

CONVERGENCE OF THE J -FLOW ON TORIC MANIFOLDS

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Abstract

We show that on a Kähler manifold whether the J -flow converges or not is independent of the chosen background metric in its Kähler class. On toric manifolds we give a numerical characterization of when the J -flow converges, verifying a conjecture in [19] in this case. We also strengthen existing results on more general inverse σ_k equations on Kähler manifolds.

1. Introduction

Let (M, α) be a compact Kähler manifold of dimension n , and suppose that Ω is a Kähler class on M , unrelated to α . The J -flow, introduced by Donaldson [10] and Chen [6, 7] is the parabolic equation

$$\frac{\partial}{\partial t} \omega_t = -\sqrt{-1} \partial \bar{\partial} \Lambda_{\omega_t} \alpha,$$

with initial condition $\omega_0 \in \Omega$. It was shown by Song–Weinkove [21] that this flow converges whenever there exists a metric $\omega \in \Omega$ satisfying $\Lambda_{\omega} \alpha = c$, where the constant c only depends on the classes $[\alpha], \Omega$, and is determined by

$$(1) \quad \int_M c \omega^n - n \omega^{n-1} \wedge \alpha = 0.$$

In addition, Song–Weinkove [21] showed that such an ω exists if and only if there is a metric $\chi \in \Omega$ such that

$$(2) \quad c \chi^{n-1} - (n-1) \chi^{n-2} \wedge \alpha > 0,$$

in the sense of positivity of $(n-1, n-1)$ -forms. Unfortunately, in practice it seems to be almost as difficult to produce a metric χ with this positivity property, as solving the equation $\Lambda_{\omega} \alpha = c$, except in the case when $n = 2$, when the condition reduces to the class $c\Omega - [\alpha]$ being Kähler. In particular when $n > 2$, it was unknown whether the existence of a solution to the equation depends only on the classes $\Omega, [\alpha]$, or if it depends on the choice of α in its class. Our first main result settles this problem by showing that solvability depends only on the class $[\alpha]$.

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Theorem 1. *Suppose that there is a metric $\omega \in \Omega$ such that $\Lambda_\omega \alpha = c$, and $\beta \in [\alpha]$ is another Kähler metric. Then there exists an $\omega' \in \Omega$ such that $\Lambda_{\omega'} \beta = c$.*

This result still leaves open the problem of finding effective necessary and sufficient conditions for solvability of the equation, or equivalently, for convergence of the J -flow. In this direction, the second author and Lejmi [19] proposed the following conjecture for when the equation can be solved.

Conjecture 2. *There exists an $\omega \in \Omega$ satisfying $\Lambda_\omega \alpha = c$, with c defined by Equation (1), if and only if for all irreducible subvarieties $V \subset M$ with $0 < p = \dim V < n$ we have*

$$\int_V c\chi^p - p\chi^{p-1} \wedge \alpha > 0,$$

for $\chi \in \Omega$.

It is natural to think of metrics χ satisfying the positivity condition (2) as subsolutions for the equation $\Lambda_\omega \alpha = c$. The result of Song–Weinkove [21] then says that we can solve the equation whenever a subsolution exists, whereas Conjecture 2 provides a numerical criterion for the existence of a subsolution. In this sense it is somewhat analogous to the result of Demailly–Paun [9] characterizing the Kähler cone. At the same time as shown in [19], the conjecture is related to a circle of ideas in Kähler geometry, relating the existence of special Kähler metrics to algebro-geometric stability conditions, such as the Yau–Tian–Donaldson conjecture [32, 25, 11] on the existence of constant scalar curvature Kähler metrics.

Our next result is that the conjecture holds for toric manifolds (see Yao [30] for prior results on toric manifolds). In fact we prove the following more general result.

Theorem 3. *Let M be a compact toric manifold of dimension n , with two Kähler metrics α, χ . Suppose that the constant $c > 0$ satisfies*

$$(3) \quad \int_M c\chi^n - n\chi^{n-1} \wedge \alpha \geq 0,$$

and for all irreducible toric subvarieties $V \subset M$ of dimension $0 < p \leq n - 1$ we have

$$\int_V c\chi^p - p\chi^{p-1} \wedge \alpha > 0.$$

Then there is a metric $\omega \in [\chi]$ such that

$$(4) \quad \Lambda_\omega \alpha + d \frac{\alpha^n}{\omega^n} = c,$$

for a suitable constant $d \geq 0$. In particular if in (3) we have equality, then necessarily $\Lambda_\omega \alpha = c$, and so Conjecture 2 holds for toric manifolds M .

The advantage of this more general result is that the hypotheses are amenable to an inductive argument. The result for $(n - 1)$ -dimensional manifolds can be used to construct a suitable barrier function near the union $D \subset M$ of the toric divisors, that allows us to reduce the problem to obtaining a priori estimates on compact subsets of $M \setminus D$. Theorem 1 allows us to work with torus invariant data, which on $M \setminus D$ means that the equation reduces to an equation for convex functions on \mathbf{R}^n . Equation 4 is of the form

$$(5) \quad \text{Tr}((D^2u)^{-1}) + \frac{d}{\det(D^2u)} = 1,$$

where $u : \mathbf{R}^n \rightarrow \mathbf{R}$ is convex, although we have to deal with variable coefficients as well. The main estimate is an upper bound for D^2u on compact sets, which we obtain in Proposition 25. Using the Legendre transform and the constant rank theorem of Bian–Guan [1], this can be reduced to obtaining a priori $C^{2,\alpha}$ estimates for convex solutions of the equation

$$(6) \quad \Delta h + d \det(D^2h) = 1.$$

The difficulty is that the operator $M \mapsto \text{Tr}(M) + d \det(M)$ is neither concave, nor convex for $d > 0$, and so the standard Evans–Krylov theory does not apply. In addition the level sets $\{M : \det(M) = 1 - t\}$ are not uniformly convex as $t \rightarrow 1$, and so the results of Caffarelli–Yuan [5] also do not apply directly. Instead we obtain the interior $C^{2,\alpha}$ estimates by showing that $\det(D^2h)^{1/n}$ is a supersolution for the linearized equation, to which the techniques of [5] can be applied. This result may be of independent interest.

Many of our techniques apply to more general equations than Equation (4), of the form $F(A) = c$, where A is the matrix $A_j^i = \alpha^{i\bar{k}} \omega_{j\bar{k}}$ and $F(A)$ is a symmetric function of the eigenvalues of A satisfying certain structural conditions (see Section 3 for details). In particular we can consider general inverse σ_k equations of the form

$$(7) \quad \sum_{k=1}^n c_k \binom{n}{k} \alpha^k \wedge \omega^{n-k} = c \omega^n,$$

where $c_i \geq 0$ are given non-negative constants, and $c \geq 0$ is determined by the c_i by integrating the equation over M .

The question of looking at general equations of this form was raised by Chen [6], and some special cases beyond the J -flow were treated by Fang–Lai–Ma [14], and also by Guan–Sun [15], Sun [23] on Hermitian manifolds. More general non-linear flows related to the inverse σ_k -equations were investigated by Fang–Lai [13]. The particular Equation (4) was studied by Zheng [33]. In [14] it was shown that a solution to these special cases of equation (7) exist if and only if there is a metric

$\chi \in \Omega$ satisfying a positivity condition analogous to (2). We show that such a result holds for the general equation too, as was conjectured by Fang–Lai–Ma.

Theorem 4. *Equation (7) has a solution $\omega \in \Omega$ if and only if we can find a metric $\chi \in \Omega$ such that*

$$(8) \quad c\chi^{n-1} - \sum_{k=1}^{n-1} c_k \binom{n-1}{k} \chi^{n-k-1} \wedge \alpha^k > 0,$$

in the sense of positivity of $(n-1, n-1)$ -forms.

Note that when $c_k = 0$ for $k < n$, then Equation (7) is simply a complex Monge–Ampère equation, which can always be solved by Yau’s Theorem [31]. In this case the positivity condition (8) is always satisfied, so Theorem 4 is a generalization of Yau’s Theorem. We expect that our methods can be used to generalize Theorems 1 and 3 to more general equations of this form, but we will leave a detailed study of this to future work, except for the following result that is needed in reducing Theorem 3 to the case of torus invariant α . The proof of this is similar to, but simpler than that of Theorem 1.

Theorem 5. *Suppose that Equation (7) has a solution, and $c_n > 0$. Then the equation can also be solved if α is replaced by any other metric $\beta \in [\alpha]$.*

A brief summary of the contents of the paper is as follows. In Section 2 we recall some basic convexity properties of the elementary symmetric functions, which play a key role in the later calculations. In Section 3 we generalize the C^2 -estimates of Song–Weinkove [21] and prove Theorem 4. While the basic ideas are similar to those in [21] and also Fang–Lai–Ma [14], Fang–Lai [13] we hope that our more streamlined proof highlights the required structural conditions for the equation. In Section 4 we prove Theorems 1 and 5. A key ingredient is a smoothing construction, based on work of Blocki–Kolodziej [2] on regularizing plurisubharmonic functions. The remainder of the paper is concerned with the proof of Theorem 3. We expect that our inductive method of proof will be helpful in resolving Conjecture 2 on non-toric manifolds as well, although there are certainly new difficulties in the general case.

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2. Convexity properties

In this section we collect some calculations relating to the inverse σ_k -operator. Most of these results are well known, and available in the literature (for instance see Spruck [22]), but for the reader's convenience we present the calculations here.

For an n -tuple of numbers λ_i , denote by

$$S_k(\lambda_i) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k},$$

the elementary symmetric function of degree k . We have $S_0 = 1$ and $S_{-1} = 0$. For distinct indices i_1, \dots, i_l let us write

$$S_{k; i_1, \dots, i_l}(\lambda_i) = S_k(\lambda_i) |_{\lambda_{i_1} = \dots = \lambda_{i_l} = 0},$$

while if i_1, \dots, i_l are not distinct, then $S_{k; i_1, \dots, i_l}(\lambda_i) = 0$. Given an $n \times n$ matrix A we will also write $S_k(A)$ for the elementary symmetric function of the eigenvalues of A , and in addition if A is diagonal then we define $S_{k; i_1, \dots, i_l}(A)$ by letting $\lambda_i = A_{ii}$.

Lemma 6. *The derivatives of S_k at a diagonal matrix A are given by*

$$\begin{aligned} \partial_{ij} S_k(A) &= \begin{cases} S_{k-1; i}(A), & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \\ \partial_{ij} \partial_{rs} S_k(A) &= \begin{cases} S_{k-2; i, r}(A), & \text{if } i = j, r = s, i \neq r, \\ -S_{k-2; i, j}(A), & \text{if } i \neq j, r = j, s = i, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Here ∂_{ij} means partial derivative with respect to the ij -component.

Proof. This result follows from the fact that $S_k(A)$ is the coefficient of $(-x)^{n-k}$ in $\det(A - xI)$. q.e.d.

Using this, we can compute the derivatives of the inverse σ_k operators.

Lemma 7. *For $0 \leq k \leq n$ let us write $F(A) = S_k(A^{-1}) = \frac{S_{n-k}(A)}{S_n(A)}$. At a diagonal matrix A , with eigenvalues λ_i , we have*

$$\begin{aligned} \partial_{ij} F(A) &= \begin{cases} -\frac{S_{n-k; i}(A)}{\lambda_i S_n(A)}, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases} \\ \partial_{ii} \partial_{jj} F(A) &= \frac{S_{n-k; i, j}(A)}{\lambda_i \lambda_j S_n(A)}, & \text{if } i \neq j, \\ \partial_{ij} \partial_{ji} F(A) &= \frac{S_{n-k; i}(A) + \lambda_i S_{n-k-1; i, j}(A)}{\lambda_i \lambda_j S_n(A)}, & \text{if } i \neq j, \\ \partial_{ii} \partial_{ii} F(A) &= 2 \frac{S_{n-k; i}(A)}{\lambda_i^2 S_n(A)}, \\ \partial_{ij} \partial_{rs} F(A) &= 0, & \text{otherwise.} \end{aligned}$$

The following expresses a strong convexity property of the inverse σ_k operators on positive definite matrices.

Lemma 8. *Let us write $F(A) = S_k(A^{-1})$ as above. Then $A \mapsto F(A)$ is convex on the space of positive definite Hermitian matrices A , and in fact if A is diagonal with eigenvalues $\lambda_i > 0$, then for any matrix B_{ij} we have*

$$(9) \quad \sum_{p,q,r,s} B_{rs} \overline{B_{qp}} (\partial_{pq} \partial_{rs} F(A)) + \sum_{i,j} |B_{ij}|^2 \frac{\partial_{ii} F(A)}{\lambda_j} \geq 0.$$

Proof. Suppose that A is diagonal. Given B_{ij} Hermitian, using the previous result we can compute

$$\begin{aligned} \sum_{p,q,r,s} B_{rs} \overline{B_{qp}} (\partial_{pq} \partial_{rs} F(A)) &= \sum_{i,j} B_{ii} \overline{B_{jj}} \frac{S_{n-k;i,j}(A) + \delta_{ij} S_{n-k;i}(A)}{\lambda_i \lambda_j S_n(A)} \\ &\quad + \sum_{i,j} |B_{ij}|^2 \frac{S_{n-k;i}(A) + \lambda_i S_{n-k-1;i,j}(A)}{\lambda_i \lambda_j S_n(A)}. \\ &\geq \sum_{i,j} B_{ii} \overline{B_{jj}} \frac{S_{n-k;i,j}(A) + \delta_{ij} S_{n-k;i}(A)}{\lambda_i \lambda_j S_n(A)} \\ &\quad - \sum_{i,j} |B_{ij}|^2 \frac{\partial_{ii} F(A)}{\lambda_j}. \end{aligned}$$

It remains to show that the matrix $M_{ij} = S_{n-k;i,j}(A) + \delta_{ij} S_{n-k;i}(A)$ is non-negative. This is shown in Fang–Lai–Ma [14] as follows. For any $(n-k)$ -tuple $I = \{i_1, \dots, i_{n-k}\} \subset \{1, \dots, n\}$, denote by $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_{n-k}}$, and let E_I be the matrix whose entries are

$$(E_I)_{ij} = \begin{cases} 1, & \text{if } i, j \notin I, \\ 0, & \text{otherwise.} \end{cases}$$

Then the matrix M is non-negative because

$$M = \sum_{|I|=n-k} \lambda_I E_I. \quad \text{q.e.d.}$$

The following is an even stronger convexity property of the map $A \mapsto S_1(A^{-1})$, on the set of matrices with eigenvalues bounded away from zero.

Lemma 9. *Given $\delta > 0$, let \mathcal{M} be the set of positive Hermitian matrices with eigenvalues $\lambda_i > \delta$. For $\epsilon > 0$ we let $F(A) = S_1(A^{-1}) - \epsilon S_n(A^{-1})$. If ϵ is sufficiently small (depending on δ), then we have*

- 1) $F(A) > 0$ and $\partial_{ii} F(A) < 0$ for diagonal $A \in \mathcal{M}$.
- 2) F is convex on \mathcal{M} , and in fact F satisfies the inequality (9) for diagonal $A \in \mathcal{M}$.

Proof. 1) If each eigenvalue is greater than δ , then $S_n(A^{-1}) < \delta^{-(n-1)}S_1(A^{-1})$, and so $F(A) > 0$ for sufficiently small ϵ . In addition we have

$$\partial_{ii}F(A) = \frac{-S_{n-1;i}(A) + \epsilon}{\lambda_i S_n(A)}.$$

Since $S_{n-1;i} > \delta^{n-1}$, we have $\partial_{ii}F(A) < 0$ for sufficiently small ϵ .

2) Using the computation in the previous lemma, we have, if A is diagonal with eigenvalues λ_i , that

$$\begin{aligned} \sum_{p,q,r,s} B_{rs} \overline{B_{pq}} (\partial_{pq} \partial_{rs} F(A)) &= \sum_{i,j} B_{ii} \overline{B_{jj}} \frac{\delta_{ij} S_{n-1;i}(A) - \epsilon(1 + \delta_{ij})}{\lambda_i \lambda_j S_n(A)} \\ &\quad + \sum_{i,j} |B_{ij}|^2 \frac{S_{n-1;i}(A) + \lambda_i S_{n-2;i,j}(A) - \epsilon}{\lambda_i \lambda_j S_n(A)}. \\ &\geq \frac{1}{S_n(A)} \sum_i \frac{|B_{ii}|^2}{\lambda_i^2} S_{n-1;i}(A) - \frac{\epsilon}{S_n(A)} \left| \sum_i \frac{B_{ii}}{\lambda_i} \right|^2 \\ &\quad - \frac{\epsilon}{S_n(A)} \sum_i \frac{|B_{ii}|^2}{\lambda_i^2} - \sum_{i,j} |B_{ij}|^2 \frac{\partial_{ii}F(A)}{\lambda_j}. \end{aligned}$$

If $A \in \mathcal{M}$, then $S_{n-1;i}(A) \geq \delta^{n-1}$, and so we get the required inequality (9) if ϵ is sufficiently small. q.e.d.

3. C^2 -estimates

In this section we prove an analog of the C^2 -estimates obtained by Song–Weinkove [21] for the J -flow, and Fang–Lai–Ma [14], Fang–Lai [13] for a more general class of inverse σ_k flows. We will need the corresponding estimates also for manifolds with boundary, analogous to results of Guan–Sun [15].

We will work with general operators

$$F(A) = f(\lambda_1, \dots, \lambda_n),$$

on the space of positive Hermitian matrices, where f is a symmetric function of the eigenvalues of A . We will require certain structural conditions to hold for F . We do not expect that these conditions are optimal, but they are sufficient for our needs. Also, in contrast with the standard setup in Hessian type nonlinear PDE, we work with a convex, decreasing operator, rather than a concave, increasing one. This is just a matter of changing signs, but we found it more convenient to think about the convex, decreasing operator $\text{Tr}(A^{-1})$ rather than the concave, increasing $-\text{Tr}(A^{-1})$.

We require that there are constants $K, C > 0$ (with $K = \infty$ allowed), and a connected component \mathcal{M} of the set $\{F(A) < K\}$, such that if $A \in \mathcal{M}$ then in coordinates such that A is diagonal, we have

- 1) $F(A) > 0$, and $\partial_{ii}F(A) < 0$ for all i ,
- 2) For any matrix B_{ij} we have

$$\sum_{i,j,r,s} B_{ij} \overline{B_{sr}} (\partial_{ij} \partial_{rs} F(A)) + \sum_{i,j} \frac{\partial_{ii} F(A)}{\lambda_j} |B_{ij}|^2 \geq 0,$$

so in particular F is convex on the set \mathcal{M} .

- 3) We have

$$C^{-1}F(A) < \sum_k -\lambda_k \partial_{kk} F(A) < CF(A).$$

- 4) If λ_1 denotes the smallest eigenvalue, then $-\lambda_1 \partial_{11} F(A) \geq -C^{-1} \lambda_i \partial_{ii} F(A)$ for all i .
- 5) The function $g(x_1, \dots, x_n) = f(x_1^{-1}, \dots, x_n^{-1})$ extends to a smooth function on the orthant $\{x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$.

Note that in most of our situations we will be able to take $K = \infty$, and \mathcal{M} the set of all positive Hermitian matrices. We will only need the greater generality that we are allowing in Section 4. Let us denote by A the matrix $A_j^i = \alpha^{i\bar{k}} g_{j\bar{k}}$. This matrix is Hermitian with respect to the inner product defined by α .

Lemma 10. *Suppose that we work at a point in normal coordinates for the metric α , such that g is diagonal. In addition assume that $A \in \mathcal{M}$. Then we have*

$$\partial_1 \partial_{\bar{1}} F(A) \geq - \sum_p -\partial_{pp} F(A) g_{1\bar{1}} \partial_p \partial_{\bar{p}} \log g_{1\bar{1}} - CF(A),$$

where C depends on F , α .

Proof. We compute, using Lemma 7 and Lemma 8:

$$\begin{aligned} \partial_1 \partial_{\bar{1}} F(A) &= \partial_1 (\partial_{\bar{1}} A_q^p \partial_{pq} F(A)) \\ &= \partial_1 \partial_{\bar{1}} (A_q^p) \partial_{pq} F(A) + (\partial_1 A_s^r) (\partial_{\bar{1}} A_q^p) \partial_{pq} \partial_{rs} F(A) \\ &= \left[(\partial_1 \partial_{\bar{1}} \alpha^{p\bar{p}}) g_{p\bar{p}} + \partial_1 \partial_{\bar{1}} g_{p\bar{p}} \right] \partial_{pp} F(A) \\ &\quad + (\partial_1 g_{s\bar{r}}) (\partial_{\bar{1}} g_{q\bar{p}}) \partial_{pq} \partial_{rs} F(A) \\ &= \left[(\partial_1 \partial_{\bar{1}} \alpha^{p\bar{p}}) g_{p\bar{p}} + g_{1\bar{1}} \partial_p \partial_{\bar{p}} \log g_{1\bar{1}} \right. \\ &\quad \left. + g^{1\bar{1}} (\partial_1 g_{p\bar{1}}) (\partial_{\bar{1}} g_{1\bar{p}}) \right] \partial_{pp} F(A) + (\partial_1 g_{p\bar{q}}) (\partial_{\bar{1}} g_{q\bar{p}}) \partial_{pq} \partial_{rs} F(A) \\ &\geq - \sum_p -\partial_{pp} F(A) g_{1\bar{1}} \partial_p \partial_{\bar{p}} \log g_{1\bar{1}} - C \sum_p -\lambda_p d_{pp} F(A), \end{aligned}$$

where we used assumption (2) with $B_{ij} = \partial_1 g_{i\bar{j}}$. The result follows by assumption (3) above. q.e.d.

Suppose now that M is a compact manifold and that α, ω_0 are Kähler metrics on M . Suppose that $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$ satisfies the equation

$$(10) \quad \begin{cases} \frac{\partial\varphi_t}{\partial t} &= -F(A_t), \\ \varphi_0 &= 0, \end{cases}$$

where A_t is the matrix $(A_t)_j^i = \alpha^{i\bar{k}}g_{t,j\bar{k}}$ as above, with $g_{t,j\bar{k}}$ being the components of ω_t . By our assumptions on F , this equation is parabolic on the set of metrics satisfying $A \in \mathcal{M}$. Our goal is to show the long time existence of this flow, generalizing the result of Chen [7] for the J -flow and Fang–Lai [13] for more general flows.

The following shows that if $A_0 \in \mathcal{M}$, then we will have $F(A_t) < K$ for as long as the flow exists, and so in particular $A_t \in \mathcal{M}$ as well.

Lemma 11. *We have that $F(A_t) \leq F(A_0)$ for as long as the flow exists, and $|\varphi_t| < C(t+1)$ for some constant C .*

Proof. Differentiating the equation with respect to t we have

$$\frac{\partial\dot{\varphi}_t}{\partial t} = -\partial_{ij}F(A_t)\alpha^{i\bar{k}}\dot{\varphi}_{t,j\bar{k}}.$$

The maximum principle then implies that $\inf_M \dot{\varphi}_t \geq \inf_M \dot{\varphi}_0$, i.e., $F(A_t) \leq F(A_0)$ as long as the flow is parabolic. In particular, if $A_0 \in \mathcal{M}$ the flow is parabolic for as long as it exists. Similarly we can bound $\sup_M \dot{\varphi}_t$ which in turn allows us to bound φ_t . q.e.d.

To show the long time existence of the flow, we will obtain time dependent C^2 -estimates along the flow.

Proposition 12. *There is a constant C such that*

$$\Lambda_\alpha\omega_t < C(t+1)$$

along the flow as long as it exists.

Proof. Consider the function

$$f(x, \xi, t) = \log |\xi|_{g_t}^2 - N_1 t - N_2 \varphi_t,$$

where $x \in M$ and $\xi \in T_x M$ has unit length with respect to α , and N_1, N_2 are large constants to be chosen later. Given $T > 0$, suppose that f achieves its maximum on $[0, T]$, at (x, ξ, t_0) , with $t_0 > 0$. Choose normal coordinates at x for α such that g_{t_0} is diagonal and $\xi = \partial/\partial z^1$. If we define

$$h(z, t) = \log \frac{g_{1\bar{1}}}{\alpha_{1\bar{1}}} - N_1 t - N_2 \varphi_t,$$

then at the origin $z = 0$ and $t = t_0$ we must have

$$\begin{aligned}
0 &\leq \partial_t h(z, t) - \sum_p -\partial_{pp} F(A_t) \partial_p \partial_{\bar{p}} h(z, t) \\
&= -g^{1\bar{1}} F(A_t)_{1\bar{1}} - N_1 - \sum_p -\partial_{pp} F(A_t) \partial_p \partial_{\bar{p}} [\log g_{1\bar{1}} - \log \alpha_{1\bar{1}} - N_2 \varphi_t] \\
&\leq -N_1 + C \sum_p -\partial_{pp} F(A_t) + N_2 \sum_p -\partial_{pp} F(A_t) g_{t, p\bar{p}} \\
&\quad - N_2 \sum_p -\partial_{pp} F(A_t) g_{0, p\bar{p}}.
\end{aligned}$$

Choosing N_2 sufficiently large, we will have

$$0 \leq -N_1 + N_2 \sum_p -\partial_{pp} F(A_t) g_{t, p\bar{p}} < -N_1 + CN_2 F(A_t),$$

using property (3) of F . Since $F(A_t) < K$, if N_1 is chosen sufficiently large, we will have a contradiction. It follows that then $f(x, \xi, t)$ achieves its maximum at $t = 0$, from which the result follows by the bound we already have for φ_t . q.e.d.

Using the convexity of F we can apply the Evans–Krylov theorem to obtain $C^{2,\alpha}$ -estimates for φ_t as long as the flow exists, and higher order estimates follow from standard Schauder estimates. It follows that the flow exists for all time.

Proposition 13. *There is a solution of Equation (10) for all $t > 0$.*

We will make use of this long time existence result in Section 4. While one could pursue the existence of solutions to the equation $F(A) = c$ by studying the convergence of the flow, we will instead primarily use the continuity method. For this we need elliptic C^2 -estimates, assuming the existence of a subsolution of the equation in a certain sense. We will also need to use the case when M is a manifold with boundary.

Suppose, therefore, that $(M, \partial M)$ is a compact Kähler manifold with possibly empty boundary, and suppose that ω, α are metrics on M satisfying

$$F(A) = c,$$

for a constant c . For simplicity we will assume that in the structural conditions for F we can take $K = \infty$.

To define the notion of subsolution that we use, define the function

$$\tilde{f}(\lambda_1, \dots, \lambda_{n-1}) = \lim_{\lambda_n \rightarrow \infty} f(\lambda_1, \dots, \lambda_n) = g(\lambda_1^{-1}, \dots, \lambda_{n-1}^{-1}, 0),$$

in terms of the function g in property (5). For a Hermitian matrix B we will write

$$\tilde{F}(B) = \max \tilde{f}(\lambda_1, \dots, \lambda_{n-1}),$$

where the max runs over all $(n - 1)$ -tuples $(\lambda_1, \dots, \widehat{\lambda}_i, \dots, \lambda_n)$ of eigenvalues of B , where $1 \leq i \leq n$.

REMARK. Trudinger [27] studied the Dirichlet problem (over the reals) for equations of the eigenvalues of the Hessian satisfying certain structural conditions, which for example allow for treating the equation

$$(11) \quad S_1((D^2u)^{-1}) = c,$$

in open domains $\Omega \subset \mathbf{R}^n$. Writing the equation as $f(\lambda_1, \dots, \lambda_n) = c$ in terms of the eigenvalues of D^2u , a key role in the estimates is played by the function

$$f_\infty(\lambda_1, \dots, \lambda_{n-1}) = \lim_{\lambda_n \rightarrow \infty} f(\lambda_1, \dots, \lambda_n),$$

which is the same as our function \tilde{f} above (our equation is the reciprocal of that studied by Trudinger). In our situation the function \tilde{f} is not only relevant in deriving estimates, but it is also used to define the notion of subsolution. We also remark that for the equation to fit into the framework of Caffarelli–Nirenberg–Spruck [4], one would need $f_\infty = 0$.

For technical reasons we will need a notion of viscosity subsolution which we give now.

Definition 14. Suppose that χ is a Kähler current with continuous local potential, i.e., in local charts U we can write $\chi = \sqrt{-1}\partial\bar{\partial}f$ with $f \in C^0(U)$. We say that χ satisfies $\tilde{F}(\alpha^{i\bar{p}}\chi_{j\bar{p}}) \leq c$ in the viscosity sense, if the following holds: suppose that $p \in M$, and $h : U \rightarrow \mathbf{R}$ is a C^2 function on a neighborhood U of p , where $\chi = \sqrt{-1}\partial\bar{\partial}f$. If $h - f$ has a local minimum at p , then $\tilde{F}(\alpha^{i\bar{p}}\partial_j\partial_{\bar{p}}h) \leq c$ at p .

It is clear from the monotonicity of F (and, therefore, \tilde{F}), i.e., structural condition (1), that if χ is a smooth metric satisfying $\tilde{F}(\alpha^{i\bar{p}}\chi_{j\bar{p}}) \leq c$, then this inequality is also satisfied in the viscosity sense. We will need the following, which is a special case of the general fact that a maximum of a family of viscosity subsolutions is a viscosity subsolution (see Caffarelli–Cabré [3]).

Lemma 15. *Suppose that in an open set U we have smooth metrics $\chi_k = \sqrt{-1}\partial\bar{\partial}f_k$ for $k = 1, \dots, N$, satisfying $\tilde{F}(\alpha^{i\bar{p}}\chi_{k,j\bar{p}}) \leq c$. Then $\chi = \sqrt{-1}\partial\bar{\partial}\max\{f_k\}$ satisfies $\tilde{F}(\alpha^{i\bar{p}}\chi_{j\bar{p}}) \leq c$ in the viscosity sense.*

Proof. Fix a point $p \in U$ and suppose that h is a smooth function such that $h - \max\{f_k\}$ has a local minimum at p . Without loss of generality we can assume that $\max\{f_k(p)\} = f_1(p)$, and then $h - f_1$ also has a local minimum at p . By assumption $\tilde{F}(\alpha^{i\bar{p}}\partial_j\partial_{\bar{p}}f_1)(p) \leq c$, and so the monotonicity of \tilde{F} implies that $\tilde{F}(\alpha^{i\bar{p}}\partial_j\partial_{\bar{p}}h)(p) \leq c$. q.e.d.

Proposition 16. *Suppose that $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$ is smooth, and satisfies $F(A) = c$, where $A_j^i = \alpha^{i\bar{p}}\omega_{j\bar{p}}$, and F satisfies the structural conditions with $K = \infty$. Suppose that we have a strict viscosity subsolution $\chi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi$, i.e., that $\tilde{F}(\alpha^{i\bar{p}}\chi_{j\bar{p}}) \leq c - \delta$ in the viscosity sense, for some $\delta > 0$. Here $\psi \in C^0(M)$. Then we have an estimate*

$$\Lambda_\alpha\omega < Ce^{N(\varphi - \inf \varphi)},$$

where the constants C, N depend on the given data, including χ, δ , as well as the maximum of $\Lambda_\alpha\omega$ on ∂M .

Proof. Consider the function

$$f(x, \xi) = \log |\xi|_g^2 - N\varphi(x) + N\psi(x),$$

where $x \in M$ and $\xi \in T_x M$ has unit length with respect to α , and N is a large constant to be chosen later. If this function achieves its maximum on ∂M , then we will have

$$\Lambda_\alpha\omega < Ce^{N(\varphi - \inf \varphi)},$$

where C depends on $\sup_{\partial M} \Lambda_\alpha\omega$ and the given data.

Suppose that f achieves its maximum at (x, ξ) , where $x \in M$ is in the interior. We can choose normal coordinates at x for α such that g is diagonal, and $\xi = \partial/\partial z^1$. This means that the function

$$h(z) = \log \frac{g_{1\bar{1}}}{\alpha_{1\bar{1}}} - N\varphi + N\psi$$

has a maximum at the origin. Define

$$\psi_1 = \varphi - N^{-1} \log \frac{g_{1\bar{1}}}{\alpha_{1\bar{1}}}.$$

Then ψ_1 is smooth, and $\psi_1 - \psi$ has a local minimum at the origin. If we define $\chi' = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_1$, then the definition of viscosity subsolution means that $\tilde{F}(\alpha^{i\bar{p}}\chi'_{j\bar{p}}) \leq c - \delta$. In addition the function

$$\tilde{h}(z) = \log \frac{g_{1\bar{1}}}{\alpha_{1\bar{1}}} - N\varphi + N\psi_1$$

is identically zero. It follows that at the origin

$$\begin{aligned} 0 &= \sum_p -\partial_{pp}F(A)\partial_p\partial_{\bar{p}}\tilde{h} \\ &= \sum_p -\partial_{pp}F(A)\partial_p\partial_{\bar{p}}\log g_{1\bar{1}} - \sum_p -\partial_{pp}F(A)\partial_p\partial_{\bar{p}}\log \alpha_{1\bar{1}} \\ &\quad - N \sum_p -\partial_{pp}F(A)[g_{p\bar{p}} - \chi'_{p\bar{p}}] \\ &\geq -g^{1\bar{1}}CF(A) - C \sum_p -\partial_{pp}F(A) - N \sum_p -\partial_{pp}F(A)(g_{p\bar{p}} - \chi'_{p\bar{p}}), \end{aligned}$$

where we used Lemma 10. We can assume without loss of generality that $g_{1\bar{1}} > 1$. By assumption (3) on F , we know that $\sum_p -\partial_{pp}F(A)g_{p\bar{p}}$ is bounded below. In addition $\chi'_{p\bar{p}}$ also has a fixed positive lower bound by the assumption that χ is a Kähler current. It follows that for any $\epsilon > 0$ we can choose N so large, that we obtain

$$0 \geq -N(1 + \epsilon) \sum_p -\partial_{pp}F(A)g_{p\bar{p}} + N(1 - \epsilon) \sum_p -\partial_{pp}F(A)\chi'_{p\bar{p}}.$$

Rearranging this and changing ϵ slightly, for sufficiently large N we will have

$$(12) \quad \sum_p -\partial_{pp}F(A)\chi'_{p\bar{p}} \leq (1 + \epsilon) \sum_p -\partial_{pp}F(A)g_{p\bar{p}}.$$

Let us now change notation slightly. Write λ_i for the eigenvalues of g , and μ_i for the eigenvalues of χ' . In addition suppose that $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and $\mu_1 \geq \dots \geq \mu_n$. The convexity of f implies that then $-f_i(\lambda) \leq -f_j(\lambda)$ for $i < j$. In addition from the Schur–Horn theorem we have that the diagonal entries $\chi'_{p\bar{p}}$ form an n -tuple in the convex hull of the permutations of the μ_i , and so in particular

$$\sum_p -f_p(\lambda)\mu_p \leq \sum_p -f_p(\lambda)\chi'_{p\bar{p}}.$$

For simplicity of notation let us also suppose that $c = 1$. We then have the following:

$$(13) \quad \begin{aligned} f(\lambda_1, \dots, \lambda_n) &= 1, \\ \tilde{f}(\mu_1, \dots, \mu_{n-1}) &\leq 1 - \delta, \\ \sum_p -f_p(\lambda_1, \dots, \lambda_n)\mu_p &\leq (1 + \epsilon) \sum_p -f_p(\lambda_1, \dots, \lambda_n)\lambda_p. \end{aligned}$$

Using assumption (3) for F and the positive lower bound for the μ_i , we obtain an upper bound for $-f_1(\lambda_i)$. Together with assumption (4) for F , this implies a positive lower bound for the lowest eigenvalue λ_1 .

By assumption (5), there is a number K such that if $\lambda_n > K$, then

$$\begin{aligned} \tilde{f}(\lambda_1, \dots, \lambda_{n-1}) &> 1 - \tau, \\ f_p(\lambda_1, \dots, \lambda_n) - \tau &< \tilde{f}_p(\lambda_1, \dots, \lambda_{n-1}) < f_p(\lambda_1, \dots, \lambda_n) + \tau, \end{aligned}$$

since $f_p(\lambda_i) = -\lambda_p^{-2}g_p(\lambda_i^{-1})$, and the function $x_p^2g(x_i)$ is uniformly continuous on compact subsets of the orthant $\{x_j \geq 0 \text{ for all } 1 \leq j \leq n\}$. The convexity of the map $\lambda_i \rightarrow \tilde{f}(\lambda_i)$ implies that if we denote

$$r(t) = \tilde{f}(\lambda_i + t(\mu_i - \lambda_i)),$$

then $r(1) \geq r(0) + r'(0)$. This means

$$\tilde{f}(\mu_i) \geq \tilde{f}(\lambda_i) + \sum_{p=1}^{n-1} (\mu_p - \lambda_p) \tilde{f}_p(\lambda_i).$$

From this we get

$$\delta - \tau \leq \sum_{p=1}^{n-1} -\tilde{f}_p(\mu_p - \lambda_p) \leq \sum_{p=1}^n -f_p(\lambda_i) \mu_p - \sum_{p=1}^{n-1} -\tilde{f}_p(\lambda_i) \lambda_p.$$

From assumption (5) we also have that $\lambda_p f_p \rightarrow 0$ uniformly as $\lambda_p \rightarrow \infty$, since $\lambda_p f_p(\lambda_i) = -\lambda_p^{-1} g_p(\lambda_i^{-1})$, and $x_p g(x_i)$ is uniformly continuous on compact subsets of $\{x_j \geq 0 \text{ for all } 1 \leq j \leq n\}$. So we can choose a constant K' such that $-f_p \lambda_p < \tau$ if $\lambda_p > K'$. In addition we take K above so that $K > K'$. We then have

$$\begin{aligned} \delta - \tau &\leq \sum_{p=1}^n -f_p \mu_p - \sum_{p=1}^n -f_p \lambda_p + n\tau + \sum_{p=1}^{n-1} (\tilde{f}_p - f_p) K' \\ &\leq \epsilon \sum_{p=1}^n -f_p \lambda_p + n\tau + (n-1)\tau K' \\ &\leq C\epsilon F(A) + n\tau + (n-1)\tau K'. \end{aligned}$$

We can choose τ so small (i.e., K above so large), that

$$(n+1)\tau + (n-1)\tau K' < \frac{\delta}{2}.$$

We will then have

$$\frac{\delta}{2} \leq C\epsilon F(A) = C\epsilon.$$

If now ϵ is sufficiently small (i.e., the constant N above is chosen sufficiently large), then this will be a contradiction.

It follows that if the constant N before is chosen sufficiently large, then at the maximum of our function f we have a bound $g_{1\bar{1}} < K$ for some large K . From this it follows that we have an inequality of the form

$$\Lambda_\alpha g < C e^{N(\varphi - \inf \varphi)},$$

which is what we wanted to prove.

q.e.d.

We now prove Theorem 4 under the more general assumption that χ satisfies the positivity condition (8) in the viscosity sense. For this let us write

$$(14) \quad F(A) = \sum_{k=1}^n c_k S_k(A^{-1}),$$

for constants $c_k \geq 0$. Note that if χ is smooth and $B_j^i = \alpha^{i\bar{p}} \chi_{j\bar{p}}$ then $\tilde{F}(B) \leq c - \delta$ for some $\delta > 0$ is equivalent to the positivity of $(n-1, n-1)$ forms

$$(15) \quad c\chi^{n-1} - \sum_{k=1}^{n-1} c_k \binom{n-1}{k} \chi^{n-k-1} \wedge \alpha^k > 0.$$

In particular, Theorem 4 follows from the following.

Theorem 17. *Suppose that we have a Kähler current $\chi \in \Omega$ satisfying $\tilde{F}(\alpha^{i\bar{p}} \chi_{j\bar{p}}) \leq c - \delta$ in the viscosity sense, for some $\delta > 0$. Suppose that the constant c satisfies*

$$\sum_{k=1}^n c_k \binom{n}{k} \int_M \alpha^k \wedge \omega_0^{n-k} = c \int_M \omega_0^n.$$

Then there is an $\omega \in \Omega$ satisfying the equation $F(\alpha^{i\bar{p}} \omega_{j\bar{p}}) = c$, i.e.,

$$(16) \quad \sum_{k=1}^n c_k \binom{n}{k} \alpha^k \wedge \omega^{n-k} = c\omega^n.$$

Proof. We will use the continuity method to solve the equation

$$(17) \quad F_d(A) = F(A) + d \frac{\alpha^n}{\omega^n} = c_d,$$

for $d \in [0, \infty)$, where the constant c_d is determined by d by integrating the equation with respect to ω^n over M . In particular $c_d \geq c$. According to Lemma 18 below, $F_d(A)$ satisfies the structural conditions required by the C^2 -estimates (with $K = \infty$ and \mathcal{M} the space of all positive Hermitian matrices). In addition $\tilde{F}_d = \tilde{F}$, so χ is a strict viscosity subsolution for the equation $F_d(A) = c_d$, for all $d \geq 0$. We will, therefore, be able to use Proposition 16 to obtain C^2 -estimates.

Let $I = \{d \in [0, \infty) : (17) \text{ has a solution}\}$. By Yau's theorem [31] we can solve the equation $\alpha^n/\omega^n = c_\infty$ for a suitable constant c_∞ . The implicit function theorem then implies that we can solve (17) for sufficiently large d , and that I is open.

To see that I is closed, suppose that $\omega_k = \omega_0 + \sqrt{-1} \partial\bar{\partial} \varphi_k$ are solutions, with corresponding $d_k \rightarrow d$. Proposition 16 then implies that

$$\Lambda_\alpha \omega_k < C e^{N(\varphi_k - \inf \varphi_k)},$$

for uniform C, N . Normalizing so that $\sup \varphi_k = 0$, it follows from Weinkove [29, Lemma 3.4] (see also [28, Proposition 4.2]), that we have $\Lambda_\alpha \omega_k < C$ for a uniform C . The equation then implies a lower bound $\omega_k > C^{-1} \alpha$ as well. Since F_d is convex, the Evans–Krylov theorem [12, 18] (see Tosatti–Wang–Weinkove–Yang [26] for a general version adapted to complex geometry), together with Schauder estimates can be used to obtain higher order estimates for the ω_k , allowing us to

pass to a limit as $k \rightarrow \infty$. This shows that I is closed, and so $0 \in I$, which was our goal. q.e.d.

Lemma 18. *The map $F(A)$ in Equation (14) satisfies the structural conditions at the beginning of Section 3, on the whole space of positive Hermitian matrices.*

Proof. It is clear that $F(A) > 0$, and from Lemma 7 we have, at a diagonal A with eigenvalues $\lambda_i > 0$, that

$$\partial_{ii}F(A) = - \sum_{k=1}^n c_k \frac{S_{n-k;i}(A)}{\lambda_i S_n(A)},$$

which is negative. The required convexity property (2) follows from Lemma 8.

For property (3), note that

$$\begin{aligned} \sum_i -\lambda_i \partial_{ii}F(A) &= \sum_{k,i} c_k \frac{S_{n-k;i}(A)}{S_n(A)} \\ &= \sum_{k=1}^n k c_k \frac{S_{n-k}(A)}{S_n(A)} = \sum_{k=1}^n k c_k S_k(A^{-1}), \end{aligned}$$

using the identity $\sum_{i=1}^n S_{l;i}(A) = (n-l)S_l(A)$. It follows that

$$F(A) \leq - \sum_i \lambda_i \partial_{ii}F(A) \leq nF(A).$$

To show property (4), recall that

$$-\lambda_i \partial_{ii}F(A) = \sum_{k=1}^n c_k \frac{S_{n-k;i}(A)}{S_n(A)},$$

and note that if λ_1 is the smallest eigenvalue, then $S_{n-k;1}(A) \geq S_{n-k;i}(A)$ for all i . It follows that $-\lambda_1 \partial_{11}F(A) \geq -\lambda_i \partial_{ii}F(A)$.

Finally, property (5) is clear since we have

$$g(x_1, \dots, x_n) = f(x_1^{-1}, \dots, x_n^{-1}) = \sum_{k=1}^n c_k S_k(x_1, \dots, x_n),$$

which extends smoothly to the orthant $\{x_i \geq 0 \text{ for all } 1 \leq i \leq n\}$. q.e.d.

4. The proof of Theorem 1

In the proof of Theorem 1 a key role is played by the parabolic equation (10). In this section we will consider the operator

$$F_\epsilon(A) = S_1(A^{-1}) - \epsilon S_n(A^{-1}),$$

for small $\epsilon > 0$. Note that this operator is not convex on the space of all positive definite Hermitian matrices. The following lemma shows that it is still convex, however, on a suitable set of matrices A .

- Lemma 19.** 1) For all $Q > 0$, if ϵ is sufficiently small, then F_ϵ satisfies the structural conditions in Section 3 on the set where $S_1(A^{-1}) < Q$.
- 2) For all $K > 0$, if ϵ is sufficiently small, then F_ϵ satisfies the structural conditions in Section 3 on the connected component \mathcal{M} of the set $\{F_\epsilon(A) < K\}$ containing the set $\{S(A^{-1}) < K\}$.

Proof. For statement (1), note that if $S_1(A^{-1}) < Q$, then all eigenvalues of A are greater than $1/Q$. By Lemma 9, for sufficiently small ϵ , the map F_ϵ satisfies the structural conditions (1) and (2) on the set of matrices with eigenvalues greater than $1/Q$. For structural condition (3), we have

$$\begin{aligned} \sum_k -\lambda_k \partial_{kk} F_\epsilon(A) &= \sum_k \frac{S_{n-1;k}(A) - \epsilon}{S_n(A)} \\ &= \frac{S_{n-1}(A)}{S_n(A)} - \frac{n\epsilon}{S_n(A)} = S_1(A^{-1}) - n\epsilon S_n(A^{-1}). \end{aligned}$$

The required inequality follows since $S_n(A^{-1}) < Q^{n-1} S_1(A^{-1})$, and so if ϵ is sufficiently small, we will have

$$\frac{1}{2} F_\epsilon(A) < \sum_k -\lambda_k \partial_{kk} F(A) < CF(A).$$

Although we do not actually need structural assumptions (4) and (5) below (since we will only use Proposition 13 which does not use them), they are also easy to check, (5) being immediate. For (4), note that

$$-\lambda_i \partial_{ii} F_\epsilon(A) = \frac{S_{n-1;i}(A) - \epsilon}{S_n(A)},$$

and so for sufficiently small ϵ we will have

$$\frac{1}{2} (-\lambda_i \partial_{ii} S_1(A^{-1})) \leq -\lambda_i \partial_{ii} F_\epsilon(A) \leq \lambda_i \partial_{ii} S_1(A^{-1}),$$

so (4) follows from the corresponding property of the map $A \mapsto S_1(A^{-1})$, which we have shown in Lemma 18.

For statement (2) let us take $Q = K + 1$, and take ϵ sufficiently small for (1) to apply. In addition, note that the AM-GM inequality implies that $S_n(A^{-1}) \leq n^{-n} S_1(A^{-1})^n$, so if $S_1(A^{-1}) = K + 1$, then $S_n(A^{-1}) < K'$ for some constant K' . Let us choose ϵ even smaller, so that $K + 1 - \epsilon K' > K$. This means that if $S_1(A^{-1}) = K + 1$, then

$$F_\epsilon(A) \geq K + 1 - \epsilon K' > K.$$

Suppose now that A_1 is in the connected component of $\{F_\epsilon(A) < K\}$ containing $\{S_1(A^{-1}) < K\}$, i.e., we have positive Hermitian matrices A_t for $t \in [0, 1]$ such that $F_\epsilon(A_t) < K$, and $S_1(A_0^{-1}) < K$. Then from the above we have $S_1(A_t^{-1}) < K + 1$ for all t , and in particular $S_1(A_1^{-1}) < K + 1$. It follows from (1) that F_ϵ satisfies the structural conditions on this connected component. q.e.d.

Consider now the evolution equation

$$(18) \quad \begin{cases} \frac{\partial \varphi_t}{\partial t} &= c_\epsilon - F_\epsilon(A_t), \\ \varphi_0 &= 0, \end{cases}$$

where $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$, and $(A_t)_j^i = \alpha^{i\bar{k}}g_{t,j\bar{k}}$. In addition the constant c_ϵ is chosen so that

$$\int_M c_\epsilon \omega^n = \int_M F_\epsilon(A) \omega^n.$$

From the previous lemma we have that if $F(A_0) < K$, and ϵ is chosen sufficiently small, then F_ϵ satisfies the structural conditions on the connected component of $\{F_\epsilon(A) < K\}$ containing A_0 . Proposition 13 then implies that the flow exists for all time. Note that in addition the proof of Lemma 19 also shows that we can assume $S_1(A_t^{-1}) < K + 1$ along the flow.

The parabolic equation (18) is the negative gradient flow of the functional \mathcal{J}_ϵ on the Kähler class $[\omega]$, defined by the variational formula

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{J}_\epsilon(\omega + t\sqrt{-1}\partial\bar{\partial}\varphi) = \int_M \varphi(F_\epsilon(A) - c_\epsilon)\omega^n,$$

and normalized so that $\mathcal{J}_\epsilon(\omega_0) = 0$ for a fixed choice of ω_0 . Note that \mathcal{J}_0 is the J -functional considered in Song–Weinkove [21].

Definition 20. We say that the function \mathcal{J}_0 is proper, if there are constants $C, \delta > 0$ such that if $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$, then

$$\begin{aligned} \mathcal{J}_0(\omega) &\geq -C + \delta \int_M \varphi(\omega_0^n - \omega^n) \\ &= -C + \delta \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge (\omega_0^{n-1} + \dots + \omega^{n-1}), \end{aligned}$$

for all $\omega \in \Omega$.

The main ingredient in the proof of Theorem 1 is the following.

Proposition 21. *Suppose that \mathcal{J}_0 is proper. Then we can find $\omega \in \Omega$ such that $A_j^i = \alpha^{i\bar{k}}\omega_{j\bar{k}}$ satisfies $S_1(A^{-1}) = c$, i.e., $\Lambda_\alpha\omega = c$.*

Proof. For simplicity of notation let us normalize the class Ω and α so that $\int_M \omega^n = \int_M \alpha^n = 1$. It follows that

$$c_\epsilon = c - \epsilon \int_M \frac{\alpha^n}{\omega^n} \omega^n = c - \epsilon,$$

for any $\epsilon > 0$. If $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi$, then we have

$$(\mathcal{J}_\epsilon - \mathcal{J}_0)(\omega) = \epsilon \int_0^1 \int_M \varphi(\omega_t^n - \alpha^n),$$

where $\omega_t = \omega_0 + t\sqrt{-1}\partial\bar{\partial}\varphi$. Using Yau's Theorem [31], we can assume that we chose our base point ω_0 so that $\omega_0^n = \alpha^n$, so

$$\begin{aligned} \mathcal{J}_\epsilon(\omega) &= \mathcal{J}_0(\omega) + \epsilon \int_0^1 \int_M \varphi(\omega_t^n - \omega_0^n) \\ &\geq -C + \delta \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge (\omega_0^{n-1} + \dots + \omega_0^{n-1}) \\ &\quad - \epsilon \int_0^1 \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge (\omega_t^{n-1} + \dots + \omega_0^{n-1}). \end{aligned}$$

We claim that the third term above is controlled by the second term. To see this observe that $\omega_t = (1-t)\omega_0 + t\omega$, and so

$$\omega_t^{n-1-k} \wedge \omega_0^k = \sum_{i=0}^k p_i(t) \omega_0^{n-1-k-i} \wedge \omega_0^{k+i},$$

where $p_i(t)$ is a polynomial in t depending only on n, k . The claim clearly follows, and so if ϵ is sufficiently small, then \mathcal{J}_ϵ will be proper.

Choose ϵ even smaller if necessary so that Equation (18) has a solution $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$ for all $t > 0$. Since this flow is the negative gradient flow of \mathcal{J}_ϵ , and this functional is bounded from below, we can find a sequence of metrics ω_k along the flow such that

$$\lim_{k \rightarrow \infty} \int_M (F_\epsilon(A_k) - c_\epsilon)^2 \omega_k^n = 0.$$

Since we have $S_1(A_t^{-1}) < K + 1$ along the flow for a uniform constant K , we have a uniform lower bound $\omega_k > \kappa\alpha$. It follows that

$$(19) \quad \lim_{k \rightarrow \infty} \|F_\epsilon(A_k) - c_\epsilon\|_{L^2(\alpha)} = 0,$$

and from Lemma 11 we know that $F_\epsilon(A_k) < K$. Choosing ϵ even smaller, using Lemma 9 we can assume that F_ϵ satisfies the structural conditions on the set of matrices with eigenvalues bounded below by $\frac{\kappa}{2}$. The importance of this is that this is a convex set.

Let us write $\psi_k = \varphi_k - \sup_M \varphi_k$, so that $\omega_k = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_k$, and $\sup_M \psi_k = 0$. The properness of \mathcal{J}_ϵ implies that we have a uniform

constant C such that

$$\int_M \psi_k(\omega_0^n - \omega_k^n) < C.$$

A standard argument using the inequality $\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_k > 0$ together with $\sup_M \psi_k = 0$ implies that $\int_M \psi_k \omega_0^n$ is bounded below uniformly. It follows that we have a uniform bound

$$(20) \quad - \int_M \psi_k \omega_k^n < C.$$

Choosing a subsequence we can assume that the ω_k converge weakly to a current $\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_\infty$, and $\psi_k \rightarrow \psi_\infty$ in L^1 . From Guedj–Zeriahi [16, Corollary 1.8, Corollary 2.7], the bound (20) implies that ψ_∞ has zero Lelong numbers.

We now use the technique of Blocki–Kolodziej [2] to mollify the metrics ω_k , in order to obtain pointwise bounds from the integral bound (19). The fact that ψ_∞ has zero Lelong numbers will ensure that we can perform this mollification uniformly in k .

Fix a small number $\tau > 0$, and choose a finite open cover $\{W_i\}$ of M such that on each W_i we have local coordinates z^j , in which the matrix of components of α satisfies

$$(21) \quad (1 - \tau)\delta_{ij} < \alpha_{i\bar{j}} < (1 + \tau)\delta_{ij}.$$

Let $V_i \subset U_i \subset W_i$ be relatively compact so that the V_i still cover M . On each W_i we have $\omega_0 = \sqrt{-1}\partial\bar{\partial}f_i$ for local potentials f_i , and so we have the plurisubharmonic functions $u_i^{(k)} = f_i + \psi_k$, which are local potentials for the ω_k . We allow $k = \infty$ here. For sufficiently small $\delta > 0$ (depending on the distance between the boundaries of U_i, W_i , and so on τ) we can define plurisubharmonic functions $u_{i,\delta}^{(k)}$ on U_i by

$$u_{i,\delta}^{(k)}(z) = \int_{\mathbf{C}^n} u_i^{(k)}(z - \delta w) \rho(w) dw,$$

where $\rho : \mathbf{C}^n \rightarrow \mathbf{R}$ is a standard mollifier: $\rho \geq 0$, $\rho(w) = 0$ for $|w| > 1$, and $\int \rho(w) dw = 1$.

For each i we choose $\eta_i : U_i \rightarrow \mathbf{R}$ such that $\eta_i \leq 0$, and in addition $\eta_i = 0$ on V_i and $\eta_i = -1$ on ∂U_i . Fix $\gamma > 0$ to be sufficiently small (depending on τ) such that $\gamma|\sqrt{-1}\partial\bar{\partial}\eta_i| < \tau\alpha$ on U_i for all i .

Consider the function

$$(22) \quad \psi_{k,\delta}(z) = \max_i \{u_{i,\delta}^{(k)}(z) - f_i(z) + \gamma\eta_i(z)\},$$

where the maximum is taken over all i for which $z \in U_i$. The results in Blocki–Kolodziej [2] (see the proof of Theorem 2) imply that if δ is sufficiently small (depending on τ), then

$$|(u_{i,\delta}^{(\infty)} - f_i) - (u_{j,\delta}^{(\infty)} - f_j)| < \frac{\gamma}{4},$$

on $U_i \cap U_j$, since ψ_∞ has zero Lelong numbers. The L^1 -convergence $\psi_k \rightarrow \psi_\infty$ implies that we have uniform convergence $u_{i,\delta}^{(k)} \rightarrow u_{i,\delta}^{(\infty)}$ of the mollifications as $k \rightarrow \infty$, and so once k is chosen sufficiently large we will have

$$|(u_{i,\delta}^{(k)} - f_i) - (u_{j,\delta}^{(k)} - f_j)| < \frac{\gamma}{2},$$

on the set $U_i \cap U_j$. This implies that if $z \in \partial U_i \cap V_j$, then

$$(u_{i,\delta}^{(k)} - f_i + \gamma\eta_i)(z) < (u_{j,\delta}^{(k)} - f_j + \gamma\eta_j)(z),$$

so all $z \in M$ have a neighborhood U such that in the definition of $\psi_{k,\delta}$ the maximum can be taken over those j for which $u_{j,\delta}^{(k)} - f_j + \gamma\eta_j$ is defined on U (i.e., for which $U \subset U_j$). In particular $\psi_{k,\delta}$ is continuous. We will now see that in addition for large k and small δ , the form $\chi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_{k,\delta}$ satisfies $\tilde{F}(\alpha^{i\bar{p}}\chi_{j\bar{p}}) \leq c - \delta'$ in the viscosity sense for small $\delta' > 0$, and so by Theorem 17 we can solve the equation $\Lambda_\omega\alpha = c$. Using Lemma 15 it is enough to show that for large k and small δ the metric $\chi_i = \omega_0 + \sqrt{-1}\partial\bar{\partial}(u_{i,\delta}^{(k)} + \gamma\eta_i)$ on U_i satisfies $\tilde{F}(\alpha^{p\bar{m}}\chi_{i,q\bar{m}}) \leq c - \delta'$. Note that $F(A) = S_1(A^{-1})$ here.

Let us work at a point $z \in U_i$. Define the following four matrix valued functions, defined in U_i :

$$\begin{aligned} A_q^p &= \alpha^{p\bar{m}}\partial_q\partial_{\bar{m}}u_i^{(k)} = \alpha^{p\bar{m}}\omega_{k,q\bar{m}}, \\ B_q^p &= \partial_q\partial_{\bar{p}}u_i^{(k)}, \\ C_q^p &= \partial_q\partial_{\bar{p}}u_{i,\delta}^{(k)}, \\ D_q^p &= \alpha^{p\bar{m}}\partial_q\partial_{\bar{m}}(u_{i,\delta}^{(k)} + \gamma\eta_i). \end{aligned}$$

Because of (21), the definition of the mollification $u_{i,\delta}^{(k)}$ and our bound on γ , the eigenvalues of all these matrices are at least $\kappa/2$, and we can choose eigenvectors so that the corresponding eigenvalues are all as close to each other as we like, if τ is sufficiently small. In particular the same holds for the reciprocals of the eigenvalues. In what follows, let us denote by $h(\tau)$ a function such that $h(\tau) \rightarrow 0$ as $\tau \rightarrow 0$, and which may change from line to line. By the structural assumption (5) for F_ϵ , we will then have

$$F_\epsilon(B) < F_\epsilon(A) + h(\tau).$$

The convexity of F_ϵ implies that

$$F_\epsilon(C) \leq \int_{\mathbf{C}^n} F_\epsilon(B(z - \delta w))\rho(w) dw \leq \int_{\mathbf{C}^n} F_\epsilon(A(z - \delta w))\rho(w) dw + h(\tau).$$

It follows that

$$F_\epsilon(C) - c_\epsilon \leq C_\tau \|F_\epsilon(A) - c_\epsilon\|_{L^2(\alpha)} + h(\tau),$$

where the constant C_τ blows up as $\tau \rightarrow 0$. Note that in our notation here A is the same as A_k in Equation (19). Finally interchanging C with D will only introduce a small error, again by structural assumption (5), so we have

$$F_\epsilon(D) - c_\epsilon \leq C_\tau \|F_\epsilon(A) - c_\epsilon\|_{L^2(\alpha)} + h(\tau).$$

Let us write out what this means at z in normal coordinates for α , such that $\sqrt{-1}\partial\bar{\partial}(u_{i,\delta}^{(k)} + \gamma\eta_i)$ is diagonal with eigenvalues $\lambda_1, \dots, \lambda_n$ (so that these are the eigenvalues of D). We have

$$\frac{1}{\lambda_1} + \dots + \frac{1}{\lambda_n} - \frac{\epsilon}{\lambda_1 \cdots \lambda_n} - c + \epsilon \leq C_\tau \|F_\epsilon(A) - c_\epsilon\|_{L^2} + h(\tau).$$

Since $\lambda_i > \kappa/2$, we can first choose ϵ sufficiently small so that $\epsilon(\lambda_1 \cdots \lambda_n)^{-1} < 1/\lambda_i$ for all i . We then choose τ so small that $h(\tau) < \epsilon/4$, and finally, according to (19) we can choose k sufficiently large, so that $C_\tau \|F_\epsilon(A_k) - c_\epsilon\|_{L^2} < \epsilon/4$. Combining these we have for each i , that

$$\sum_{j \neq i} \frac{1}{\lambda_j} < c - \frac{\epsilon}{2}.$$

But this means that $\chi_i = \omega_0 + \sqrt{-1}\partial\bar{\partial}(u_{i,\delta}^{(k)} + \gamma\eta_i)$ satisfies $\tilde{F}(\alpha^{p\bar{m}}\chi_{i,q\bar{m}}) < c - \epsilon/2$. This implies that $\chi = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_{k,\delta}$ satisfies $\tilde{F}(\alpha^{i\bar{p}}\chi_{j\bar{p}}) \leq c - \epsilon/2$ in the viscosity sense, and so by Theorem 17 there is a metric $\omega \in \Omega$ solving $\Lambda_\alpha\omega = c$. q.e.d.

REMARK. In the proof above, one could try to find a smooth metric χ satisfying $\tilde{F}(\alpha^{i\bar{p}}\chi_{j\bar{p}}) \leq c - \epsilon/2$, by taking a regularized maximum in Equation (22). It is not at all clear, however, that the regularized maximum will satisfy the required subsolution property. This is why we take the maximum instead and work with subsolutions in the viscosity sense. In the proof of Theorem 24 we will face a similar problem, and need to consider viscosity subsolutions.

The proof of Theorem 1 now follows from Proposition 22 below, which is essentially contained in the work of Song–Weinkove [21] (see also [24] and Li–Shi–Yao [20] for similar arguments). Let us denote by \mathcal{J}_α the functional \mathcal{J}_0 above, and by \mathcal{J}_β the same functional with $\beta \in [\alpha]$ replacing α .

Proposition 22. *If there is a metric $\chi \in \Omega$ satisfying $\Lambda_\chi\alpha = c$, then \mathcal{J}_α is proper. In addition if \mathcal{J}_α is proper, then \mathcal{J}_β is proper.*

Proof. Suppose that we can find an χ such that $\Lambda_\chi\alpha = c$. For small $\delta > 0$ the form $\alpha - \delta\chi$ is positive, and we have $\Lambda_\chi(\alpha - \delta\chi) = c - n\delta$. Song–Weinkove [21] showed that in this case the corresponding functional $\mathcal{J}_{\alpha-\delta\chi}$ is bounded below. This functional is given, up to

adding a constant, by

$$\mathcal{J}_{\alpha-\delta\chi}(\omega) = \mathcal{J}_\alpha(\omega) - n\delta \int_0^1 \int_M \varphi(\chi^n - \omega_t^n),$$

where $\omega_t = \chi + t\sqrt{-1}\partial\bar{\partial}\varphi$ and $\omega = \omega_1$. It follows that

$$\mathcal{J}_\alpha(\omega) \geq -C + n\delta \int_0^1 t \int_M \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge (\chi^{n-1} + \dots + \omega_t^{n-1}),$$

which implies that \mathcal{J}_α is proper.

If $\beta = \alpha + \sqrt{-1}\partial\bar{\partial}\psi$, and ω_t is as above then we have

$$\begin{aligned} \mathcal{J}_\beta(\omega_1) - \mathcal{J}_\alpha(\omega_1) &= \int_0^1 \int_M \varphi(\beta - \alpha) \wedge \omega_t^{n-1} dt \\ &= \int_0^1 \int_M \psi \sqrt{-1}\partial\bar{\partial}\varphi \wedge \omega_t^{n-1} dt \\ &= \frac{1}{n} \int_0^1 \int_M \psi \frac{d}{dt} \omega_t^n dt \\ &= \frac{1}{n} \int_M \psi(\omega_1^n - \omega_0^n). \end{aligned}$$

It follows that $|\mathcal{J}_\beta - \mathcal{J}_\alpha| < C$ for some constant depending on ψ , so if \mathcal{J}_α is proper, then so is \mathcal{J}_β . q.e.d.

The proof of Theorem 5 is very similar to the above, but simpler. For small $\kappa, \epsilon > 0$ we consider the operator

$$F_{\kappa,\epsilon}(A) = \sum_{k=1}^n c_k S_k(A^{-1}) + \kappa S_1(A^{-1}) - \epsilon S_n(A^{-1}).$$

We are assuming that $c_n > 0$, so that for sufficiently small ϵ this operator will satisfy the structural conditions on the space of all positive Hermitian matrices. In particular for any initial metric $\omega_0 \in \Omega$, the flow

$$\frac{\partial \varphi_t}{\partial t} = c_{\kappa,\epsilon} - F_{\kappa,\epsilon}(A_t)$$

has a solution for all time, with $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t$, $\varphi_0 = 0$, and $(A_t)_j^i = \alpha^{i\bar{k}} \omega_{t,j\bar{k}}$. The constant $c_{\kappa,\epsilon}$ is chosen so that

$$\int_M c_{\kappa,\epsilon} \omega^n = \int_M F_{\kappa,\epsilon}(A) \omega^n.$$

The flow is the negative gradient flow of the function $\mathcal{J}_{\kappa,\epsilon}$, defined by

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{J}_{\kappa,\epsilon}(\omega + t\sqrt{-1}\partial\bar{\partial}\varphi) = \int_M \varphi(F_{\kappa,\epsilon}(A) - c_{\kappa,\epsilon}) \omega^n,$$

normalized so that $\mathcal{J}_{\kappa,\epsilon}(\omega_0) = 0$.

The following result is not stated in this generality in Fang–Lai–Ma [14], but it follows using exactly the same argument (see also Song–Weinkove [21]), together with the perturbation method in the proof of Proposition 22.

Proposition 23. *Suppose that there is a metric ω such that $F_{\kappa,\epsilon}(\alpha^{i\bar{k}}\omega_{j\bar{k}}) = c_{\kappa,\epsilon}$. Then $\mathcal{J}_{\kappa,\epsilon}$ is proper.*

Proof of Theorem 5. We are assuming that there is a metric ω such that $F(A) = c$, where we are writing $F = F_{0,0}$ and $c = c_{0,0}$, and $A_j^i = \alpha^{i\bar{k}}\omega_{j\bar{k}}$. Using the implicit function theorem, we can also solve $F_{\kappa,\epsilon}(A) = c_{\kappa,\epsilon}$, for sufficiently small $\kappa, \epsilon > 0$, and this implies that $\mathcal{J}_{\kappa,\epsilon}$ is proper. Let us write $\mathcal{J}'_{\kappa,\epsilon}$ for the functional defined in the same way as $\mathcal{J}_{\kappa,\epsilon}$ but with α replaced by a metric $\beta \in [\alpha]$. Just as in Proposition 22 we obtain that $\mathcal{J}'_{\kappa,\epsilon}$ is also proper. We can then use the negative gradient flow to obtain a sequence of metrics ω_k , such that the matrices B_k defined by $(B_k)_j^i = \beta^{i\bar{p}}\omega_{k,j\bar{p}}$ satisfy

$$\lim_{k \rightarrow \infty} \int_M (F_{\kappa,\epsilon}(B_k) - c_{\kappa,\epsilon})^2 \omega_k^n = 0.$$

In addition we have $F_{\kappa,\epsilon}(B_k) < C$ along the flow, for a uniform constant C , and so using that $F_{\kappa,\epsilon}(B_k) > \kappa S_1(B_k^{-1})$, we obtain a uniform lower bound $\omega_k > C^{-1}\beta$, where C also depends on κ .

Performing the same mollification argument as in the proof of Proposition 21, for any $\delta > 0$ we can obtain a Kähler current $\chi \in [\omega]$ with continuous local potentials, satisfying

$$F_{\kappa,\epsilon}(\beta^{i\bar{p}}\chi_{j\bar{p}}) \leq c_{\kappa,\epsilon} + \delta,$$

in the viscosity sense. We have

$$\int_M c_{\kappa,\epsilon}\omega^n = \int_M F_{\kappa,\epsilon}\omega^n = \int_M c\omega^n + \int_M \left[\kappa S_1(A^{-1}) - \epsilon S_n(A^{-1}) \right] \omega^n,$$

and so $c_{\kappa,\epsilon} = c + \kappa d_1 - \epsilon d_2$ for some positive constants $d_1, d_2 > 0$, so

$$F_{\kappa,\epsilon}(\beta^{i\bar{p}}\chi_{j\bar{p}}) \leq c + \delta + \kappa d_1 - \epsilon d_2.$$

Choosing κ sufficiently small so that $\kappa d_1 - \epsilon d_2 < 0$, and then δ sufficiently small, we will have

$$F_{\kappa,\epsilon}(\beta^{i\bar{p}}\chi_{j\bar{p}}) \leq c - \delta',$$

for some small $\delta' > 0$. The definition of $F_{\kappa,\epsilon}$ then implies

$$\sum_{k=1}^{n-1} c_k S_k(B^{-1}) \leq c - \delta',$$

in the viscosity sense, where $B_j^i = \beta^{i\bar{p}}\omega_{j\bar{p}}$. Theorem 17 then implies that we can find a metric $\eta \in [\omega]$ satisfying $F(\beta^{i\bar{p}}\eta_{j\bar{p}}) = c$. q.e.d.

5. Toric manifolds

In the remainder of this article we will work on a toric manifold M . Our goal is to prove Theorem 3. By Theorems 1 and 5 it is sufficient to work with torus invariant metrics. We restate the theorem here.

Theorem 24. *Let M be a toric manifold, and α, χ torus invariant Kähler metrics on M . Suppose that $c > 0$ is such that*

$$(23) \quad \int_M c\chi^n - n\chi^{n-1} \wedge \alpha \geq 0,$$

and for all irreducible toric subvarieties $V \subset M$ of dimension $p = 1, 2, \dots, n-1$ we have

$$\int_V c\chi^p - p\chi^{p-1} \wedge \alpha > 0.$$

Then there exists a torus invariant metric $\omega \in [\chi]$ such that

$$(24) \quad \Lambda_\omega \alpha + b \frac{\alpha^n}{\omega^n} = c,$$

for some constant $b \geq 0$, depending on the choice of c . In particular either $\Lambda_\omega \alpha = c$, i.e., we have a solution of the J -equation, or $\Lambda_\omega \alpha < c$, depending on whether we have equality in (23).

Proof. The proof proceeds by induction on the dimension of M , the result being straightforward when $\dim M = 1$. Let us assume that we already know the result for dimensions less than n , and suppose that $\dim M = n$. We solve Equation (24), by the continuity method just as in the proof of Theorem 4.

Using Yau's theorem, we can find $\omega \in [\chi]$ such that $\frac{\alpha^n}{\omega^n}$ is constant, which corresponds to the limit $c \rightarrow \infty$. From the implicit function theorem it then follows that there is some c' , such that we can solve Equation (24) for all $\hat{c} \in (c', \infty)$ with b depending on \hat{c} . Define c_0 to be the infimum of all such $c' > 0$. Our goal is to show that if $c_0 \geq c$, and $c_k \rightarrow c_0$, then we have uniform estimates $C^{\ell, \alpha}$ estimates for the solutions ω_k of the equations

$$\Lambda_{\omega_k} \alpha + b_k \frac{\alpha^n}{\omega_k^n} = c_k.$$

Let us denote by $D = \cup_{i=1}^N D_i$ the union of all toric divisors on M .

We use the inductive hypothesis to build a suitable subsolution for the equation in a neighborhood of D . For each D_i , the inductive hypothesis implies that there is some $b_i > 0$, and a form $\omega_i \in [\chi]$, such that the restriction of ω_i to D_i is positive, and satisfies

$$\Lambda_{\omega|_{D_i}} \alpha_i|_{D_i} + b_i \frac{\alpha|_{D_i}^{n-1}}{\omega_i|_{D_i}^{n-1}} = c.$$

This follows from the inductive hypothesis by a simple extension argument. Though we will not need it, we remark that we can even assume that the form ω_i is Kähler on M [8]. We define

$$\chi_i = \omega_i + A\sqrt{-1}\partial\bar{\partial}(\gamma(d_i)|d_i|^2),$$

where d_i denotes the distance from D_i , A is a large constant, and $\gamma : \mathbf{R} \rightarrow \mathbf{R}$ is a cutoff function supported near 0. If A is chosen sufficiently large, then χ_i will be positive in a small neighborhood U_i of D_i and will satisfy $\Lambda_\alpha \chi_i < c - \kappa$ for some small $\kappa > 0$.

Fixing a reference form ω_0 , we can write $\chi_i = \omega_0 + \sqrt{-1}\partial\bar{\partial}\psi_i$ for each i . Consider now the functions

$$\tilde{\psi}_i = \psi_i - B_i + \delta \sum_{j < i} \gamma(d_j) \log d_j,$$

where $B_i, \delta > 0$ are constants, and let $\tilde{\chi}_i = \omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\psi}_i$. We can choose δ sufficiently small, so that on a neighborhood of $D_i \setminus \cup_{j < i} D_j$, the form $\tilde{\chi}_i$ is positive definite and satisfies $\Lambda_{\tilde{\chi}_i} \alpha < c - \kappa$. We choose B_i inductively, for $i = N, N-1, \dots, 1$, starting with $B_N = 0$, so that on D_j , for all $j > i$ we have $\psi_i - B_i < \psi_j - B_j$.

Suppose that $x \in M$ is in a neighborhood V of an intersection $D_{i_1} \cap \dots \cap D_{i_k}$, where $i_1 < \dots < i_k$, and suppose that

$$\tilde{\psi}_a(x) = \max_j \tilde{\psi}_j(x).$$

If V is sufficiently small, then we must have $a \leq i_1$, since $\tilde{\psi}_j = -\infty$ along D_i for $i < j$. Together with the choice of the B_i , this implies that we must have $a = i_1$, if the neighborhood V is sufficiently small. We can also assume that V is disjoint from D_j for $j < i_1$, so that we have $\Lambda_{\tilde{\chi}_{i_1}} \alpha < c - \kappa$ on V . Using Lemma 15 we have that on a sufficiently small neighborhood U of $D = \cup_i D_i$ the Kähler current $\chi = \omega_0 + \max \tilde{\psi}_i$ satisfies $\tilde{F}(\alpha^{i\bar{p}} \chi_{j\bar{p}}) \leq c - \kappa$ in the viscosity sense, where $F(A) = S_1(A^{-1})$. We can, therefore, apply Proposition 16, reducing C^2 -estimates on U to the boundary ∂U .

In Proposition 25 below, we will show that we have uniform $C^{l,\alpha}$ estimates for the ω_k outside the neighborhood U of D , and so we can apply Proposition 16 to the closure of U to obtain bounds of the form

$$\Lambda_\alpha \omega_k < C e^{N(\varphi_k - \inf \varphi_k)},$$

on U , where $\omega_k = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_k$. Since we already have estimates outside of U , the same inequality is true globally on M . Just as in the proof of Theorem 4, we can use Weinkove [28, Proposition 4.2] to obtain the estimate $\Lambda_\alpha \omega_k < C$, and as in the proof of Theorem 17 the Evans–Krylov theorem and Schauder estimates can be used to obtain higher order estimates. q.e.d.

5.1. Interior estimates. In this section we work on a toric manifold M with torus invariant Kähler metrics α, ω , satisfying the equation

$$(25) \quad S_1(A) + bS_n(A) = c,$$

where $A_j^i = \omega^{i\bar{p}}\alpha_{j\bar{p}}$, and $b \geq 0$ and $c > 0$ are constants. Our goal is to obtain estimates for ω in terms of α, c, b , away from the torus invariant divisors of M .

Proposition 25. *Suppose that α, ω are torus invariant metrics on M satisfying Equation (25). Then on any compact set $K \subset M$ disjoint from the torus invariant divisors, we have bounds*

$$\begin{aligned} \omega &> C^{-1}\alpha, \\ \|\omega\|_{C^\ell, \alpha} &< C_\ell, \end{aligned}$$

for C, C_ℓ depending on M, α , bounds on c, b , and the Kähler class $[\omega]$, and C_ℓ depends in addition on $\ell \geq 2$.

The proof of this proposition will occupy the rest of this section. To obtain these estimates, we write our equation in terms of convex functions on \mathbf{R}^n , corresponding to the dense complex torus $\mathbf{R}^n \times (S^1)^n$ in M . Suppose that $\alpha = \sqrt{-1}\partial\bar{\partial}f$ and $\omega = \sqrt{-1}\partial\bar{\partial}g$, where $f, g : \mathbf{R}^n \rightarrow \mathbf{R}$ are convex. We are assuming that f, g satisfy the equation

$$S_1(A) + bS_n(A) = c,$$

where $A_j^i = g^{i\bar{p}}f_{j\bar{p}}$. The function f is fixed, and we want to derive estimates for the function g on compact sets $K \subset \mathbf{R}^n$. By adding an affine linear function to g , we can assume that $g(0) = 0$ and $\nabla g(0) = 0$.

Lemma 26. *For any compact $K \subset \mathbf{R}^n$ there exists a $C > 0$ such that $\sup_K |g| < C$.*

Proof. The image of ∇g is a convex polytope P , determined by the Kähler class of ω up to translation by adding affine linear functions to g . Our normalization ensures that $0 \in P$, and in particular we get a gradient bound on g . The result follows immediately. q.e.d.

We next prove a C^2 -estimate for g on compact sets by a contradiction argument.

Proposition 27. *Suppose that $f, g : B \rightarrow \mathbf{R}$ are convex functions on the unit ball, satisfying*

$$f_{ij}g^{ij} + b\frac{\det(D^2 f)}{\det(D^2 g)} = c,$$

with $b \geq 0$ as above, and suppose $\inf_B g = g(0) = 0$. Then there is a $C > 0$ depending on $\sup_B |g|$, bounds on c, b , $C^{3, \alpha}$ bounds on f and a positive lower bound on the Hessian of f , such that

$$\sup_{\frac{1}{2}B} |g_{ij}| < C.$$

Proof. We can assume that $c = 1$ by scaling f . We argue by contradiction. Suppose that we have sequences f_k, g_k satisfying the hypotheses, including $|g_k| < N$, but $|\partial^2 g_k(x_k)| > k$ for some $x_k \in \frac{1}{2}B$. Note that the equation implies that

$$g_{k,ij} > f_{k,ij} > \tau \delta_{ij},$$

for some fixed $\tau > 0$, i.e., we have a uniform lower bound on the Hessians of the g_k .

Let $h_k : U_k \rightarrow \mathbf{R}$ be the Legendre transform of g_k . By shrinking the ball a bit, we can assume that $g_k \rightarrow g$ uniformly for some strictly convex $g : B \rightarrow \mathbf{R}$. Lemma 28 below implies that for sufficiently large k we have $U_k \supset \nabla g(0.9B)$, and so $\nabla g(0.8B)$ is of a definite distance from ∂U_k for large k . In addition, h_k satisfies the equation

$$(26) \quad \sum_{i,j} f_{k,ij}(\nabla h_k(y)) h_{k,ij}(y) + b_k \det(D^2 f_k(\nabla h_k)) \det(D^2 h_k(y)) = 1.$$

In addition from the normalization we get $h_k(0) = \nabla h_k(0) = 0$. We use Proposition 29 below together with the Schauder estimates, to obtain uniform $C^{l,\alpha}$ bounds on each h_k , on $\nabla g(0.8B)$, so we can take a limit $h_\infty : \nabla g(0.8B) \rightarrow \mathbf{R}$, satisfying an equation of the form

$$\sum_{ij} f_{\infty,ij}(\nabla h_\infty(y)) h_{\infty,ij}(y) + b_\infty \det(D^2 f_\infty(\nabla h_\infty(y))) \det(D^2 h_\infty(y)) = 1.$$

There are two cases:

- 1) We have a positive lower bound on the Hessian of h_∞ . This implies a lower bound on $\text{Hess } h_k$ for large k , on $\nabla g(0.8B)$, i.e., we get an upper bound on $\text{Hess } g_k$ at all points $x \in B$ for which $\nabla g_k(x) \in \nabla g(0.8B)$. But by Lemma 28, $\nabla g_k(0.7B) \subset \nabla g(0.8B)$ for large k , so we get an upper bound on $\text{Hess } g_k$ on $0.7B$, which contradicts our assumption.
- 2) The Hessian of h_∞ is degenerate somewhere. Then we can apply the constant rank theorem of Bian–Guan [1, Theorem 1.1]. Indeed, for a fixed value of ∇h_∞ , the equation is of the form

$$F(A) = \text{Tr}(BA) + c \det(A) - 1 = 0,$$

for a positive definite matrix B and positive constant c . The assumptions of Theorem 1.1 in [1] are satisfied, since the map $A \mapsto F(A^{-1})$ is convex in A , according to Lemma 8. It follows that if the Hessian of h_∞ is degenerate at a point, then it must be degenerate everywhere, and so

$$\int_{\nabla g(0.8B)} \det(h_{\infty,ij}) = 0.$$

This contradicts the fact that $\nabla g_k(0.7B) \subset \nabla g(0.8B)$ for large k , and

$$\int_{\nabla g_k(0.7B)} \det(h_{k,ij}) = \text{Vol}(0.7B),$$

but $h_k \rightarrow h_\infty$ in $C^{2,\alpha}$. q.e.d.

Lemma 28. *Suppose that $f_k : B \rightarrow \mathbf{R}$ are convex, with $f_{k,ij} > \tau\delta_{ij}$, such that they converge uniformly to $f : B \rightarrow \mathbf{R}$. If $B_1 \subset B_2 \subset B_3 \subset B$ are relatively compact balls, then for sufficiently large k the gradient maps satisfy*

$$\nabla f_k(B_1) \subset \nabla f(B_2) \subset \nabla f_k(B_3).$$

Proof. From Gutiérrez [17, Lemma 1.2.2] we have that

$$\limsup_{k \rightarrow \infty} \nabla f_k(K) \subset \nabla f(K),$$

for any compact $K \subset B$. Suppose that $B_1 \subset B' \subset B_2$. The strict convexity of f implies that $\nabla f(\partial B_2)$ is a positive distance from $\nabla f(\overline{B'})$. In particular for each $x \in \nabla f(\partial B_2)$, there is a k_x , such that $x \notin \nabla f_k(\overline{B'})$ for all $k > k_x$. The strict convexity then implies that there is a (fixed) radius $\delta > 0$, such that $B_\delta(x)$ is disjoint from $\nabla f_k(\overline{B_1})$ for all $k > k_x$. Since $\nabla f(\partial B_2)$ is compact by [17, Lemma 1.1.3], we can find some N such that $\nabla f_k(\overline{B_1})$ is disjoint from $\nabla f(\partial B_2)$ for all $k > N$. But this implies

$$\nabla f_k(B_1) \subset \nabla f(B_2),$$

for $k > N$.

For the other inclusion we use that for any compact $K \subset B$ and open set $U \supset K$ with $\overline{U} \subset B$, we have

$$\nabla f(K) \subset \liminf_{k \rightarrow \infty} \nabla f_k(U).$$

Now choose an intermediate ball B' with $B_2 \subset B' \subset B_3$. For any $x \in \nabla f(\overline{B_2})$, we have a k_x such that $x \in \nabla f_k(B')$ for all $k > k_x$. By the strict convexity we have some $\delta > 0$ such that $B_\delta(x) \in \nabla f_k(B_3)$. Using that $\nabla f(\overline{B_2})$ is compact, we can again cover by finitely many such balls, and we get an N such that $\nabla f(\overline{B_2}) \subset \nabla f_k(B_3)$ for all $k > N$. q.e.d.

The higher order estimates required by Proposition 25 follow from standard elliptic theory. We now show the $C^{2,\alpha}$ -estimates for equation (26). The difficulty is that the operator is neither concave, nor convex, and the result of Caffarelli–Yuan [5] also does not apply directly.

Proposition 29. *Suppose that $h : B \rightarrow \mathbf{R}$ is a smooth convex function on the unit ball in \mathbf{R}^n satisfying the equation*

$$(27) \quad \sum_{i,j} a_{ij}(\nabla h)h_{ij} + b(\nabla h) \det(D^2h) = 1,$$

where $a_{ij}, b \in C^{1,\alpha}$, $b \geq 0$ and $\lambda < a_{ij} < \Lambda$. Then we have $\|h\|_{C^{2,\alpha}(\frac{1}{2}B)} < C$ for a constant C depending on λ, Λ and $C^{1,\alpha}$ bounds for a_{ij}, b and C^1 bounds for h .

As a first step we prove a priori $C^{2,\alpha}$ estimates for the constant coefficient equation.

Proposition 30. *Suppose that $h : B \rightarrow \mathbf{R}$ is a smooth convex function on the unit ball in \mathbf{R}^n satisfying the equation*

$$(28) \quad \Delta h + b \det(D^2 h) = 1,$$

where $b \geq 0$ is a non-negative constant. Then we have $\|h\|_{C^{2,\alpha}(\frac{1}{2}B)} < C$ for a constant C depending on b and $|h|_{C^{1,\alpha}(B)}$.

Proof. We will assume $b > 0$, since $b = 0$ is standard. Let $f = \det(D^2 h)^{1/n}$, and denote the linearized operator by L , which acts on a smooth function g by

$$Lg = \Delta g + b \det(D^2 h) h^{ij} g_{ij},$$

where we use summation convention for repeated indices. We now compute Lf . We work at a point where $D^2 h$ is diagonal. We also write $S_n = \det(D^2 h)$ to simplify notation. First, differentiating the equation we have for each k

$$\begin{aligned} h_{iik} + b S_n h^{ij} h_{ijk} &= 0, \\ h_{iik} (1 + b S_n h^{ii}) + b S_n h^{pp} h_{ppk} h^{qq} h_{qqk} - b S_n h^{pp} h^{qq} h_{ppk}^2 &= 0. \end{aligned}$$

Also, differentiating f , we have

$$\begin{aligned} f_k &= \frac{1}{n} S_n^{1/n} h^{ij} h_{ijk}, \\ f_{kk} &= \frac{1}{n} S_n^{1/n} h^{ii} h_{iik} + \frac{1}{n^2} S_n^{1/n} h^{pp} h_{ppk} h^{qq} h_{qqk} - \frac{1}{n} S_n^{1/n} h^{pp} h^{qq} h_{ppk}^2. \end{aligned}$$

We now compute Lf :

$$\begin{aligned} Lf &= (1 + b S_n h^{kk}) f_{kk} \\ &= \frac{S_n^{1/n}}{n^2} (1 + b S_n h^{kk}) \left[n h^{ii} h_{iik} + h^{pp} h^{qq} (h_{ppk} h_{qqk} - n h_{ppk}^2) \right] \\ &= \frac{S_n^{1/n}}{n} \left[h^{ii} b S_n h^{pp} h^{qq} (h_{ppk}^2 - h_{ppk} h_{qqk}) \right] \\ &\quad + \frac{S_n^{1/n}}{n^2} (1 + b S_n h^{ii}) h^{pp} h^{qq} (h_{ppk} h_{qqk} - n h_{ppk}^2) \\ &= \frac{S_n^{1/n}}{n^2} h^{ii} h^{pp} h^{qq} h_{ppi} h_{qqi} (-n b S_n + b S_n) \\ &\quad + \frac{S_n^{1/n}}{n^2} \sum_{i,p,q} h^{pp} h^{qq} h_{ppi} h_{qqi} - \frac{S_n^{1/n}}{n} \sum_{i,p,q} h^{pp} h^{qq} h_{ppk}^2. \end{aligned}$$

We have

$$\begin{aligned} \sum_{p,q} h^{pp} h^{qq} h_{ppi} h_{qqi} &= \left(\sum_p h^{pp} h_{ppi} \right)^2 \\ &\leq n \sum_p (h^{pp} h_{ppi})^2 \leq n \sum_{p,q} h^{pp} h^{qq} h_{ppi}^2. \end{aligned}$$

It follows from this that $Lf \leq 0$.

At this point we can follow the argument in Caffarelli–Yuan [5] closely. The only difference in the argument is that in [5] the function $e^{K\Delta u}$ is a subsolution of the linearized equation for a sufficiently large constant K . This does not appear to be the case for our equation, but instead we can use that $\det(D^2u)^{1/n}$ is a supersolution according to our calculation above. In [5] this supersolution property is only used in dealing with “Case 2” in the proof of their Proposition 1. We will see that the same argument works in our situation as well.

As in [5], we fix $\rho, \xi, \delta, k_0 > 0$ to be determined and we set

$$s_k := \sup_{x \in B_{1/2^k}} \Delta u(x), \quad 1 \leq k \leq k_0.$$

From the equation we know that $s_k \leq 1$. Define

$$E_k := \left\{ x \in B_{1/2^k} \mid \Delta u(x) \leq s_k - \xi \right\},$$

and as in Case 2 in [5], assume that for all $1 \leq k \leq k_0$ we have $|E_k| > \delta |B_{1/2^k}|$. Let us define

$$w_k(x) = 2^{2k} u\left(\frac{x}{2^k}\right).$$

We apply the above computation to conclude that

$$L \left[(1 - \Delta w_k)^{1/n} - (1 - s_k)^{1/n} \right] \leq 0.$$

Since $(1 - \Delta w_k)^{1/n} - (1 - s_k)^{1/n} \geq 0$ on B_1 , we can apply the weak Harnack inequality to obtain

$$\begin{aligned} (1 - s_{k+1})^{1/n} &= \inf_{B_{1/2}} (1 - \Delta w_k)^{1/n} \\ &\geq (1 - s_k)^{1/n} + c \|(1 - \Delta w_k)^{1/n} - (1 - s_k)^{1/n}\|_{L^{p_0}(B_1)}, \end{aligned}$$

on $B_{1/2}$, for uniform constants $c, p_0 > 0$. The right hand side can be estimated using the assumption on the measure $|E_k|$, and we get

$$\inf_{B_{1/2}} (1 - \Delta w_k)^{1/n} \geq (1 - s_k)^{1/n} + c\delta^{1/p_0} \frac{\xi}{n},$$

since we have

$$(1 - s_k + \xi)^{1/n} - (1 - s_k)^{1/n} \geq \frac{\xi}{n}.$$

Since $1 - s_k \leq 1$, there can only be at most a bounded number k_0 of such steps. The remainder of the argument is identical to Caffarelli–Yuan [5].
q.e.d.

We can now prove the interior $C^{2,\alpha}$ estimates for the general equation in Proposition 29 by using a blow-up argument. We begin by proving a Liouville rigidity theorem for convex solutions of our equation;

Lemma 31. *Suppose that $u : \mathbf{R}^n \rightarrow \mathbf{R}$ is a smooth, convex function satisfying*

$$(29) \quad \Delta u + b \det(D^2 u) = 1,$$

then u is a quadratic polynomial.

Proof. The lemma follows from a simple rescaling argument, which is essentially the same as the proof of Gutiérrez [17, Theorem 4.3.1]. By subtracting a plane we may assume that $u(0) = 0$ and $\nabla u(0) = 0$. Since u is convex, the equation implies that $|D^2 u| \leq \sqrt{n}$. By integration we obtain the bound $|\nabla u|(x) \leq \sqrt{n}|x|$. We consider the function $v_R(x) := R^{-2}u(Rx)$. By the above, for $x \in B_1$ there holds

$$v_R(0) = 0, \quad |\nabla v_R|(x) \leq \sqrt{n}, \quad D^2 v_R(x) = D^2 u(Rx),$$

and so $v_R(x)$ is uniformly bounded in $C^2(B_1)$. Moreover, $v_R(x)$ solves equation (29) on B_1 , and so by the interior estimates in Proposition 30 we have a uniform bound for $|D^2 v_R|_{C^\alpha(B_{1/2})}$. Writing this in terms of u , we have

$$R^\alpha |D^2 u|_{C^\alpha(B_{2^{-1}R})} \leq C,$$

where C is independent of R . Taking the limit as $R \rightarrow \infty$ we see that we must have $D^2 u = D^2 u(0)$ a constant. Hence u is a quadratic polynomial.
q.e.d.

Proof of Proposition 29. To deal with the case of varying coefficients, we use a blowup argument to reduce to the Liouville result in the constant coefficient case.

Suppose then that h satisfies equation (27) on B . Let

$$N_h = \sup_{x \in B} d_x |D^3 h(x)|,$$

where $d_x = d(x, \partial B)$ is the distance to the boundary of B . Our goal is to bound N_h from above, so we can assume $N_h > 1$ say. Let us assume that the supremum is achieved at a point $x = x_0 \in B$. We define the function

$$\tilde{h}(z) = d_{x_0}^{-2} N_h^2 h(x_0 + d_{x_0} N_h^{-1} z) - A - A_i z_i,$$

where A, A_i are constants chosen so that

$$(30) \quad \tilde{h}(0) = 0, \quad \nabla \tilde{h}(0) = 0.$$

The function $\tilde{h}(z)$ is defined on the ball $B_{N_h}(0)$ around the origin. By direct computation we have

$$D^2\tilde{h}(z) = D^2h(x_0 + d_{x_0}N_h^{-1}z), \quad D^3\tilde{h}(z) = d_{x_0}N_h^{-1}D^3h(x_0 + d_{x_0}N_h^{-1}z),$$

and so $|D^3\tilde{h}(z)| \leq 2$ on $B_{2^{-1}N_h}(0)$. Moreover, since $|D^2\tilde{h}| = |D^2h| < C$, the normalization (30) implies that we have a bound

$$\|\tilde{h}\|_{C^3(B_{2^{-1}N_h})} < C,$$

for a uniform constant C . In addition \tilde{h} satisfies an equation of the form

$$(31) \quad \sum_{i,j} \tilde{a}_{ij}(\nabla\tilde{h})\tilde{h}_{ij} + \tilde{b}(\nabla\tilde{h}) \det(D^2\tilde{h}) = 1,$$

for coefficients $\tilde{a}_{ij}, \tilde{b}$ satisfying the same bounds as a_{ij}, b , but

$$(32) \quad \begin{aligned} \sup |\nabla\tilde{a}_{ij}| &\leq d_{x_0}N_h^{-1} \sup |\nabla a_{ij}|, \\ \sup |\nabla\tilde{b}| &\leq d_{x_0}N_h^{-1} \sup |\nabla b|. \end{aligned}$$

Differentiating equation (31), we obtain a linear elliptic equation for the derivatives \tilde{h}_p , with C^α coefficients (using also our C^3 bound for \tilde{h}). The standard Schauder theory then implies $C^{2,\alpha}$ bounds on $\nabla\tilde{h}$ on compact subsets of $B_{N_h/4}$.

For the sake of obtaining a contradiction we suppose that we have a sequence of convex functions h_k on B satisfying (27), such that the corresponding constants $N_{h_k} > 4k$. Then the rescaled functions \tilde{h}_k are defined on $B_{4k}(0)$ and have uniform $C^{3,\alpha}$ bounds on $B_k(0)$, and satisfy $|D^3\tilde{h}_k(0)| = 1$. By taking a diagonal subsequence, we can extract a convex limit $\tilde{h}_\infty : \mathbf{R}^n \rightarrow \mathbf{R}$ in $C^{3,\alpha/2}$, satisfying $|D^3\tilde{h}_\infty(0)| = 1$, and equation (31) with constant coefficients because of (32). Since \tilde{h}_∞ is convex, $C^{3,\alpha/2}$ on \mathbf{R}^n and satisfies the constant coefficient equation (29) (after a linear change of coordinates), we easily obtain that \tilde{h} is in fact smooth. In particular, we can apply the Liouville rigidity result in Lemma 31 to conclude that \tilde{h}_∞ is a quadratic polynomial. But this contradicts $|D^3\tilde{h}_\infty(0)| = 1$. q.e.d.

References

- [1] B. BIAN AND P. GUAN, *A microscopic convexity principle for nonlinear partial differential equations*, Invent. Math., 177 (2009), pp. 307–335, MR2511744, Zbl 1168.35010.
- [2] Z. BŁOCKI AND S. KOŁODZIEJ, *On regularization of plurisubharmonic functions on manifolds*, Proc. Amer. Math. Soc., 135 (2007), pp. 2089–2093, MR2299485, Zbl 1116.32024.
- [3] L. CAFFARELLI AND X. CABRÉ, *Fully nonlinear elliptic equations*, vol. 43 of American Mathematical Society Colloquium Publications, American Mathematical Society, Providence, RI, 1995, MR1351007, Zbl 0834.35002.

- [4] L. CAFFARELLI, L. NIRENBERG, AND J. SPRUCK, *The Dirichlet problem for nonlinear second-order elliptic equations III: Functions of the eigenvalues of the Hessian*, Acta Math., 155 (1985), pp. 261–301, MR0806416, Zbl 0654.35031.
- [5] L. CAFFARELLI AND Y. YUAN, *A priori estimates for solutions of fully nonlinear equations with convex level set*, Indiana Univ. Math. J., 49 (2000), pp. 681–695, MR1793687, Zbl 0965.35045.
- [6] X. X. CHEN, *On the lower bound of the Mabuchi energy and its application*, Int. Math. Res. Notices, 12 (2000), pp. 607–623, MR1772078, Zbl 0980.58007.
- [7] X. X. CHEN, *A new parabolic flow in Kähler manifolds*, Comm. Anal. Geom., 12 (2004), pp. 837–852, MR2104078, Zbl 1073.53089.
- [8] D. COMAN, V. GUEDJ, AND A. ZERIAHI, *Extension of plurisubharmonic functions with growth control*, J. Reine Angew. Math., 676 (2013), pp. 33–49, MR3028754, Zbl 1269.32018.
- [9] J.-P. DEMAILLY AND M. PAUN, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. of Math. (2), 159 (2004), pp. 1247–1274, MR2113021, Zbl 1064.32019.
- [10] S. K. DONALDSON, *Moment maps and diffeomorphisms*, Asian J. Math., 3 (1999), pp. 1–15, MR1701920, Zbl 0999.53053.
- [11] S. K. DONALDSON, *Scalar curvature and stability of toric varieties*, J. Differential Geom., 62 (2002), pp. 289–349, MR1988506, Zbl 1074.53059.
- [12] L. C. EVANS, *Classical solutions of fully nonlinear, convex, second order elliptic equations*, Comm. Pure Appl. Math., 25 (1982), pp. 333–363, MR0649348, Zbl 0469.35022.
- [13] H. FANG AND M. LAI, *Convergence of general inverse σ_k -flow on Kähler manifolds with Calabi ansatz*, Trans. Amer. Math. Soc., 365 (2013), pp. 6543–6567, MR3105762, Zbl 1288.53058.
- [14] H. FANG, M. LAI, AND X. MA, *On a class of fully nonlinear flows in Kähler geometry*, J. Reine Angew. Math., 653 (2011), pp. 189–220, MR2794631, Zbl 1222.53070.
- [15] B. GUAN AND W. SUN, *On a class of fully nonlinear elliptic equations on Hermitian manifolds*, Calc. Var. Partial Differential Equations, 54 (2015), pp. 901–916, MR3385185, Zbl 06481192.
- [16] V. GUEDJ AND A. ZERIAHI, *The weighted Monge–Ampère energy of quasi-plurisubharmonic functions*, J. Funct. Anal., 250 (2007), pp. 442–482, MR2352488, Zbl 1143.32022.
- [17] C. E. GUTIÉRREZ, *The Monge–Ampère equation*, vol. 44 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston Inc., 2001, MR1829162, Zbl 0989.35052.
- [18] N. V. KRYLOV, *Boundedly inhomogeneous elliptic and parabolic equations*, Izvestia Akad. Nauk. SSSR, 46 (1982), pp. 487–523, MR0661144.
- [19] M. LEJMI AND G. SZÉKELYHIDI, *The J-flow and stability*, Adv. Math., 274, (2015), pp. 404–431, MR3318155.
- [20] H. LI, Y. SHI, AND Y. YAO, *A criterion for the properness of the K-energy in a general Kähler class*, Math. Ann., 361 (2015), pp. 135–156, MR3302615, Zbl 06399404.
- [21] J. SONG AND B. WEINKOVE, *On the convergence and singularities of the J-flow with applications to the Mabuchi energy*, Comm. Pure Appl. Math., 61 (2008), pp. 210–229, MR2368374, Zbl 1135.53047.

- [22] J. SPRUCK, *Geometric aspects of the theory of fully nonlinear elliptic equations*, in Global theory of minimal surfaces, vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 283–309, MR2167264, Zbl 1151.53345.
- [23] W. SUN, *On a class of fully nonlinear elliptic equations on closed Hermitian manifolds*, arXiv:1310.0362.
- [24] G. SZÉKELYHIDI, *Greatest lower bounds on the Ricci curvature of Fano manifolds*, Compos. Math., 147 (2011), pp. 319–331, MR2771134, Zbl 1222.32046.
- [25] G. TIAN, *Kähler–Einstein metrics with positive scalar curvature*, Invent. Math., 137 (1997), pp. 1–37, MR1471884, Zbl 0892.53027.
- [26] V. TOSATTI, Y. WANG, B. WEINKOVE, AND X. YANG, *$C^{2,\alpha}$ estimates for nonlinear elliptic equations in complex and almost complex geometry*, Calc. Var. Partial Differential Equations, 54 (2015), pp. 431–453, MR3385166, Zbl 06481173.
- [27] N. TRUDINGER, *On the Dirichlet problem for Hessian equations*, Acta Math., 175 (1995), pp. 151–164, MR1368245, Zbl 0887.35061.
- [28] B. WEINKOVE, *Convergence of the J -flow on Kähler surfaces*, Comm. Anal. Geom., 12 (2004), pp. 949–965, MR2104082, Zbl 1073.53092.
- [29] B. WEINKOVE, *On the J -flow in higher dimensions and the lower boundedness of the Mabuchi energy*, J. Differential Geom., 73 (2006), pp. 351–358, MR2226957, Zbl 1107.53048.
- [30] Y. YAO, *The J -flow on toric manifolds*, arXiv:1407.1168.
- [31] S.-T. YAU, *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation I.*, Comm. Pure Appl. Math., 31 (1978), pp. 339–411, MR0480350, Zbl 0369.53059
- [32] S.-T. YAU, *Open problems in geometry*, Proc. Symposia Pure Math., 54 (1993), pp. 1–28, MR1216573, Zbl 0801.53001.
- [33] K. ZHENG, *I -properness of Mabuchi’s K -energy*, arXiv:1410.1821.

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