f\mapsto\left\{2^{\nu\frac{n}{2}}\langle f,\Psi^{\nu}_{Gm}\rangle\right\} \tag{47}$$

is an isomorphic map from $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ onto $\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}$. Moreover, if in addition $\max(p^+,q)<\infty$, then $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ is an unconditional basis in $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

(2) *Let $p^+<\infty$, $f\in\mathcal{S}'(\mathbb{R}^n)$ and*

$$k>\max(\sigma_{p,q}-\alpha_1,\alpha_2)$$

in (41) and (42). Then $f\in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ if, and only if, it can be represented as (45) with $\lambda\in\tilde{f}^{\boldsymbol{w}}_{p(\cdot),q}$ with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any space $F^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$ with $\sup_{x\in\mathbb{R}^n}\frac{\varrho_j(x)}{w_j(x)}\to 0$ for $j\to\infty$. The representation

(45) *is unique, we have* (46) *and* (47) *is an isomorphic map from* $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ *onto* $\tilde{f}^{\boldsymbol{w}}_{p(\cdot),q}$. *Moreover, if in addition* $q<\infty$, *then* $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ *is an unconditional basis in* $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

Proof. *Step I.* Let $f\in\mathcal{S}'(\mathbb{R}^n)$ be given by (45). Then by the support properties we have that $\{2^{-\nu\frac{n}{2}}\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ are $[k,k,M]$ molecules for every $M>0$. From Theorem 3.13 and (44) we obtain $f\in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ and

$$\left\| f|\, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\| \leqslant c\left\|\lambda|\,\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}\right\| \tag{48}$$

with $c>0$ independent of $\lambda\in\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}$.

Step II. Let $f\in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ then we can apply Theorem 4.6 with $k_{\nu m}=2^{\nu\frac{n}{2}}\Psi^\nu_{Gm}$. Since all conditions on $k_{\nu m}$ are fulfilled by (44) and the compact support of the wavelets we get

$$\left\|\lambda(f)|\,\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}\right\| \leqslant c\left\| f|\, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\|. \tag{49}$$

Step III. For $\max(p^+,q)<\infty$ we get the unconditional convergence of (45) in $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ by (48) and the properties of the sequence spaces $\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}$.

Let $p^+<\infty$ and $q=\infty$, then we get the convergence in $B^{\boldsymbol{\varrho}}_{p(\cdot),\infty}(\mathbb{R}^n)$ for all $\{\varrho_j\}_{j\in\mathbb{N}_0}$ by using (48) again and $\sup_{x\in\mathbb{R}^n}\frac{\varrho_j(x)}{w_j(x)}\to 0$ for $j\to\infty$.

To obtain the convergence for $p^+=\infty$ we have to compensate the behavior at infinity by introducing a weaker weight sequence $\boldsymbol{\varrho}$ with $\frac{\varrho_\nu(x)}{w_\nu(x)}\to 0$ for $|x|\to\infty$ for all $\nu\in\mathbb{N}_0$ and $\sup_{x\in\mathbb{R}^n}\frac{\varrho_\nu(x)}{w_\nu(x)}\to 0$ for $\nu\to\infty$. Then we get the unconditional convergence in $B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$ by using (48) again.

A simple example of such a weaker weight sequence is given for every $\varepsilon>0$ by

$$\varrho_\nu(x)=2^{-\nu\varepsilon}(1+2^\nu|x|)^{-\varepsilon}w_\nu(x) \tag{50}$$

which belongs to $\mathcal{W}^{\alpha+\varepsilon}_{\alpha_1-2\varepsilon,\alpha_2+\varepsilon}$.

Step IV. Now, we want to prove the uniqueness of the coefficients. We define

$$g=\sum_{\nu=0}^{\infty}\sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n}\lambda^\nu_{Gm}2^{-\nu\frac{n}{2}}\Psi^\nu_{Gm} \tag{51}$$

where λ^ν_{Gm} is given by (46). We want to show that $g=f$, or

$$\langle g,\varphi\rangle=\langle f,\varphi\rangle \qquad \text{for every } \varphi\in\mathcal{S}(\mathbb{R}^n).$$

From the first step we have $g\in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$. The third step tells us that (51) converges at least in $B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$ for all $\boldsymbol{\varrho}$ given by (50) for some $\varepsilon>0$. Since $\Psi^{\nu'}_{G'm'}$ are the Daubechies wavelets, they still belong to the dual space $(B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n))'$.

Because of the convergence in $B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$,the dual pairing and the orthonormality of $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ in $L_2(\mathbb{R}^n)$ we get

$$\left\langle g, \Psi^{\nu'}_{G'm'}\right\rangle = \left\langle f, \Psi^{\nu'}_{G'm'}\right\rangle. \tag{52}$$

This holds also for finite linear combinations of $\Psi^{\nu'}_{G'm'}$. For a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have the unique $L_2(\mathbb{R}^n)$ representation

$$\varphi = \sum_{\nu,G,m} 2^{-\nu\frac{n}{2}} \langle \varphi, \Psi^\nu_{Gm}\rangle \Psi^\nu_{Gm}.$$

Since $\mathcal{S}(\mathbb{R}^n)$ is a subspace in every Besov space considered this representation converges in $(B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n))'$ and by (52) we get

$$\langle g, \varphi\rangle = \langle f, \varphi\rangle.$$

Final Step. Hence, $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ if, and only if, it can be represented by (45). This representation is unique with coefficients (46). By (48), (51), with $g = f$, and (49) it follows

$$\left\| \lambda(f) | \, \tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}\right\| \sim \left\| f | \, B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)\right\|.$$

Hence, I in (47) is an isomorphic map from $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ into $\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}$. It remains to prove that this map is onto. Let $\lambda \in \tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}$. Then by the above considerations it follows that

$$f = \sum_{\nu,G,m} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \Psi^\nu_{Gm} \in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n).$$

By the same reasoning as in the fourth step this representation is unique and $\lambda^\nu_{Gm} = \lambda^\nu_{Gm}(f)$. This proves that I is a map onto.

The proof for the Triebel-Lizorkin spaces follows the same line of arguments as the proof for the Besov spaces. We use $k_{\nu m} = 2^{\nu\frac{n}{2}}\Psi^\nu_{Gm}$ as kernels of local means with $A = B = k$ and $C > 0$ arbitrary and that $\{2^{-\nu\frac{n}{2}}\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ are $[k,k]$-atoms. In the proof we replace the used Besov space theorems with the corresponding F-parts in Theorem 3.13 and Theorem 4.6. Also the part on convergence gets easier, because $p(\cdot)$ is by definition smaller than infinity. ■

Now, we present a wavelet decomposition theorem with the help of Meyer wavelets, described in Proposition 4.1. We have ψ_M, $\psi_F \in \mathcal{S}(\mathbb{R}^n)$ and we have infinitely many moment conditions on ψ_M. We lose the compact support property for the wavelets and we use the molecular decomposition. The proof is the same as in Theorem 4.8. We use again our wavelets as molecules and also as kernels from Definition 4.3 where the technicalities turn out to be easier because A, B, C are infinite.

Theorem 4.9. *Let* $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ *be the Meyer wavelets according to Proposition 4.1. Further, let* $\boldsymbol{w}\in\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0<q\leqslant\infty$, $p(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$ *and let* $f\in\mathcal{S}'(\mathbb{R}^n)$.

(1) *We have* $f\in B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f=\sum_{\nu=0}^{\infty}\sum_{G\in G^\nu}\sum_{m\in\mathbb{Z}^n}\lambda^\nu_{Gm}2^{-\nu\frac{n}{2}}\Psi^\nu_{Gm}\qquad\text{with }\ \lambda\in\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q},\tag{53}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and in any space* $B^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$ *with* $\frac{\varrho_\nu(x)}{w_\nu(x)}\to 0$ *for* $|x|\to\infty$ *and all* ν *and also* $\sup_{x\in\mathbb{R}^n}\frac{\varrho_\nu(x)}{w_\nu(x)}\to 0$ *for* $\nu\to\infty$. *The representation* (53) *is unique, we have*

$$\lambda^\nu_{Gm}=\lambda^\nu_{Gm}(f)=2^{\nu\frac{n}{2}}\left\langle f,\Psi^\nu_{Gm}\right\rangle\tag{54}$$

and

$$I:f\mapsto\left\{2^{\nu\frac{n}{2}}\left\langle f,\Psi^\nu_{Gm}\right\rangle\right\}\tag{55}$$

is an isomorphic map from $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ *onto* $\tilde{b}^{\boldsymbol{w}}_{p(\cdot),q}$. *Moreover, if in addition* $\max(p^+,q)<\infty$, *then* $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ *is an unconditional basis in* $B^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

(2) *Let* $p^+<\infty$ *then* $f\in F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ *if, and only if, it can be represented as* (53) *with* $\lambda\in\tilde{f}^{\boldsymbol{w}}_{p(\cdot),q}$ *with unconditional convergence in* $\mathcal{S}'(\mathbb{R}^n)$ *and in any space* $F^{\boldsymbol{\varrho}}_{p(\cdot),q}(\mathbb{R}^n)$ *with* $\sup_{x\in\mathbb{R}^n}\frac{\varrho_j(x)}{w_j(x)}\to 0$ *for* $j\to\infty$. *The representation* (53) *is unique, we have* (54) *and* (55) *is an isomorphic map from* $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ *onto* $\tilde{f}^{\boldsymbol{w}}_{p(\cdot),q}$. *Moreover, if in addition* $q<\infty$, *then* $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ *is an unconditional basis in* $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$.

Remark 4.10. The wavelet characterization of $F^{\boldsymbol{w}}_{p(\cdot),q}(\mathbb{R}^n)$ is not restricted to the two wavelet systems presented in Proposition 4.1. The proof of Theorem 4.8 also applies to all wavelet systems $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ which satisfy that $\{2^{-\nu\frac{n}{2}}\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ are $[K,K,M]$ molecules with

$$K>\max(\sigma_p-\alpha_1,\alpha_2)\qquad\text{and}\qquad M>K+\sigma_p+2n+2\alpha,$$

in the B-case and with $\sigma_{p,q}$ replacing σ_p in the F-case.

The proofs can easily be extended to bi-orthogonal wavelet bases (see [24] for details).

5. The case of $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$

In this section we present the decomposition of functions in $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ by atoms, molecules and wavelets. The proofs above can be modified to this situation. First, we define the spaces under consideration.

Definition 5.1. *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity. Further, let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leqslant p^+ < \infty$ and $0 < q^- \leqslant q^+ \leqslant \infty$. The space $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is defined by*

$$F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \left\{ f \in S' : \left\| f \right| F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \big\|_{\varphi} < \infty \right\},$$

where

$$\left\| f \right| F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \big\|_{\varphi} = \left\| \left(\sum_{j=0}^{\infty} \left| (\varphi_j \hat{f})^{\vee}(x) w_j(x) \right|^{q(x)} \right)^{1/q(x)} \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|.$$

This definition was given in [22] where also a characterization by local means was proved. This local means characterization shows the independence of the start function φ_0 of the resolution of unity if $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $p^+, q^+ < \infty$.

Remark that due to Lemma 2.6 this definition is a generalization of the spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ from [12].

We also have to define the modified sequence spaces.

Definition 5.2. *Let $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $0 < p^- \leqslant p^+ < \infty$ and $0 < q^- \leqslant q^+ \leqslant \infty$.*

(1) *We define*

$$f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} := \left\{ \lambda = \{\lambda_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C} : \left\| \lambda \right| f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \right\| < \infty \right\}$$

where

$$\left\| \lambda \right| f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \right\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |w_\nu(2^{-\nu}m)\lambda_{\nu m}|^{q(x)} \chi_{\nu m}(x) \right)^{1/q(x)} \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|.$$

(2) *We define*

$$\tilde{f}^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} := \left\{ \lambda = \{\lambda^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^\nu, m\in\mathbb{Z}^n} \subset \mathbb{C} : \left\| \lambda \right| \tilde{f}^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \right\| < \infty \right\}$$

where

$$\left\| \lambda \right| \tilde{f}^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \right\| = \left\| \left(\sum_{\nu=0}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} |w_\nu(2^{-\nu}m)\lambda^{\nu}_{Gm}|^{q(x)} \chi_{\nu m}(x) \right)^{1/q(x)} \right| L_{p(\cdot)}(\mathbb{R}^n) \right\|.$$

Since the maximal operator is not bounded on $L_{p(\cdot)}(\ell_{q(\cdot)})$ (see [12, Section 5]) if $q(\cdot)$ is not constant, we have to work with another tool. We use a convolution inequality from [12]. Therefore we introduce the functions $\eta_{\nu,R}(x) = 2^{n\nu}(1 + 2^\nu|x|)^{-R}$ for $\nu \in \mathbb{N}_0$ and $R > 0$.

Lemma 5.3 (Theorem 3.2 in [12]). *Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p^- \leqslant p^+ < \infty$ and $1 < q^- \leqslant q^+ < \infty$. Then the inequality*

$$\big\| \big\| \eta_{\nu,R} * f_\nu | \, \ell_{q(\cdot)} \big\| \big| \, L_{p(\cdot)}(\mathbb{R}^n) \big\| \leqslant c \, \big\| f_\nu | \, L_{p(\cdot)}(\ell_{q(\cdot)}) \big\|$$

holds for every sequence $\{f_\nu\}_{\nu \in \mathbb{N}_0}$ of $L_1^{loc}(\mathbb{R}^n)$ functions and constant $R > n$.

Further we need a generalized version of Lemma 3.8 which is proved in [22].

Lemma 5.4 (Lemma 4.2 in [22]). *Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $0 < q^- \leqslant q^+ \leqslant \infty$ and $0 < p^- \leqslant p^+ \leqslant \infty$. For any sequence $\{g_j\}_{j \in \mathbb{N}_0}$ of nonnegative measurable functions on $\mathbb{R}^n$ and $\delta > 0$ let*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|\delta} g_k(x) \qquad \text{for all } x \in \mathbb{R}^n \text{ and } j \in \mathbb{N}_0.$$

Then with constant $c = c(p, q, \delta)$ we have

$$\big\| \{G_j\}_{j \in \mathbb{N}_0} | \, L_{p(\cdot)}(\ell_{q(\cdot)}) \big\| \leqslant c \, \big\| \{g_j\}_{j \in \mathbb{N}_0} | \, L_{p(\cdot)}(\ell_{q(\cdot)}) \big\|.$$

Since the maximal operator is of no use in the case when q is a variable function, we have to give a modified version of the heavily used Lemma 3.7, which is not hard to prove.

Lemma 5.5. *Let $0 < t \leqslant 1$, $j, \nu \in \mathbb{N}_0$ and $\{h_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ be positive real numbers then we have for $R > 0$ and $x \in \mathbb{R}^n$, that*

$$\sum_{m \in \mathbb{Z}^n} h_{\nu m} (1 + 2^j |x - 2^{-\nu} m|)^{-R}$$
$$\leqslant c \max(1, 2^{(\nu - j)R}) \left(\left[\eta_{\nu, Rt} * \left(\sum_{m \in \mathbb{Z}^n} h_{\nu m}^t \chi_{\nu m}(\cdot) \right) \right] (x) \right)^{1/t}.$$

It is now very easy to follow the proofs on the previous pages and replace Lemma 3.8 by Lemma 5.4, Lemma 3.7 by Lemma 5.5 and instead of the boundedness of the maximal operator we use Lemma 5.3. Furthermore, we use [22, Corollary 4.7] as the local means characterization and we obtain the decompositions by atoms, molecules and wavelets for the Triebel-Lizorkin spaces with variable $p(\cdot)$ and $q(\cdot)$.

Corollary 5.6. *Let $\{w_j\}_{j \in \mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1, \alpha_2}$ and let $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $0 < p^- \leqslant p^+ < \infty$ and $0 < q^- \leqslant q^+ < \infty$. Furthermore, let $K, L \in \mathbb{N}_0$ with*

$$K > \alpha_2 \qquad \text{and} \qquad L > \sigma_{p,q} - \alpha_1.$$

(1) *If $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ and $\{a_{\nu m}\}_{\nu \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ are $[K, L]$-atoms centered at $Q_{\nu m}$, then f represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \qquad \text{convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{56}$$

belongs to the space $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and there exists a constant* $c > 0$ *with*

$$\left\| f | F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \leqslant c \left\| \lambda | f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \right\| .$$

The constant $c > 0$ *is universal for all* λ *and* $a_{\nu m}$*.*

(2) *For every* $f \in F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, there exists* $\lambda \in f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$ *and* $[K, L]$*-atoms centered at* $Q_{\nu m}$ *such that there exists a representation* (58) *converging in* $\mathcal{S}'(\mathbb{R}^n)$*, with*

$$\left\| \lambda | f^{\boldsymbol{w}}_{p(\cdot),q(\cdot)} \right\| \leqslant c \left\| f | F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| ,$$

where the constant $c > 0$ *is universal for all* $f \in F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*.*

Corollary 5.7. *Let* $\boldsymbol{w} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ *and let* $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $0 < p^- \leqslant p^+ < \infty$ *and* $0 < q^- \leqslant q^+ < \infty$*. Let* $f \in \mathcal{S}'(\mathbb{R}^n)$ *and* $\{\Psi^{\nu}_{Gm}\}_{\nu \in \mathbb{N}_0, G \in G^{\nu}, m \in \mathbb{Z}^n}$ *be the Daubechies wavelets with*

$$k > \max(\sigma_{p,q} - \alpha_1, \alpha_2)$$

in (41) *and* (42)*. Then* $f \in F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{G \in G^{\nu}} \sum_{m \in \mathbb{Z}^n} \lambda^{\nu}_{Gm} 2^{-\nu \frac{n}{2}} \Psi^{\nu}_{Gm} \qquad \text{with} \;\; \lambda \in \tilde{f}^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}, \tag{57}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and in* $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*. The representation* (57) *is unique, we have*

$$\lambda^{\nu}_{Gm} = \lambda^{\nu}_{Gm}(f) = 2^{\nu \frac{n}{2}} \left\langle f, \Psi^{\nu}_{Gm} \right\rangle$$

and

$$I : f \mapsto \left\{ 2^{\nu \frac{n}{2}} \left\langle f, \Psi^{\nu}_{Gm} \right\rangle \right\}$$

is an isomorphic map from $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *onto* $\tilde{f}^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}$*.*

Here the part on convergence gets easier since $p^+, q^+ < \infty$. The first is natural but the restriction $q^+ < \infty$ comes from the use of Lemma 5.5.

Above we stated explicitly the two most important results but it is also possible to give analogues of Theorems 3.13 and 4.9 for the spaces $F^{\boldsymbol{w}}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ where the proofs get modified as described.

Remark 5.8. Quiet recently in [2] Almeida and Hästö introduced Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, where all three parameters are variable. To that end they presented a convolution inequality as Lemma 5.3 for the mixed spaces $\ell_{q(\cdot)}(L_{p(\cdot)}(\mathbb{R}^n))$; see [2, Lemma 4.7] for details.

It seems that it is possible to use this convolution inequality as a replacement for the boundedness of the maximal operator and to receive a wavelet characterization for $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, also with the smoothness function $2^{js(\cdot)}$ replaced by an admissible weight sequence $\boldsymbol{w}$, as done above in the F-case. Since the definition of the spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is very advanced and directly working with the modular, we leave that for future research.

6. Results for spaces of variable smoothness

This section collects the important results from the previous sections and we write it down for the special case of function spaces with variable smoothness. These spaces were introduced in [12] and attracted a lot of attention. The definition of the spaces $B^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$, $F^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and its corresponding sequence spaces is easy: just use $w_j(x) = 2^{js(x)}$ in the previous definitions. Then it is easy to verify $\alpha_1 = \inf(s(x)) = s^-$ and $\alpha_2 = \sup(s(x)) = s^+$. Using Lemma 2.6 we get that $s(\cdot) \in L_\infty \cap C^{\log}_{loc}(\mathbb{R}^n)$ defines an admissible weight sequence. Therefore, the following corollaries are covered by the previous results.

Corollary 6.1. *Let $s : \mathbb{R}^n \to \mathbb{R}$ be in $L_\infty \cap C^{\log}_{loc}(\mathbb{R}^n)$ and $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$. The symbol A stands for B or F and so does a symbolize b or f respectively.*

(1) *Let $0 < q \leqslant \infty$ ($p^+ < \infty$ in the F-case) and $K, L \in \mathbb{N}_0$ with*

$$K > s^+ \qquad \text{and} \qquad L > \sigma_p - s^- \quad (\sigma_{p,q} \text{ in the F-case}).$$

(a) *If $\lambda \in a^{s(\cdot)}_{p(\cdot),q}$ and $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K,L]$-atoms centered at $Q_{\nu m}$, then*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu m} \quad \text{convergence in } \mathcal{S}'(\mathbb{R}^n), \tag{58}$$

belongs to the space $A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ and there exists a constant $c > 0$ with

$$\left\| f \middle| A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n) \right\| \leqslant c \left\| \lambda \middle| a^{s(\cdot)}_{p(\cdot),q} \right\|.$$

The constant $c > 0$ is universal for all λ and $a_{\nu m}$.

(b) *For every $f \in A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$, there exists $\lambda \in a^{s(\cdot)}_{p(\cdot),q}$ and $[K,L]$-atoms centered at $Q_{\nu m}$ such that there exists a representation (58), converging in $\mathcal{S}'(\mathbb{R}^n)$, with*

$$\left\| \lambda \middle| a^{s(\cdot)}_{p(\cdot),q} \right\| \leqslant c \left\| f \middle| A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n) \right\|,$$

where the constant $c > 0$ is universal for all $f \in A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$.

(2) *Let $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ with $q^+ < \infty$. Further, let $p^+ < \infty$ and $K, L \in \mathbb{N}_0$ with*

$$K > s^+ \qquad \text{and} \qquad L > \sigma_{p,q} - s^-.$$

(a) *If $\lambda \in f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ and $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ are $[K,L]$-atoms centered at $Q_{\nu m}$, then f represented by (58) belongs to the space $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and there exists a constant $c > 0$ with*

$$\left\| f \middle| F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\| \leqslant c \left\| \lambda \middle| f^{s(\cdot)}_{p(\cdot),q(\cdot)} \right\|.$$

The constant $c > 0$ is universal for all λ and $a_{\nu m}$.

(b) *For every* $f \in F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*, there exists* $\lambda \in f^{s(\cdot)}_{p(\cdot),q(\cdot)}$ *and* $[K,L]$*-atoms centered at* $Q_{\nu m}$ *such that there exists a representation* (58)*, converging in* $\mathcal{S}'(\mathbb{R}^n)$*, with*

$$\left\| \lambda |\, f^{s(\cdot)}_{p(\cdot),q(\cdot)} \right\| \leqslant c \left\| f |\, F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) \right\|,$$

where the constant $c > 0$ *is universal for all* $f \in F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*.*

And finally we mention the decomposition with Daubechies wavelets for these spaces.

Corollary 6.2. *Let* $s(\cdot) \in L_\infty \cap C^{\log}_{loc}(\mathbb{R}^n)$ *and* $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$*. The symbol* A *stands for* B *or* F *and so does* a *symbolize* b *or* f *respectively.*

(1) *Let* $0 < q \leqslant \infty$ *(*$p^+ < \infty$ *in the F-case),* Ψ^ν_{Gm} *be the Daubechies wavelets and*

$$k > \max(\sigma_p - s^-, s^+) \quad (\sigma_{p,q} \text{ in the F-case})$$

in (41) *and* (42)*. Then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ *if, and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \Psi^\nu_{Gm} \qquad \text{with } \lambda \in \tilde{a}^{s(\cdot)}_{p(\cdot),q}, \tag{59}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ *and in any space* $A^{\sigma(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$*, where* $\sigma(x) < s(x)$ *with* $\inf(s(x)-\sigma(x)) > 0$ *and* $\sigma(x)/s(x) \to 0$ *for* $|x| \to \infty$*. The representation* (59) *is unique, we have*

$$\lambda^\nu_{Gm} = \lambda^\nu_{Gm}(f) = 2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle$$

and

$$I : f \mapsto \left\{ 2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle \right\}$$

is an isomorphic map from $A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ *onto* $\tilde{a}^{s(\cdot)}_{p(\cdot),q}$*. Moreover, if in addition* $\max(p^+, q) < \infty$*, then* $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^\nu, m\in\mathbb{Z}^n}$ *is an unconditional basis in* $A^{s(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$*.*

(2) *Let* $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$ *with* $0 < p^- \leqslant p^+ < \infty$*,* $0 < q^- \leqslant q^+ < \infty$ *and let*

$$k > \max(\sigma_{p,q} - s^-, s^+)$$

in (41) *and* (42)*. Then* $f \in \mathcal{S}'(\mathbb{R}^n)$ *belongs to* $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *if, and only if, it can be represented as* (59) *with* $\lambda \in \tilde{f}^{s(\cdot)}_{p(\cdot),q(\cdot)}$*, with unconditional convergence in* $\mathcal{S}'(\mathbb{R}^n)$ *and in* $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*. The representation* (59) *is unique, we have*

$$\lambda^\nu_{Gm} = \lambda^\nu_{Gm}(f) = 2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle$$

and

$$I : f \mapsto \left\{ 2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle \right\}$$

is an isomorphic map from $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *onto* $\tilde{f}^{s(\cdot)}_{p(\cdot),q(\cdot)}$*.*

Appendix

In this appendix we present the proofs of Lemmas 3.7 and 3.11.

Proof of Lemma 3.7. Remember that $0 < t \leqslant 1$ and $R > n/t$. We set $\delta = \frac{R}{n} - \frac{1}{t} > 0$ and for each $k \in \mathbb{N}$ we define

$$\Omega_k = \{m \in \mathbb{Z}^n : 2^{k-1} < 2^{\min(\nu,j)}|2^{-\nu}m - x| \leqslant 2^k\} \tag{60}$$

and

$$\Omega_0 = \{m \in \mathbb{Z}^n : 2^{\min(\nu,j)}|2^{-\nu}m - x| \leqslant 1\}.$$

Step I: $\nu \leqslant j$. If $x \in Q_{jl}$, then

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n} |h_{\nu m}|(1+2^j|x-2^{-\nu}m|)^{-R} &\leqslant \sum_{k=0}^{\infty}\sum_{m\in\Omega_k} |h_{\nu m}|(1+2^\nu|x-2^{-\nu}m|)^{-R} \\ &\leqslant c\sum_{k=0}^{\infty}\sum_{m\in\Omega_k} |h_{\nu m}|2^{-\delta nk-\frac{nk}{t}} \leqslant c' \sup_{k\in\mathbb{N}_0} 2^{-\frac{kn}{t}} \sum_{m\in\Omega_k}|h_{\nu m}| \\ &\leqslant c''\left(\sup_{k\in\mathbb{N}_0} 2^{-kn}\sum_{m\in\Omega_k}|h_{\nu m}|^t\right)^{1/t} \\ &= c''\left(\sup_{k\in\mathbb{N}_0} 2^{n(\nu-k)}\int_{\bigcup_{m\in\Omega_k}}\left(\sum_{m\in\Omega_k}|h_{\nu m}|\chi_{\nu m}(y)\right)^t dy\right)^{1/t}.\end{aligned} \tag{61}$$

We set $Q_k = \bigcup_{\{m:2^{\min(\nu,j)}|2^{-\nu}m-x|\leqslant 2^k\}} Q_{\nu m}$ and we have $|Q_k| \sim 2^{(k-\nu)n}$ and $\bigcup_{m\in\Omega_k} Q_{\nu m} = Q_k \setminus Q_{k-1}$. Then for all $k \in \mathbb{N}_0$ we obtain

$$\begin{aligned}(\mathcal{M}f)(x) = \sup_{x\in Q}\frac{1}{|Q|}\int_Q |f(y)|dy &\geqslant \frac{1}{|Q_k|}\frac{|Q_k\setminus Q_{k-1}|}{|Q_k\setminus Q_{k-1}|}\int_{Q_k}|f(y)|dy \\ &\geqslant c\frac{1}{|Q_k\setminus Q_{k-1}|}\int_{Q_k\setminus Q_{k-1}}|f(y)|dy \\ &= c\frac{1}{|\bigcup_{m\in\Omega_k}Q_{\nu m}|}\int_{\bigcup_{m\in\Omega_k}Q_{\nu m}}|f(y)|dy.\end{aligned}$$

Therefore, we get together with (61)

$$\begin{aligned}\sum_{m\in\mathbb{Z}^n}|h_{\nu m}|(1+2^j|x-2^{-\nu}m|)^{-R} &\leqslant c\left(\sup_{k\in\mathbb{N}_0}\left[\mathcal{M}_t\left(\sum_{m\in\Omega_k}|h_{\nu m}|\chi_{\nu m}\right)(x)\right]^t\right)^{1/t} \\ &\leqslant c\mathcal{M}_t\left(\sum_{m\in\mathbb{Z}^n}|h_{\nu m}|\chi_{\nu m}\right)(x).\end{aligned}$$

Step II: $j < \nu$. We use the same decomposition of $\mathbb{R}^n$ as in (60) and use the same $\delta > 0$ as above. The only change is that $|\bigcup_{m\in\Omega_k} Q_{\nu m}| \sim 2^{n(k-j)}$ and this explains

$$\sum_{m\in\mathbb{Z}^n} |h_{\nu m}|(1+2^j|x-2^{-\nu}m|)^{-R} \leqslant c2^{\frac{n}{t}(\nu-j)}\mathcal{M}_t\left(\sum_{m\in\mathbb{Z}^n}|h_{\nu m}|\chi_{\nu m}\right)(x). \qquad \blacksquare$$

Proof of Lemma 3.11. First, we use the embedding results. We choose $t < \min(p^-, 1)$ appropriately and get from Propositions 3.9 and 3.10 that

$$b^{\boldsymbol{w}}_{p(\cdot),q} \hookrightarrow b^{\boldsymbol{\varrho}}_{p(\cdot)/t,q} \hookrightarrow b^{\boldsymbol{\varrho}}_{p(\cdot)/t,\infty}$$

and

$$f^{\boldsymbol{w}}_{p(\cdot),q} \hookrightarrow f^{\boldsymbol{\varrho}}_{p(\cdot)/t,q} \hookrightarrow b^{\boldsymbol{\varrho}}_{p(\cdot)/t,\infty},$$

where $\varrho_j(x) = w_j(x)2^{-j\frac{t}{p(x)}\sigma_t}$. We have that $2^{-j\frac{t}{p(x)}\sigma_t} =: 2^{-js(x)\sigma_t}$ is again a 2-microlocal weight sequence with $s(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $0 \leqslant s(x) < 1$. Therefore, $\boldsymbol{\varrho}$ is also an admissible weight sequence and it is easy to show that $\boldsymbol{\varrho} \in \mathcal{W}^{\beta}_{\alpha_1-\sigma_t,\alpha_2}$ with $\beta \geqslant 0$ large enough. We have to prove that the limes

$$\lim_{r\to\infty}\sum_{\nu=0}^{r}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(x) \qquad \text{exists in } \mathcal{S}'(\mathbb{R}^n).$$

We can assume, that $\lambda \in b^{\boldsymbol{\varrho}}_{p(\cdot)/t,\infty}$, see the above embeddings. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we get from the moment conditions (15) for fixed $\nu \in \mathbb{N}_0$

$$\begin{aligned}\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy &= \int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(y)\varrho_\nu(y) \\ &\quad\times\left(\varphi(y)-\sum_{|\beta|<L}\frac{D^\beta\varphi(2^{-\nu}m)}{\beta!}(y-2^{-\nu}m)^\beta\right)\varrho_\nu^{-1}(y)\frac{\langle y\rangle^\kappa}{\langle y\rangle^\kappa}dy,\end{aligned} \tag{62}$$

where $\kappa > 0$ will be specified later on. We use Taylor expansion of φ up to the order L and get with ξ on the line segment joining y and $2^{-\nu}m$. By using the properties of the weight sequence and $\langle y\rangle^\kappa \leqslant \langle y-2^{-\nu}m\rangle^\kappa\langle\xi\rangle^\kappa$ we derive

$$\begin{aligned}&|\mu_{\nu m}(y)|\left|\varphi(y)-\sum_{|\gamma|<L}\frac{D^\gamma\varphi(2^{-\nu}m)}{\gamma!}(y-2^{-\nu}m)^\gamma\right|\varrho_\nu^{-1}(y)\frac{\langle y\rangle^\kappa}{\langle y\rangle^\kappa} \\ &\quad\leqslant c'2^{-\nu(L+\alpha_1-\sigma_t)}(1+2^\nu|y-2^{-\nu}m|)^{L+\kappa-M}\langle y\rangle^{\beta-\kappa}\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^\kappa\sum_{|\gamma|=L}\frac{|D^\gamma\varphi(\xi)|}{\gamma!},\end{aligned}$$

where $M>0$ and $\kappa>0$ are at our disposal. Hence, we derive from (62)

$$\left|\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy\right| \leqslant c2^{-\nu(L+\alpha_1-\sigma_t)}\|\varphi\|_{\kappa,L}$$
$$\times\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\varrho_\nu(2^{-\nu}m)(1+2^\nu|y-2^{-\nu}m|)^{L+\beta+\kappa-M}\langle y\rangle^{\beta-\kappa}dy.$$

We estimate now the integral and from the Hölder's inequality for $L_{p(\cdot)}(\mathbb{R}^n)$ (see [23]) we get with $p(\cdot)/t>1$ and choosing κ large enough

$$\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\varrho_\nu(2^{-\nu}m)(1+2^\nu|y-2^{-\nu}m|)^{L+\beta+\kappa-M}\langle y\rangle^{\beta-\kappa}dy$$
$$\leqslant c'\left\|\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\varrho_\nu(2^{-\nu}m)(1+2^\nu|y-2^{-\nu}m|)^{L+\beta+\kappa-M}\right|L_{p(\cdot)/t}(\mathbb{R}^n)\right\|.$$

Finally, we use Lemma 3.7 with M large enough and the boundedness of the maximal operator and obtain

$$\left|\int_{\mathbb{R}^n}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy\right|$$
$$\leqslant c''\|\varphi\|_{\kappa,L}2^{-\nu(L+\alpha_1-\sigma_t)}\left\|\mathcal{M}\left(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\varrho_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot)\right)\right|L_{p(\cdot)/t}(\mathbb{R}^n)\right\|$$
$$\leqslant c'''\|\varphi\|_{\kappa,L}2^{-\nu(L+\alpha_1-\sigma_t)}\left\|\lambda|\,b^{\boldsymbol{\varrho}}_{p(\cdot)/t,\infty}\right\|.$$

Since $L>\sigma_t-\alpha_1>\sigma_p-\alpha_1$ and $\lambda\in b^{\boldsymbol{\varrho}}_{p(\cdot)/t,\infty}$, the convergence of (24) in $\mathcal{S}'(\mathbb{R}^n)$ follows. ■

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