Functiones et Approximatio
41.1 (2009), 7–13

COINCIDENCE POINT RESULTS IN NONCONVEX DOMAINS OF q-NORMED SPACES

Hemant Kumar Nashine, Mohammad Saeed Khan†

Abstract: Coincidence points results for families of four relatively nonexpansive mappings on nonconvex domains in q-normed spaces have been obtained in the present work. As applications, best approximation results have been given. These results extend and generalize previously known results to a more general class of non commuting relatively nonexpansive mappings in a space which is not necessarily locally convex.

Keywords: Best approximant, common fixed point, compatible maps, contractive jointly continuous family, q-normed space, relatively nonexpansive mappings.

1. Introduction

The concept of relatively nonexpansive maps for pair of maps was given by Park [14]. It was extended by Jungck [8] for families of four self maps (noncontinuous). By using this concept he proved the coincidence and fixed points results for starshaped domain and generalized the results of Dotson [2].

Fixed point theorems have been applied in the field of invariant approximation theory for last four decades and several interesting and valuable results have been studied.

Meinardus [10] was the first to employ a fixed-point theorem of Schauder to establish the existence of an invariant approximation. Further, Brosowski [1] obtained a celebrated result and generalized the Meinardus's result. Later, several results [5, 15] have been proved in the direction of Brosowski [1]. In the year 1988, Sahab et al. [13] extended the result of Hicks and Humpheries [5] and Singh [15] by considering one linear and the other nonexpansive mappings.

In this context, it may be mentioned that Dotson [2] proved the existence of fixed point for nonexpansive mapping. He further extended his result without starshapedness under non-convex condition [3]. Mukherjee and Som [11] used it to prove existence of fixed point and further applied it for proving existence of best

1991 Mathematics Subject Classification: 41A50, 47H10, 54H25.
† corresponding author

approximant. This result was an extension of Singh [15] without starshapedness condition.

In this paper, coincidence point results of four relatively nonexpansive self mappings on nonconvex domains in q-normed spaces have been obtained. These results extend and improve the results of Jungck [8]. As applications, best approximation results have also been established. These results extend and generalize various existing known results in the literature to a more general class of non commuting relatively nonexpansive mappings in a space which is not necessarily locally convex space

2. Preliminaries

Let $\mathcal{X}$ be a linear space. A q-norm on $\mathcal{X}$ is a real-valued function $\|.\|_q$ on $\mathcal{X}$ with $0 < q \leqslant 1$, satisfying the following conditions:

(a) $\|x\|_q \geqslant 0$ and $\|x\|_q = 0$ iff $x = 0$,
(b) $\|\lambda x\|_q = |\ \lambda\ |^q \|x\|_q$,
(c) $\|x + y\|_q \leqslant \|x\|_q + \|y\|_q$,

for all $x, y \in \mathcal{X}$ and all scalars λ. The pair $(\mathcal{X}, \|.\|_q)$ is called a q-normed spaces. It is a metric space with $d_q(x, y) = \|x - y\|_q$ for all $x, y \in \mathcal{X}$, defining a translation invariant metric d_q on $\mathcal{X}$. If $q = 1$, we obtain the concept of a normed linear space. It is well-known that the topology of every Hausdorff locally bounded topological linear space is given by some q-norm, $0 < q \leqslant 1$. The spaces l_q and $\mathcal{L}_q[0, 1]$, $0 < q \leqslant 1$ are q-normed spaces. A q-normed space is not necessarily a locally convex space. Recall that, if $\mathcal{X}$ is a topological linear space, then its continuous dual space $\mathcal{X}^*$ is said to separate the points of $\mathcal{X}$, if for each $x \neq 0$ in $\mathcal{X}$, there exists an $\mathcal{I} \in \mathcal{X}^*$ such that $\mathcal{I}x \neq 0$. In this case the weak topology on $\mathcal{X}$ is well-defined. We mention that, if $\mathcal{X}$ is not locally convex, then $\mathcal{X}^*$ need not separate the points of $\mathcal{X}$. For example, if $\mathcal{X} = \mathcal{L}_q[0, 1]$, $0 < q < 1$, then $\mathcal{X}^* = \{0\}$ [12, pp. 36–37]. However, there are some non-locally convex spaces (such as the q-normed space l_q, $0 < q < 1$) whose dual separates the points [6].

Let $\mathcal{X}$ be a metric space and let $\mathcal{C}$ be a nonempty subset of $\mathcal{X}$. Let $x \in \mathcal{X}$. An element $y \in \mathcal{C}$ is called a best $\mathcal{C}$-approximant to $x \in \mathcal{X}$ if

$$d(x, y) = dist(x, \mathcal{C}) = \inf\{d(x, z) : z \in \mathcal{C}\}.$$

The set of best $\mathcal{C}$-approximants to x is denoted by $\mathcal{P}_\mathcal{C}(x_0)$ and is defined as $\mathcal{P}_\mathcal{C}(x_0) = \{y \in \mathcal{C} : d(x, y) = dist(x, \mathcal{C})\}$. Let $\mathcal{I}, \mathcal{T} : \mathcal{C} \to \mathcal{C}$ be two mappings. A mapping $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ is called an $(\mathcal{I}, \mathcal{T})$-contraction if there exists $0 \leqslant k < 1$ such that $d(\mathcal{A}x, \mathcal{A}y) \leqslant kd(\mathcal{I}x, \mathcal{T}y)$ for any $x, y \in \mathcal{C}$. If $k = 1$, then $\mathcal{A}$ is called $(\mathcal{I}, \mathcal{T})$-nonexpansive. Also if $\mathcal{T} = \mathcal{I}$, we say that $\mathcal{A}$ is called $\mathcal{I}$-nonexpansive. Let $\mathcal{A}, \mathcal{B}, \mathcal{I}$ and $\mathcal{T}$ be self maps of $\mathcal{C}$. Mappings $\mathcal{A}$ and $\mathcal{B}$ are nonexpansive relatively [8] to $\mathcal{I}$ and $\mathcal{T}$ iff $d(\mathcal{A}x, \mathcal{B}y) \leqslant d(\mathcal{I}x, \mathcal{T}y)$ for all $x, y \in \mathcal{C}$. If $\mathcal{B} = \mathcal{A}$ and $\mathcal{T} = \mathcal{I}$, we say that $\mathcal{A}$ is $\mathcal{I}$-nonexpansive. A point $x \in \mathcal{C}$ is a common fixed point(coincidence point) of $\mathcal{I}$ and $\mathcal{T}$ if $x = \mathcal{I}x = \mathcal{T}x(\mathcal{I}x = \mathcal{T}x)$. The set of coincidence points of $\mathcal{I}$ and $\mathcal{T}$ is denoted by $\Upsilon(\mathcal{I},\mathcal{T})$. The pair $(\mathcal{I},\mathcal{T})$ is called (1) commuting if $\mathcal{I}\mathcal{T}x=\mathcal{T}\mathcal{I}x$ for all $x\in\mathcal{C}$; (2) compatible [7, 8] if $\lim_n d(\mathcal{T}\mathcal{I}x_n,\mathcal{I}\mathcal{T}x_n)=0$ when $\{x_n\}$ is a sequence such that $\lim_n \mathcal{T}x_n=\lim_n \mathcal{I}x_n=t$ for some t in $\mathcal{C}$. Every commuting pair of mappings is compatible but the converse is not true in general [7, 8]. The set of fixed points of $\mathcal{T}$(resp. $\mathcal{I}$) is denoted by $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{I})$). A subset $\mathcal{C}$ of a linear space $\mathcal{X}$ is said to be starshaped, if there exists at least one point $p\in\mathcal{C}$ such that $\lambda x+(1-\lambda)p\in\mathcal{C}$, for all $x\in\mathcal{C}$ and $0\leqslant\lambda\leqslant 1$. In this case p is called the starcenter of $\mathcal{C}$. Each convex set is starshaped with respect to each of its points, but not conversely.

Further, definition providing the notion of contractive jointly continuous family introduced by Dotson [3] may be written as:

Let $\mathcal{C}$ be a subset of metric space $\mathcal{X}$ and $\Gamma=\{f_\alpha\}_{\alpha\in\mathcal{C}}$ a family of functions from $[0,1]$ into $\mathcal{C}$ such that $f_\alpha(1)=\alpha$ for each $\alpha\in\mathcal{C}$.
The family Γ is said to be contractive, if there exists a function $\phi:(0,1)\to(0,1)$ such that for all $\alpha,\beta\in\mathcal{C}$ and all $t\in(0,1)$, we have

$$d(f_\alpha(t),f_\beta(t))\leqslant\phi(t)d(\alpha,\beta).$$

The family Γ is said to be jointly continuous if $t\to t_0$ in $[0,1]$ and $\alpha\to\alpha_0$ in $\mathcal{C}$, then $f_\alpha(t)\to f_{\alpha_0}(t_0)$.

Remark 2.1. In the light of the comment given by Dotson [3] and Khan et al. [9] that if $\mathcal{X}$ is a q-normed space, $\mathcal{C}\subseteq\mathcal{X}$ is $p-$starshaped and $f_\alpha(t)=(1-t)p+t\alpha$, $(\alpha\in\mathcal{C},\ t\in[0,1])$, then $\{f_\alpha\}_{\alpha\in\mathcal{C}}$ is a contractive jointly continuous family with $\phi(t)=t^q$. Thus, the class of subsets of $\mathcal{C}$ with the property of contractiveness and jointly continuity contains the class of starshaped sets which in turns contains the class of convex sets. If $\mathcal{C}$ is a subset of $\mathcal{X}$, there exists a contractive jointly continuous family $\Gamma=\{f_\alpha\}_{\alpha\in\mathcal{C}}$ such that $\mathcal{C}$ has the property of contractiveness and joint continuity.

To prove our results, we also use the following result due to Jungck [8]:

Theorem 2.2 ([8]). *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$ and $\mathcal{T}$ be self maps of a complete metric space $(\mathcal{X},d)$, and suppose that $\mathcal{I}$ and $\mathcal{T}$ are surjective. If there exists $r\in(0,1)$ such that for $x,y\in\mathcal{X}$;*

$$d(\mathcal{A}x,\mathcal{B}y)\leqslant rd(\mathcal{I}x,\mathcal{T}y),$$

then there exist $z,t\in\mathcal{X}$ such that $\mathcal{A}t=\mathcal{I}t=\mathcal{B}z=\mathcal{T}z$. If moreover, the pairs $(\mathcal{A},\mathcal{I})$ and $(\mathcal{B},\mathcal{T})$ are each compatible, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$ and $\mathcal{T}$ have a unique common fixed point.

3. Main results

One may now prove the following coincidence point theorem for nonconvex domain in a q-normed space.

Theorem 3.1. *Let $\mathcal{C}$ be a nonempty compact subset of a q-normed space $\mathcal{X}$ which has a contractive jointly continuous family $\Gamma = \{f_x\}_{x\in\mathcal{C}}$. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{I}$ and $\mathcal{T}$ be self mappings of $\mathcal{C}$. If $\mathcal{I}$ and $\mathcal{T}$ are surjective and if for all $x, y \in \mathcal{C}$;*

$$\|\mathcal{A}x - \mathcal{B}y\|_q \leqslant \|\mathcal{I}x - \mathcal{T}y\|_q,$$

then $\Upsilon(\mathcal{A},\mathcal{B}) \cap \Upsilon(\mathcal{I},\mathcal{T}) \neq \emptyset$. If, in addition, the pairs $(\mathcal{A},\mathcal{I})$ and $(\mathcal{B},\mathcal{T})$ are compatible, then $\mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

Proof. Choose $k_n \in (0,1)$ such that $\{k_n\} \to 1$. Then define sequences $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ as

$$\mathcal{A}_n(x) = f_{\mathcal{A}x}(k_n), \ \ \mathcal{B}_n(x) = f_{\mathcal{B}x}(k_n),$$

for all $x \in \mathcal{C}$ and for each n. $\{\mathcal{A}_n\}$ and $\{\mathcal{B}_n\}$ are well-defined maps from $\mathcal{C}$ into $\mathcal{C}$ for each n. Also, for each n, and for all $x, y \in \mathcal{C}$, we have

$$\begin{aligned}\|\mathcal{A}_n(x) - \mathcal{B}_n(y)\|_q &= \|f_{\mathcal{A}x}(k_n) - f_{\mathcal{B}y}(k_n)\|_q \\ &\leqslant [\phi(k_n)]^q \|\mathcal{A}x - \mathcal{B}y\|_q \\ &\leqslant [\phi(k_n)]^q \|\mathcal{I}x - \mathcal{T}y\|_q\end{aligned}$$

i.e.,

$$\|\mathcal{A}_n(x) - \mathcal{B}_n(y)\|_q \leqslant [\phi(k_n)]^q \|\mathcal{I}x - \mathcal{T}y\|_q$$

for all $x, y \in \mathcal{C}$.
Also, $\mathcal{C} = \mathcal{I}(\mathcal{C}) = \mathcal{T}(\mathcal{C})$ is compact and therefore complete. It follows from Theorem 2.2, there exist $x_n, y_n, p_n \in \mathcal{C}$ such that

$$\mathcal{A}_n x_n = \mathcal{I}x_n = p_n = \mathcal{B}_n y_n = \mathcal{T}y_n, \ \ for \ \ all \ \ n.$$

Also, since $\mathcal{C}$ is compact, there exists a subsequence of $\{x_n\}$, denoted by $\{x_m\}$, such that $\{\mathcal{A}x_m\}$ converging to a point $p \in \mathcal{C}$. Then by jointly continuity of Γ, $\{f_{\mathcal{A}x_m}(k_m)\}$ tends to p, too. Moreover,

$$\|\mathcal{A}x_m - \mathcal{I}x_m\|_q = \|\mathcal{A}x_m - \mathcal{A}_m x_m\|_q = \|\mathcal{A}x_m - f_{\mathcal{A}x_m}(k_m)\|_q$$

so $\mathcal{I}x_m \to p \ \ as \ \ m \to \infty$.

Now, since $\mathcal{C} = \mathcal{T}(\mathcal{C})$, $\mathcal{T}z = p$ for some $z \in \mathcal{C}$. We also have

$$\|\mathcal{B}z - p\|_q \leqslant \|\mathcal{B}z - \mathcal{A}x_m\|_q + \|\mathcal{A}x_m - p\|_q \leqslant \|\mathcal{T}z - \mathcal{I}x_m\|_q + \|\mathcal{A}x_m - p\|_q.$$

It follows that $p = \mathcal{B}z = \mathcal{T}z$. By a similar argument there exists $t \in \mathcal{C}$ such that $\mathcal{A}t = \mathcal{I}t = p$. Hence $\Upsilon(\mathcal{A},\mathcal{I}) \cap \Upsilon(\mathcal{B},\mathcal{T}) \neq \phi$.

If moreover, $\mathcal{A}$ and $\mathcal{I}$ are compatible, $\mathcal{A}t = \mathcal{I}t = p$ implies that $\mathcal{I}\mathcal{A}t = \mathcal{A}\mathcal{I}t$; i.e., $\mathcal{I}p = \mathcal{A}p$. In the same fashion, if $\mathcal{B}$ and $\mathcal{T}$ are compatible, $\mathcal{T}p = \mathcal{B}p$. Hence $\mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{T}) \neq \phi$. This completes the proof.

■

If $\mathcal{B} = \mathcal{A}$ in the Theorem 3.1 and $\mathcal{A}$ is also injective, then $\mathcal{B}z = \mathcal{A}t$ implies that $t = z$, and we have $\mathcal{A}z = \mathcal{I}z = \mathcal{T}z$. Thus, as a consequence one obtains the following:

Corollary 3.2. *Let $\mathcal{C}$ be a nonempty compact subset of a q-normed space $\mathcal{X}$ which has a contractive jointly continuous family $\Gamma = \{f_x\}_{x\in\mathcal{C}}$. Let $\mathcal{A}$, $\mathcal{I}$ and $\mathcal{T}$ be a self mappings of $\mathcal{C}$. If $\mathcal{I}$ and $\mathcal{T}$ are surjective and if for all $x, y \in \mathcal{C}$;*

$$\|\mathcal{A}x - \mathcal{A}y\|_q \leqslant \|\mathcal{I}x - \mathcal{T}y\|_q,$$

then $\Upsilon(\mathcal{A},\mathcal{I},\mathcal{T}) \neq \emptyset$ provided one of the following conditions hold:

(i) $\mathcal{A}$ is one-to-one;
(ii) $(\mathcal{A},\mathcal{I})$ and $(\mathcal{A},\mathcal{T})$ are compatible pairs.

If $\mathcal{I} = \mathcal{T}$ in the Corollary 3.2, one obtains the following:

Corollary 3.3. *Let $\mathcal{C}$ be a nonempty compact subset of a q-normed space $\mathcal{X}$ which has a contractive jointly continuous family $\Gamma = \{f_x\}_{x\in\mathcal{C}}$. Let $\mathcal{A}$ and $\mathcal{I}$ be compatible self mappings of $\mathcal{C}$. If $\mathcal{I}$ is surjective and if for all $x, y \in \mathcal{C}$;*

$$\|\mathcal{A}x - \mathcal{A}y\|_q \leqslant \|\mathcal{I}x - \mathcal{I}y\|_q,$$

then $\Upsilon(\mathcal{A},\mathcal{I}) \neq \emptyset$.

If $\mathcal{A} =Id$ in the Corollary 3.3, one obtains the following:

Corollary 3.4. *Let $\mathcal{C}$ be a nonempty compact subset of a q-normed space $\mathcal{X}$ which has a contractive jointly continuous family $\Gamma = \{f_x\}_{x\in C}$. If $\mathcal{T}$ and $\mathcal{I}$ be surjective self mappings of $\mathcal{C}$ and for all $x, y \in \mathcal{C}$;*

$$\|x - y\|_q \leqslant \|\mathcal{I}x - \mathcal{T}y\|_q,$$

then $\mathcal{C} \cap \mathcal{F}(\mathcal{I}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

As applications of Theorem 3.1, we have the following results in invariant approximation:

Theorem 3.5. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{A},\mathcal{B},\mathcal{T},\mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{C}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$ for some $x_0 \in \mathcal{X}$. Suppose that $\mathcal{P}_{\mathcal{C}}(x_0)$ is compact, has a contractive jointly continuous family $\Gamma = \{f_\alpha\}_{\alpha\in\mathcal{P}_{\mathcal{C}}(x_0)}$, $\mathcal{I}(\mathcal{P}_{\mathcal{C}}(x_0)) = \mathcal{P}_{\mathcal{C}}(x_0) = \mathcal{T}(\mathcal{P}_{\mathcal{C}}(x_0))$ and the pair $(\mathcal{T},\mathcal{I})$ is surjective. If the pair $(\mathcal{A},\mathcal{I})$ and $(\mathcal{B},\mathcal{T})$ satisfy, for all $x, y \in \mathcal{P}_{\mathcal{C}}(x_0) \cup \{x_0\}$*

$$\|\mathcal{A}x - \mathcal{B}y\|_q \leqslant \|\mathcal{I}x - \mathcal{T}y\|_q, \tag{3.1}$$

then $\mathcal{P}_{\mathcal{C}}(x_0) \cap \Upsilon(\mathcal{A},\mathcal{I},\mathcal{B},\mathcal{T}) \neq \emptyset$. If, in addition, the pairs $(\mathcal{A},\mathcal{I})$ and $(\mathcal{B},\mathcal{T})$ are compatible, then $\mathcal{P}_{\mathcal{C}}(x_0) \cap \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \cap (\mathcal{I}) \cap \mathcal{F}(\mathcal{T}) \neq \emptyset$.

Proof. Let $x \in \mathcal{P}_{\mathcal{C}}(x_0)$. Then, $x \in \mathcal{P}_{\mathcal{C}}(x_0)$. Then $\|x - x_0\|_q = dist(x_0, \mathcal{C})$. Note that for any $k \in (0, 1)$,

$$\|kx_0 + (1 - k)x - x_0\|_q = (1 - k)^q\|x - x_0\|_q < dist(x_0, \mathcal{C}).$$

It follows that the line segment $\{kx_0 + (1-k)x : 0 < k < 1\}$ and the set $\mathcal{C}$ are disjoint. Thus x is not in the interior of $\mathcal{C}$ and so $x \in \partial\mathcal{C} \cap \mathcal{C}$. Since $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$, $\mathcal{T}x$ must be in $\mathcal{C}$. Also since $\mathcal{I}x \in \mathcal{P}_{\mathcal{C}}(x_0)$, $x_0 = \mathcal{T}x_0 = \mathcal{I}x_0$ and $\mathcal{T}$ and $\mathcal{I}$ satisfy (3.1), we have

$$\|\mathcal{A}x - x_0\|_q = \|\mathcal{A}x - \mathcal{B}x_0\|_q \leqslant \|\mathcal{I}x - \mathcal{T}x_0\|_q = \|\mathcal{I}x - x_0\|_q = dist(x_0, C).$$

Thus, $\mathcal{A}x \in \mathcal{P}_{\mathcal{C}}(x_0)$. Consequently, $\mathcal{A}(\mathcal{P}_{\mathcal{C}}(x_0)) \subset \mathcal{P}_{\mathcal{C}}(x_0) = \mathcal{I}(\mathcal{P}_{\mathcal{C}}(x_0)) = \mathcal{T}(\mathcal{P}_{\mathcal{C}}(x_0))$. Similarly, we may show that $\mathcal{B}(\mathcal{P}_{\mathcal{C}}(x_0)) \subset \mathcal{P}_{\mathcal{C}}(x_0) = \mathcal{I}(\mathcal{P}_{\mathcal{C}}(x_0)) = \mathcal{T}(\mathcal{P}_{\mathcal{C}}(x_0))$. The result now follows from Theorem 3.1. ■

If $\mathcal{B} = \mathcal{A}$ and $\mathcal{A}$ is injective in the Theorem 3.5, one obtains the following:

Corollary 3.6. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{A}, \mathcal{T}, \mathcal{I} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{C}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$ for some $x_0 \in \mathcal{X}$. Suppose that $\mathcal{P}_{\mathcal{C}}(x_0)$ is compact, has a contractive jointly continuous family $\Gamma = \{f_\alpha\}_{\alpha \in \mathcal{P}_{\mathcal{C}}(x_0)}$, $\mathcal{I}(\mathcal{P}_{\mathcal{C}}(x_0)) = \mathcal{P}_{\mathcal{C}}(x_0) = \mathcal{T}(\mathcal{P}_{\mathcal{C}}(x_0))$ and the pair $(\mathcal{T}, \mathcal{I})$ is surjective. If $\mathcal{A}$, $\mathcal{I}$ and $\mathcal{T}$ satisfy, for all $x, y \in \mathcal{P}_{\mathcal{C}}(x_0) \cup \{x_0\}$*

$$\|\mathcal{A}x - \mathcal{A}y\|_q \leqslant \|\mathcal{I}x - \mathcal{T}y\|_q, \tag{3.2}$$

then $\mathcal{P}_{\mathcal{C}}(x_0) \cap \Upsilon(\mathcal{A}, \mathcal{I}, \mathcal{T}) \neq \emptyset$ provided one of the following conditions hold:

(i) $\mathcal{A}$ is one-to-one;
(ii) $\mathcal{A}, \mathcal{I}$ and $\mathcal{A}, \mathcal{T}$ are compatible pairs.

If $\mathcal{T} = \mathcal{I}$ in the Corollary 3.6, one obtains the following:

Corollary 3.7. *Let $\mathcal{X}$ be a q-normed space and $\mathcal{A}, \mathcal{T} : \mathcal{X} \to \mathcal{X}$. Let $\mathcal{C}$ be a subset of $\mathcal{X}$ such that $\mathcal{T}(\partial\mathcal{C} \cap \mathcal{C}) \subset \mathcal{C}$ and $x_0 \in \mathcal{F}(\mathcal{A})$ for some $x_0 \in \mathcal{X}$. Suppose that $\mathcal{P}_{\mathcal{C}}(x_0)$ is compact, has a contractive jointly continuous family $\Gamma = \{f_\alpha\}_{\alpha \in \mathcal{P}_{\mathcal{C}}(x_0)}$, $\mathcal{P}_{\mathcal{C}}(x_0)$ and $\mathcal{I}$ is surjective. If the pair $(\mathcal{A}, \mathcal{I})$ satisfies, for all $x, y \in \mathcal{P}_{\mathcal{C}}(x_0) \cup \{x_0\}$*

$$\|\mathcal{A}x - \mathcal{A}y\|_q \leqslant \|\mathcal{I}x - \mathcal{I}y\|_q, \tag{3.3}$$

then $\mathcal{P}_{\mathcal{C}}(x_0) \cap \Upsilon(\mathcal{A}, \mathcal{I}) \neq \emptyset$ provided one of the following conditions hold:

(i) $\mathcal{A}$ is one-to-one;
(ii) $(\mathcal{A}, \mathcal{I})$ is compatible pair.

Remark 3.8. From Remark 2.1, our results generalize the results of Jungck [8] to q-normed space and consequently Dotson's [2].

Acknowledgments. The authors thank the referee for his helpful corrections, comments and valuable suggestions in preparation of the paper.

References

[1] B. Brosowski, *Fixpunktsätze in der Approximationstheorie*, Mathematica(Cluj) **11** (1969), 165–220.

[2] W.G. Dotson, *Fixed point theorems for nonexpasive mappings on starshaped subsets of Banach space*, J. London Math. Soc. **4**(2) (1972), 408–410.

[3] W.G. Dotson, Jr, *On fixed point of nonexpansive mappings in nonconvex sets*, Proc. Amer. Math. Soc. **38**(1) (1973), 155–156.

[4] L. Habiniak, *Fixed point theorems and invariant approximations*, J. Approx. Theory **56** (1989), 241–244.

[5] T.L. Hicks, M.D. Humpheries, *A note on fixed point theorems*, J. Approx. Theory **34** (1982), 221–225.

[6] G. Köthe, *Topological vector spaces I*, Springer-Verlag, Berlin, 1969.

[7] G. Jungck, *Compatible mappings and common fixed points*, Int. J. Math. Math. Sci. **9**(4) (1986), 771–779.

[8] G. Jungck, *Coincidence and fixed points for compatible and relatively nonexpansive maps*, Int. J. Math. Math. Sci. **16**(1) (1993), 95–100.

[9] A.R. Khan, A. Latif, A. Bano, N. Hussain, *Some results on common fixed points and best approximation*, Tamkang J. Math. **36**(1) (2005), 33–38.

[10] G. Meinardus, *Invarianze bei Linearen Approximationen*, Arch. Rational Mech. Anal. **14** (1963), 301–303.

[11] R.N. Mukherjee, T. Som, *A note on an application of a fixed point theorem in approximation theory*, Indian J. Pure Appl. Math. **16**(3) (1985), 243–244.

[12] W. Rudin, *Functional Analysis,* 2nd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1991.

[13] S.A. Sahab, M.S. Khan, S. Sessa, *A result in best approximation theory*, J. Approx. Theory **55** (1988), 349–351.

[14] S. Park, *Fixed points of f-contractive maps*, Rocky Mountain J. Math. **8**(4) (1978), 743–750.

[15] S.P. Singh, *An application of a fixed point theorem to approximation theory*, J. Approx. Theory **25** (1979), 89–90.

[16] S.P. Singh, *Application of fixed point theorems to approximation theory*, in: V. Lakshmikantam(Ed.), Applied nonlinear Analysis, Academic Press, New York, 1979.

Addresses: Hemant Kumar Nashine: Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg (Baloda Bazar Road), Mandir Hasaud, Raipur-492101(Chhattisgarh), India;
Mohammad Saeed Khan: Department of Mathematics and Statistics, College of Science, Sultan Qaboos University, P.O. Box 36, PCode 123 Al-Khod, Muscat, Sultanate of Oman.

E-mail: hemantnashine@rediffmail.com, mohammad@squ.edu.om

Received: 18 June 2007; **revised:** 16 July 2008