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COMPLETION OF THE MODULI SPACE FOR POLARIZED CALABI-YAU MANIFOLDS

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Abstract

In this paper, we construct a completion of the moduli space for polarized Calabi-Yau manifolds by using Ricci-flat Kähler-Einstein metrics and the Gromov-Hausdorff topology, which parameterizes certain Calabi-Yau varieties. We then study the algebro-geometric properties and the Weil-Petersson geometry of such completion. We show that the completion can be exhausted by sequences of quasi-projective varieties, and new points added have finite Weil-Petersson distance to the interior.

1. Introduction

A Calabi-Yau manifold X is a simply connected complex projective manifold with trivial canonical bundle $\varpi_X \cong \mathcal{O}_X$, and a polarized Calabi-Yau manifold (X, L) is a Calabi-Yau manifold X with an ample line bundle L. Let $\mathcal{M}^P$ be the moduli space of polarized Calabi-Yau manifolds (X, L) of dimension n with a fixed Hilbert polynomial $P = P(\mu) = \chi(X, L^\mu)$, i.e.

$$\mathcal{M}^P = \big\{(X, L)|P(\mu) = \chi(X, L^\mu)\big\}/\sim,$$

where $(X_1, L_1) \sim (X_2, L_2)$ if and only if there is an isomorphism $\psi : X_1 \to X_2$ such that $L_1 = \psi^* L_2$. We denote the equivalent class $[X, L] \in \mathcal{M}^P$ represented by (X, L).

The compactifications of moduli spaces were studied in various cases, for example, the Mumford's compactification of moduli spaces for curves (cf. [**27**]), the Satake compactification of moduli spaces for Abelian varieties (cf. [**38**]), and more recently the compact moduli spaces for general type stable varieties of higher dimension (cf. [**21**]). Because of the importance of Calabi-Yau manifolds in mathematics and physics (cf. [**49**]), it is also desirable to have compactifications of $\mathcal{M}^P$. The purpose of this paper is to construct a completion of $\mathcal{M}^P$ in a certain sense, which can be viewed as a partial compactification.

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There are several perspectives towards this moduli space $\mathcal{M}^P$. First of all, the Bogomolov-Tian-Todorov's unobstructedness theorem of Calabi-Yau manifolds implies that $\mathcal{M}^P$ is a complex orbifold (cf. [**41**, **43**]). The variation of Hodge structures gives a natural orbifold Kähler metric on $\mathcal{M}^P$, called the Weil-Petersson metric, which is the curvature of the first Hodge bundle with a natural Hermitian metric (cf. [**41**]). Other natural metrics were also studied in [**17**] and [**47**] etc. From the algebro-geometric point of view, Viehweg proved in [**45**] that $\mathcal{M}^P$ is a quasi-projective variety, and coarsely represents the moduli functor $\mathfrak{M}^P$ for polarized Calabi-Yau manifolds with Hilbert polynomial P. The third perspective is to understand $\mathcal{M}^P$ by considering Ricci-flat Kähler-Einstein metrics.

For a polarized Calabi-Yau manifold (X, L), Yau's theorem on the Calabi conjecture, so called Calabi-Yau theorem, asserts that there exists a unique Ricci-flat Kähler-Einstein metric ω with $\omega \in c_1(L)$, i.e. the Ricci curvature $\mathrm{Ric}(\omega) \equiv 0$ (cf. [**48**]). This theorem is obtained by showing the existence and the uniqueness of the solution of the Monge-Ampère equation

$$(1.1) \qquad (\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^n = (-1)^{\frac{n^2}{2}}\Omega \wedge \overline{\Omega}, \quad \sup_X \varphi = 0,$$

for any background Kähler metric $\omega_0 \in c_1(L)$, and letting $\omega = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi$, where Ω is a holomorphic volume form, i.e. a nowhere vanishing section of ϖ_X. We can regard $\mathcal{M}^P$ as a parameter space of Ricci-flat Calabi-Yau manifolds.

In [**10**], Gromov introduced the notion of Gromov-Hausdorff distance between metric spaces, which provides a frame to study families of Riemannian manifolds. For any two compact metric spaces A and B, the Gromov-Hausdorff distance of A and B is

$$d_{GH}(A, B) = \inf \left\{ d_H^Z(A, B) | \exists \text{ isometric embeddings } A, B \hookrightarrow Z \right\},$$

where Z is a metric space, $d_H^Z(A, B)$ is the standard Hausdorff distance between A and B regarded as subsets by the isometric embeddings, and the infimum is taken for all possible Z and isometric embeddings. We denote $\mathcal{M}et$ the space of the isometric equivalence classes of all compact metric spaces equipped with the topology, called the Gromov-Hausdorff topology, induced by the Gromov-Hausdorff distance d_{GH}. Then $\mathcal{M}et$ is a complete metric space (cf. [**10**, **34**]). The Gromov-Hausdorff topology was used to study moduli spaces for Einstein metrics by various authors (See for instance [**1**, **6**, **7**] etc.).

The Calabi-Yau theorem gives a continuous map

$$(1.2) \qquad \mathcal{CY} : \mathcal{M}^P \to \mathcal{M}et, \quad \text{by} \quad [X, L] \mapsto (X, \omega),$$

where ω is the unique Ricci-flat Kähler-Einstein metric representing $c_1(L)$. However, $\mathcal{CY}$ is not injective in general since $\mathcal{M}^P$ contains the information of complex structures.

For constructing compactifications of $\mathcal{M}^P$, Yau suggested that one uses the Weil-Petersson metric to obtain a metric completion of $\mathcal{M}^P$ first, and then tries to compactify this completion (cf. [**17**]). In [**47**], an alternative approach is proposed by using the Gromov-Hausdorff distance, instead of the Weil-Petersson metric, to construct a completion. Let $\overline{\mathcal{CY}(\mathcal{M}^P)}$ be the closure of the image $\mathcal{CY}(\mathcal{M}^P)$ in $\mathcal{M}et$. There is a natural metric space structure on $\overline{\mathcal{CY}(\mathcal{M}^P)}$ by restricting the Gromov-Hausdorff distance. A question is to understand $\overline{\mathcal{CY}(\mathcal{M}^P)}$ from the algebraic geometry and the Weil-Petersson geometry viewpoints.

A normal projective variety X is called 1-Gorenstein if the dualizing sheaf ϖ_X is an invertible sheaf, i.e. a line bundle, and is called Gorenstein if furthermore X is Cohen-Macaulay. A variety X has only canonical singularities if X is 1-Gorenstein, and for any resolution $\bar{\pi}: \bar{X} \to X$, $\bar{\pi}_*\varpi_{\bar{X}} = \varpi_X$, which is equivalent to that the canonical divisor $\mathcal{K}_X$ is Cartier, and

$$\mathcal{K}_{\bar{X}} = \bar{\pi}^*\mathcal{K}_X + \sum_E a_E E, \quad \text{and} \quad a_E \geqslant 0,$$

where E are exceptional prime divisors. If X has only canonical singularities, then the singularities are rational, and X is Cohen-Macaulay (cf. (C) of [**33**, Section 3]), which implies that X is Gorenstein. A Calabi-Yau variety X is a normal projective Gorenstein variety with trivial dualizing sheaf $\varpi_X \cong \mathcal{O}_X$, and having at most canonical singularities. A polarized Calabi-Yau variety (X, L) is a Calabi-Yau variety X with an ample line bundle L.

If $(Y, d_Y) \in \overline{\mathcal{CY}(\mathcal{M}^P)}$, then there is a sequence $\{[X_k, L_k]\} \subset \mathcal{M}^P$ such that $\mathcal{CY}([X_k, L_k]) = (X_k, \omega_k)$ converge to (Y, d_Y) in the Gromov-Hausdorff sense. Note that the diameters and the volumes satisfy that

$$\operatorname{diam}_{\omega_k}(X_k) \to \operatorname{diam}_{d_Y}(Y), \quad \operatorname{Vol}_{\omega_k}(X_k) = \frac{1}{n!} c_1(L_k)^n \equiv \text{cont.}$$

independent of k, which imply that Y is a non-collapsed limit. In [**8**], Donaldson and Sun studied the algebro-geometric structure of Y, and proved that Y is homeomorphic to a Calabi-Yau variety X_0 of dimension n. Hence loosely speaking, $\overline{\mathcal{CY}(\mathcal{M}^P)}$ can be regarded as a parameter space of certain Calabi-Yau varieties.

A degeneration of polarized Calabi-Yau manifolds $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ is a flat morphism from a variety $\mathcal{X}$ of dimension $n+1$ to a disc $\Delta \subset \mathbb{C}$ such that for any $t \in \Delta^* = \Delta \backslash \{0\}$, $X_t = \pi_\Delta^{-1}(t)$ is a Calabi-Yau manifold, the central fiber $X_0 = \pi_\Delta^{-1}(0)$ is singular, and $\mathcal{L}$ is a relative ample line bundle on $\mathcal{X}$. If we further assume that X_0 is a Calabi-Yau variety, then the total space $\mathcal{X}$ is normal as any fiber X_t is reduced and normal. Thus the relative dualizing sheaf $\varpi_{\mathcal{X}/\Delta}$ is defined, i.e. $\varpi_{\mathcal{X}/\Delta} \cong \varpi_{\mathcal{X}} \otimes \pi_{\Delta}^{*}\varpi_{\Delta}^{-1}$, and is trivial, i.e. $\varpi_{\mathcal{X}/\Delta} \cong \mathcal{O}_{\mathcal{X}}$, since every fiber is normal, Cohen-Macaulay and Gorenstein.

Let $(\pi_{\Delta} : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety X_0 as the central fiber, and ω_t be the unique Ricci-flat Kähler-Einstein metric on X_t representing $c_1(\mathcal{L}|_{X_t})$, $t \in \Delta^*$. The asymptotic behaviour of ω_t when $t \to 0$ is studied in [**36**, **35**], and it is shown that (X_t, ω_t) converges to a compact metric space of the same dimension in the Gromov-Hausdorff sense. This result, together with Donaldson-Sun's theorem, shows the equivalence between the algebro-geometric degenerating Calabi-Yau manifolds to a Calabi-Yau variety and the non-collapsing Gromov-Hausdorff convergence of Ricci-flat Kähler-Einstein metrics.

The first goal of the present paper is to investigate the algebro-geometric structure of $\overline{\mathcal{CY}(\mathcal{M}^P)}$.

Theorem 1.1. *There is a Hausdorff topological space $\overline{\mathcal{M}}^P$, and a surjection*

$$\overline{\mathcal{CY}} : \overline{\mathcal{M}}^P \to \overline{\mathcal{CY}(\mathcal{M}^P)}$$

satisfying the follows.

i) *$\mathcal{M}^P$ is an open dense subset of $\overline{\mathcal{M}}^P$, and $\overline{\mathcal{CY}}|_{\mathcal{M}^P} = \mathcal{CY}$.*
ii) *For any $p \in \overline{\mathcal{M}}^P$, $\overline{\mathcal{CY}}(p)$ is homeomorphic to a Calabi-Yau variety.*
iii) *There is an exhaustion*

$$\mathcal{M}^P \subset \mathcal{M}_{m(1)} \subset \mathcal{M}_{m(2)} \subset \cdots \subset \mathcal{M}_{m(l)} \subset \cdots \subset \overline{\mathcal{M}}^P = \bigcup_{l \in \mathbb{N}} \mathcal{M}_{m(l)},$$

where $m(l) \in \mathbb{N}$ for any $l \in \mathbb{N}$, such that $\mathcal{M}_{m(l)}$ is a quasi-projective variety, and there is an ample line bundle $\lambda_{m(l)}$ on $\mathcal{M}_{m(l)}$.

iv) *Let $(\pi_{\Delta} : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety X_0 as the central fiber. Assume that for any $t \in \Delta^*$, there is an ample line bundle L_t on X_t such that $L_t^k \cong \mathcal{L}|_{X_t}$ for a $k \in \mathbb{N}$, and $[X_t, L_t] \in \mathcal{M}^P$. Then there is a unique morphism $\rho : \Delta \to \mathcal{M}_{m(l)}$, for $l \gg 1$, such that $\overline{\mathcal{CY}}(\rho(t))$ is homeomorphic to X_t for any $t \in \Delta$, and*

$$\overline{\mathcal{CY}}(\rho(t)) \to \overline{\mathcal{CY}}(\rho(0)),$$

when $t \to 0$, in the Gromov-Hausdorff sense. Furthermore, $\rho^*\lambda_{m(l)} = \pi_{\Delta,*}\varpi_{\mathcal{X}/\Delta}^{\nu(l)}$ for a $\nu(l) \in \mathbb{N}$.

Remark 1.2. In general, we do not expect $\mathcal{M}_{m(l)} = \overline{\mathcal{M}}^P$ for some $m(l)$ because of the lack of the boundedness condition for singular Calabi-Yau varieties (cf. Section 3 in [**11**]).

When $n = 2$, a Calabi-Yau variety is a K3 orbifold, and a degeneration of K3 surfaces to a K3 orbifold is called a degeneration of type I. It is well-known that one can fill the holes in the moduli space of Kähler polarized K3 surfaces by some Kähler K3 orbifolds, and obtain a complete moduli space (cf. [**19**, **20**]). The relationship between such moduli space and the degeneration of Ricci-flat Kähler-Einstein metrics is also established in [**19**, **20**]. Theorem 1.1 is a generalization of [**19**, **20**] to higher dimensional polarized Calabi-Yau manifolds.

In [**25**], Kähler-Einstein metrics are used to construct compactifications of moduli spaces for Kähler-Einstein orbifolds, and it is proved that such compactification coincides with the standard Mumford's compactification in the case of curves. The moduli space of Fano manifolds admitting Kähler-Einstein metrics is constructed in a recent preprint [**30**], which generalizes the earlier work [**42**] for del Pezzo surfaces. The Gromov-Hausdorff compactification of such moduli space for del Pezzo surfaces of each degree is studied in [**31**], and it is proven to agree with certain algebro-geometric compactification. We can regard Theorem 1.1 as an analog result of [**31**] for the Calabi-Yau case.

Now we study the Weil-Petersson geometry of $\overline{\mathcal{M}}^P$. Note that for any flat family $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ of polarized Calabi-Yau manifolds with Hilbert polynomial P, there is a unique morphism $f : \Delta \to \mathcal{M}^P$, since $\mathcal{M}^P$ coarsely represents the moduli functor $\mathfrak{M}^P$. The Weil-Petersson metric ω_{WP} is an orbifold Kähler metric on $\mathcal{M}^P$ (cf. [**41**]) characterized by

$$f^*\omega_{WP} = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\int_{X_t}(-1)^{\frac{n^2}{2}}\Omega_t \wedge \overline{\Omega}_t,$$

where Ω_t is a relative holomorphic volume form, i.e. a nowhere vanishing section of $\varpi_{\mathcal{X}/\Delta}$. The metric ω_{WP} is the curvature of the first Hodge bundle with a natural Hermitian metric.

In [**4**], Candelas, Green and Hübsch found some nodal degenerations of Calabi-Yau 3-folds with finite Weil-Petersson distance. In general, [**46**] shows that if $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ is a degeneration of polarized Calabi-Yau manifolds, and if the central fiber X_0 is a Calabi-Yau variety, then the Weil-Petersson distance between $\{0\}$ and the interior Δ^* is finite, i.e. ω_{WP} is not complete on Δ^*. Conversely, if we assume that the Weil-Petersson distance of $\{0\}$ is finite, then after a finite base change $\pi_\Delta : \mathcal{X} \to \Delta$ is birational to a degeneration $\pi'_\Delta : \mathcal{X}' \to \Delta$ such that $\mathcal{X}\backslash X_0 \cong \mathcal{X}'\backslash X'_0$, and X'_0 is a Calabi-Yau variety by recent papers [**44**] and [**40**]. As a consequence, the algebro-geometric degenerating Calabi-Yau manifolds to a Calabi-Yau variety is equivalent to the finiteness of the Weil-Petersson distance.

Our next result shows that the points in $\overline{\mathcal{M}}^P\backslash\mathcal{M}^P$ have finite Weil-Petersson distance.

Theorem 1.3. *Let $\overline{\mathcal{M}}^P$ and $\overline{\mathcal{CY}}$ be the same as in Theorem* 1.1*.*

i) *For any point $x \in \overline{\mathcal{M}}^P \backslash \mathcal{M}^P$, there is a curve γ such that $\gamma(0) = x$, $\gamma((0,1]) \subset \mathcal{M}^P$ and the length of γ under the Weil-Petersson metric ω_{WP} is finite, i.e.*

$$\mathrm{length}_{\omega_{WP}}(\gamma) < \infty.$$

ii) *Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds such that for any $t \in \Delta^*$, $L_t^k \cong \mathcal{L}|_{X_t}$ for a $k \in \mathbb{N}$, and $[X_t, L_t] \in \mathcal{M}^P$, where L_t is an ample line bundle. If the Weil-Petersson distance between $0 \in \Delta$ and the interior Δ^* is finite, then there is a unique morphism $\varrho : \Delta \to \mathcal{M}_{m(l)}$, for $l \gg 1$, such that $\mathcal{CY}(\varrho(t))$ is homeomorphic to X_t, $t \in \Delta^*$.*

This paper is organized as follows. Section 2 studies Ricci-flat Kähler-Einstein metrics. In Section 2.1, we recall the generalized Calabi-Yau theorem in [**9**], and then in Section 2.2, we use [**8**] to improve the earlier work in [**35**, **36**], i.e. we show that along a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety as the central fiber, the Gromov-Hausdorff limit of Ricci-flat Kähler-Einstein metrics on general fibers is homeomorphic to the central fiber. All of properties about the Gromov-Hausdorff topology in Theorem 1.1 are from this section. The technique developed in this section can also be used to the unique filling-in problem for degenerations of Calabi-Yau manifolds, i.e. Corollary 2.3, which has independent interests. In Section 3, we study the algebraic geometry of the moduli space. Firstly, we recall the Viehweg's construction of quasi-projective moduli space for polarized Calabi-Yau manifolds (cf. [**45**]) in Section 3.1. Secondly, in Section 3.2, we construct an enlarged moduli space of $\mathcal{M}^P$ by using the construction of moduli spaces for varieties with at worst canonical singularities (cf. Section 8 of [**45**]). More precisely, for any $m > 0$, we construct a moduli functor $\mathfrak{M}_m$ for polarized Calabi-Yau varieties that can be embedded in $\mathbb{CP}^N$, $N = N(m)$. Then we use the results in Section 8 of [**45**] to prove that $\mathfrak{M}_m$ can be coarsely represented by a quasi-projective variety. Any $\mathcal{M}_{m(l)}$ in Theorem 1.1 comes from this construction. We prove Theorem 1.1 and Theorem 1.3 in Section 4, and finally, we give a remark for compactifications in Section 5.

In this paper, the notion scheme stands for separated schemes of finite type over $\mathbb{C}$, and the notion variety stands for either a reduced irreducible scheme or the set of its closed points with the natural analytic topology depending on the context. A point in a scheme means a closed point. For a flat family of schemes $\pi_T : \mathcal{X} \to T$ over T, we denote $X_t = \pi_T^{-1}(t)$ the fiber $\mathcal{X} \times_T \{t\}$ over a point $t \in T$. Since a Calabi-Yau manifold X is defined to be simply connected, the natural map $\mathcal{M}^P \to \mathcal{M}^{P_\mu}$ by $(X, L) \mapsto (X, L^\mu)$ for any $\mu \in \mathbb{N}$ is injective, where $P_\mu(k) = P(\mu k)$, and thus is an isomorphism. Thus, we identify $\mathcal{M}^P$ and

$\mathcal{M}^{P_\mu}$ in this paper.

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2. Ricci-flat Kähler-Einstein metrics

In this section, we study the Gromov-Hausdorff convergence of Ricci-flat Kähler-Einstein metrics along degenerations of polarized Calabi-Yau manifolds.

2.1. Singular Kähler-Einstein metric. There is a notion of Kähler metric for normal varieties (cf. [**9**, Section 5.2]). A smooth Kähler metric ω on a normal variety X is an usual Kähler metric on the regular locus X_{reg} such that for any singular point $p \in X$, there is a neighborhood U_p with an embedding $U_p \hookrightarrow \mathbb{C}^{N_p}$, and a smooth strictly pluri-subharmonic function v_p on $\mathbb{C}^{N_p}$ satisfying $\omega|_{U_p \bigcap X_{reg}} = \sqrt{-1}\partial\overline{\partial}v_p|_{U_p \bigcap X_{reg}}$. If these functions v_p are not smooth, we call ω a singular Kähler metric. A Kähler metric ω, possibly singular, defines a class $[\omega]$ in $H^1(X, \mathcal{PH}_X)$, where $\mathcal{PH}_X$ denotes the sheaf of pluri-harmonic functions on X.

If L is an ample line bundle on X, there is an $m > 0$ such that L^m is very ample, and $H^i(X, L^\mu) = \{0\}$ for any $i > 0$ and $\mu \geqslant m$. A basis $\Sigma = \{s_0, \cdots, s_N\}$ of $H^0(X, L^m)$ gives an embedding $\Phi_\Sigma : X \hookrightarrow \mathbb{CP}^N$ by $x \mapsto [s_0(x), \cdots, s_N(x)]$, which satisfies $L^m = \Phi_\Sigma^* \mathcal{O}_{\mathbb{CP}^N}(1)$, where $N = \dim_{\mathbb{C}} H^0(X, L^m) - 1$. The pullback $\omega_\Sigma = \Phi_\Sigma^*\omega_{FS}$ of the Fubini-Study metric is a smooth Kähler metric in the above sense such that $[\omega_\Sigma] = mc_1(L) \in NS_{\mathbb{R}}(X)$. The Hermitian metric h_{FS} of $\mathcal{O}_{\mathbb{CP}^N}(1)$, whose curvature is the Fubini-Study metric, restricts to an Hermitian metric $h_\Sigma = \Phi_\Sigma^* h_{FS}$ on L^m, which satisfies that $\omega_\Sigma = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log|\vartheta|^2_{h_\Sigma}$ on X_{reg} for any local section ϑ of L^m. We regard $\Phi_\Sigma(X)$ as a point in $\mathcal{H}il_N^P$, denoted still by $\Phi_\Sigma(X)$, where $\mathcal{H}il_N^P$ is the Hilbert scheme parametrizing subschemes of $\mathbb{CP}^N$ with the Hilbert polynomial $P = P(k) = \chi(X, L^{mk})$.

If $\Sigma' = \{s'_0, \cdots, s'_N\}$ is another basis of $H^0(X, L^m)$, we have a matrices $u = (u_{ij}) \in SL(N+1)$ such that $[s'_0, \cdots, s'_N] = \Big[\sum\limits_{i=0}^{N} s_i u_{i0}, \cdots,$

$\sum_{i=0}^{N} s_i u_{iN}\Big]$, denoted by $[\Sigma'] = [\Sigma] \cdot u$, and thus, $\Phi_{\Sigma'}(x) = \sigma(u, \Phi_\Sigma(x))$ for any $x \in X$, where $\sigma : SL(N+1) \times \mathbb{CP}^N \to \mathbb{CP}^N$ is the natural $SL(N+1)$-action on $\mathbb{CP}^N$. Note that σ induces an $SL(N+1)$-action on the Hilbert scheme $\mathcal{H}il_N^P$, denoted still by $\sigma : SL(N+1) \times \mathcal{H}il_N^P \to \mathcal{H}il_N^P$. We have $\Phi_{\Sigma'}(X) = \sigma(u, \Phi_\Sigma(X))$, and we denote the orbit

$$O(X, L^m) = \{\sigma(u, \Phi_\Sigma(X)) | u \in SL(N+1)\} \subset \mathcal{H}il_N^P. \tag{2.1}$$

In [**9**], a generalized Calabi-Yau theorem is obtained for polarized Calabi-Yau varieties, i.e. the existence and the uniqueness of singular Ricci-flat Kähler-Einstein metrics with bounded potentials. More precisely, for a polarized Calabi-Yau variety (X, L), Theorem 7.5 of [**9**] says that there is a unique bounded function φ satisfying the following Monge-Ampère equation

$$(\omega_\Sigma + \sqrt{-1}\partial\overline{\partial}\varphi)^n = (-1)^{\frac{n^2}{2}} \Omega \wedge \overline{\Omega}, \quad \sup_X \varphi = 0, \text{ and } \varphi \geqslant -C, \tag{2.2}$$

where Ω is a holomorphic volume form, i.e. a nowhere vanishing section of the dualizing sheaf ϖ_X. The restriction of the singular Kähler metric $\omega = \omega_\Sigma + \sqrt{-1}\partial\overline{\partial}\varphi$ on the regular locus X_{reg} is a smooth Ricci-flat Kähler-Einstein metric, and $\omega \in [\omega_\Sigma] = mc_1(L)$. Furthermore, ω is unique in $mc_1(L)$, and particularly is independent of the choice of Φ_Σ. By the boundedness of φ, we have that $h = \exp(-\varphi)h_\Sigma$ is an Hermitian metric on L^m whose curvature is ω.

We define an L^2-norm $\|\cdot\|_{L^2(h)}$ on $H^0(X, L^m)$ by

$$\|s\|^2_{L^2(h)} = \int_X |s|^2_h \omega^n = \int_X e^{-\varphi}|s|^2_{h_\Sigma}\omega^n. \tag{2.3}$$

If h' is another Hermitian metric with the same curvature ω, then $\partial\overline{\partial}\log\frac{h}{h'} \equiv 0$, i.e. $\log\frac{h}{h'}$ is a pluriharmonic function on a closed normal variety X, and thus $h = e^\varsigma h'$ for a constant ς. If $\Sigma_h = \{s_0, \cdots, s_N\}$ is an orthonormal basis of $H^0(X, L^m)$ with respect to $\|\cdot\|_{L^2(h)}$, then $\Sigma_{h'} = \{e^{-\frac{\varsigma}{2}}s_0, \cdots, e^{-\frac{\varsigma}{2}}s_N\}$ is orthonormal with respect to $\|\cdot\|_{L^2(h')}$, and furthermore Σ_h and $\Sigma_{h'}$ induce the same embedding $\Phi_{\Sigma_h} = \Phi_{\Sigma_{h'}}$.

If Σ_h and Σ'_h are two orthonormal bases of $H^0(X, L)$ with respect to $\|\cdot\|_{L^2(h)}$, there is an $u \in SU(N+1) \subset SL(N+1)$ such that $[\Sigma_h] = [\Sigma'_h] \cdot u$, $\Phi_{\Sigma'_h}(x) = \sigma(u, \Phi_{\Sigma_h}(x))$ for any $x \in X$, and thus $\Phi_{\Sigma'_h}(X) = \sigma(u, \Phi_{\Sigma_h}(X))$ in $\mathcal{H}il_N^P$. The action σ and h induce an $SU(N+1)$-orbit

$$RO(X, L^m) = \{\sigma(u, \Phi_{\Sigma_h}(X)) | u \in SU(N+1)\} \subset O(X, L^m). \tag{2.4}$$

Note that $RO(X, L^m)$ is compact, and depends only on the singular Kähler metric ω, but not on the choice of h, even the norm $\|\cdot\|_{L^2(h)}$ does.

2.2. Gromov-Hausdorff convergence of Ricci-flat Kähler-Einstein metrics. Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety X_0 as the central fiber. By taking a certain power of $\mathcal{L}$, we assume that $\mathcal{L}$ is relative very ample, and $R^i\pi_{\Delta,*}(\mathcal{L}) = \{0\}$ for $i > 0$. There is a morphism $\tilde{\Phi} : \mathcal{X} \hookrightarrow \mathbb{CP}^N \times \Delta \to \mathbb{CP}^N$ by composing an embedding and the projection such that $\mathcal{L} \cong \tilde{\Phi}^*\mathcal{O}_{\mathbb{CP}^N}(1)$. In this section, we always assume that any $\tilde{\Phi}(X_t)$ does not belong to a proper linear subspace of $\mathbb{CP}^N$ by shrinking Δ if necessary. We denote $\omega_{o,t} = \tilde{\Phi}^*\omega_{FS}|_{X_t}$, and denote ω_t the unique Ricci-flat Kähler-Einstein metric in $[\omega_{o,t}]$ for any $t \in \Delta$. Note that $\omega_t = \omega_{o,t} + \sqrt{-1}\partial\overline{\partial}\varphi_t$ for a unique bounded potential function φ_t with $\sup_{X_t}\varphi_t = 0$, which satisfies the Monge-Ampère equation (1.1) and (2.2) respectively.

The limiting behaviour of ω_t, when $t \to 0$, is studied intensively in [**37**], [**35**] and [**36**]. Theorem 1.4 in [**35**] asserts that the diameter has a uniform upper bound $D > 0$, i.e.

$$\operatorname{diam}_{\omega_t}(X_t) \leqslant D, \tag{2.5}$$

for any $t \in \Delta^*$. Furthermore, for any smooth family of embeddings $F_t : X_{0,reg} \to X_t$ with $F_0 = \mathrm{Id}$, we have

$$F_t^*\omega_t \to \omega_0, \quad \varphi_t \circ F_t \to \varphi_0, \quad \text{and} \quad \varphi_t > -C \tag{2.6}$$

for a constant $C > 0$, when $t \to 0$ in the C^∞_{loc}-sense, where φ_0 is the solution of (2.2), and ω_0 is the unique singular Ricci-flat Kähler-Einstein metric in $c_1(\mathcal{L}|_{X_0})$. In [**36**], it is proved that, when $t \to 0$, (X_t, ω_t) converges to a compact metric space X_∞ in the Gromov-Hausdorff topology, and X_∞ is the metric completion of $(X_{0,reg}, \omega_0)$. Actually, X_∞ is a Calabi-Yau variety by the following theorem due to Donaldson and Sun (cf. [**8**, **3**]).

Theorem 2.1 (Theorem 1.2 of [**8**]). *Let (X_k, L_k) be a sequence of polarized Calabi-Yau manifolds of dimension n with the same Hilbert polynomial P, and $\omega_k \in c_1(L_k)$ be the unique Ricci-flat Kähler-Einstein metric. We assume that*

$$\mathrm{Vol}_{\omega_k}(X_k) = \frac{1}{n!}c_1(L_k)^n \equiv v, \quad \operatorname{diam}_{\omega_k}(X_k) \leqslant D$$

for constants $D > 0$ and $v > 0$, and furthermore, (X_k, ω_k) converges to a compact metric space X_∞ in the Gromov-Hausdorff sense. Then we have the follows.

i) *X_∞ is homeomorphic to a Calabi-Yau variety, denoted still by X_∞.*
ii) *There are constants $m > 0$ and $\bar{N} > 0$ satisfying the following. For any k, there is an orthonormal basis Σ_k of $H^0(X_k, L_k^m)$ with*

respect to the L^2-norm induced by ω_k, which induces an embedding $\Phi_{\Sigma_k} : X_k \hookrightarrow \mathbb{CP}^{\bar{N}}$ with $L_k^m = \Phi_{\Sigma_k}^* \mathcal{O}_{\mathbb{CP}^{\bar{N}}}(1)$. And $\Phi_{\Sigma_k}(X_k)$ converges to X_∞ in the Hilbert scheme $\mathcal{H}il_{\bar{N}}^{P_m}$.

iii) *The metric space structure on X_∞ is induced by the unique singular Ricci-flat Kähler-Einstein metric $\omega \in \frac{1}{m} c_1(\mathcal{O}_{\mathbb{CP}^{\bar{N}}}(1)|_{X_\infty})$.*

By Proposition 4.15 of [**8**], X_∞ is a projective normal variety with only log-terminal singularities. Note that the holomorphic volume forms Ω_k are parallel with respect to ω_k, and converge to a holomorphic volume form Ω_∞ on the regular locus $X_{\infty,reg}$ along the Gromov-Hausdorff convergence by normalizing Ω_k if necessary. Thus the dualizing sheaf ϖ_{X_∞} is trivial, i.e. $\varpi_{X_\infty} \cong \mathcal{O}_{X_\infty}$, and X_∞ is 1–Gorenstein. Furthermore, the canonical divisor $\mathcal{K}_{X_\infty}$ is Cartier and trivial, which implies that X_∞ has at worst canonical singularities. Then X_∞ has only rational singularities, X_∞ is Cohen-Macaulay and is Gorenstein. Consequently, X_∞ is a Calabi-Yau variety.

A natural question is what's the relationship between these two Calabi-Yau varieties X_0 and X_∞ in our setting.

Lemma 2.2. *Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds with a Calabi-Yau variety X_0 as the central fiber. If ω_t, $t \in \Delta^*$, is the unique Ricci-flat Kähler metric on X_t with $\omega_t \in c_1(\mathcal{L}|_{X_t})$, then (X_t, ω_t) converges to a compact metric space X_∞ homeomorphic to X_0 in the Gromov-Hausdorff sense. As a consequence, the singular Ricci-flat Kähler metric $\omega_0 \in c_1(\mathcal{L}|_{X_0})$ induces a compact metric space structure on X_0.*

Proof. We denote $L_t = \mathcal{L}|_{X_t}$, and denote h_t the Hermitian metric on L_t, whose curvature is the Ricci-flat Kähler-Einstein metric ω_t. We apply Theorem 2.1 to a sequence $t_k \to 0$, and then X_∞ is a Calabi-Yau variety. Furthermore, there are constants $m > 0$ and $\bar{N} > 0$ such that, for any k, there is an orthonormal basis Σ_{t_k} of $H^0(X_{t_k}, L_{t_k}^m)$ with respect to the L^2-norm $\|\cdot\|_{L^2(h_{t_k}^m)}$ inducing an embedding $\Phi_{\Sigma_k} : X_k \hookrightarrow \mathbb{CP}^{\bar{N}}$ with $L_{t_k}^m = \Phi_{\Sigma_k}^* \mathcal{O}_{\mathbb{CP}^{\bar{N}}}(1)$. And $\Phi_{\Sigma_{t_k}}(X_{t_k})$ converges to X_∞ in the Hilbert scheme $\mathcal{H}il_{\bar{N}}^{P_m}$ under the natural analytic topology.

Note that $\mathcal{L}^m$ is relative very ample, and thus there is a morphism $\Psi : \mathcal{X} \hookrightarrow \mathbb{CP}^{\bar{N}} \times \Delta \to \mathbb{CP}^{\bar{N}}$ by composing an embedding and the projection such that $\mathcal{L}^m = \Psi^* \mathcal{O}_{\mathbb{CP}^{\bar{N}}}(1)$. Note that X_t, $t \in \Delta$, has the same Hilbert polynomial P, and hence $P(m) = h^0(X_t, L_t^m) = \bar{N} + 1$ for $m \gg 1$. We have that $\Psi(X_t)$, for any $t \in \Delta$, is not included in any proper linear subspace of $\mathbb{CP}^{\bar{N}}$. If we denote $\bar{h}_{FS}$ the Hermitian metric on $\mathcal{O}_{\mathbb{CP}^{\bar{N}}}(1)$ whose curvature is the Fubini-Study metric ω_{FS}, then the restriction $\bar{h}_{o,t}$ of $\Psi^* \bar{h}_{FS}$ on X_t has the curvature $\bar{\omega}_{o,t} = \Psi^* \omega_{FS}|_{X_t}$. We can choose an Hermitian metric h_t on L_t^m by $h_t^m = e^{-m\varphi_t} \bar{h}_{o,t}$, where φ_t

is the potential function for the unique Ricci-flat Kähler-Einstein metric in $c_1(L_t)$, i.e. $\omega_t = \frac{1}{m}\bar{\omega}_{o,t} + \sqrt{-1}\partial\overline{\partial}\varphi_t$ and $\sup\limits_{X_t}\varphi_t = 0$.

Let $Z_0, \cdots, Z_{\bar{N}}$ be sections of $\mathcal{O}_{\mathbb{CP}^{\bar{N}}}(1)$ such that the restrictions of $z_0 = \Psi^* Z_0, \cdots, z_{\bar{N}} = \Psi^* Z_{\bar{N}}$ on X_t form a basis of $H^0(X_t, L_t^m)$, and $z_0|_{X_0}, \cdots, z_{\bar{N}}|_{X_0}$ are orthonormal with respect to $\|\cdot\|_{L^2(h_0^m)}$. Now for any compact subset $U \subset X_{0,reg}$, by (2.6), we have

$$\int_{F_t(U)} \langle z_i, z_j\rangle_{h_t^m}\omega_t^n \to \int_U \langle z_i, z_j\rangle_{h_0^m}\omega_0^n, \quad \text{when } t \to 0, \quad \text{and}$$

$$\begin{aligned}\int_{X_t\backslash F_t(U)} |\langle z_i, z_j\rangle_{h_t^m}|\omega_t^n &\leqslant e^{mC}\sup_{X_t}|z_i|_{\bar{h}_{o,t}}|z_j|_{\bar{h}_{o,t}}\mathrm{Vol}_{\omega_t}(X_t\backslash F_t(U)) \\ &\leqslant C_1\mathrm{Vol}_{\omega_t}(X_t\backslash F_t(U)),\end{aligned}$$

for a constant $C_1 > 0$ independent of t. Since $\mathrm{Vol}_{\omega_t}(X_t) = \mathrm{Vol}_{\omega_0}(X_0) = \mathrm{Vol}_{\omega_0}(X_{0,reg})$, we obtain

$$\int_{X_t} \langle z_i, z_j\rangle_{h_t^m}\omega_t^n \to \int_{X_0} \langle z_i, z_j\rangle_{h_0^m}\omega_0^n = \delta_{ij}$$

when $t \to 0$, by taking U larger and larger, and a diagonal sequence. Thus there is a family of matrices $v_t = (v_{t,ij}) \in GL(\bar{N}+1)$ such that $\Sigma_t' = \left\{\sum\limits_{i=0}^{\bar{N}} z_i v_{t,i0}, \cdots, \sum\limits_{i=0}^{\bar{N}} z_i v_{t,i\bar{N}}\right\}$ is an orthonormal basis of $H^0(X_t, L_t^m)$ with respect to $\|\cdot\|_{L^2(h_t^m)}$, and $v_t \to$Id in $GL(\bar{N}+1)$ when $t \to 0$.

There are $u_k = (u_{k,ij}) \in U(\bar{N}+1)$ such that $\Sigma_{t_k} = \Sigma_{t_k}' \cdot u_k$, and, by passing to a subsequence, $u_k \to u_0$ in $U(\bar{N}+1)$. Thus $w_k = (\det(v_{t_k} \cdot u_k))^{-\frac{1}{\bar{N}+1}} v_{t_k} \cdot u_k \to w_0 = (\det(u_0))^{-\frac{1}{\bar{N}+1}} u_0$ in $SL(\bar{N}+1)$ when $t_k \to 0$. Under the $SL(\bar{N}+1)$-action σ, we have $\Phi_{\Sigma_{t_k}}(X_{t_k}) = \sigma(w_k, \Psi(X_{t_k}))$, and $X_\infty = \sigma(w_0, \Psi(X_0))$. We proved the conclusion that X_∞ is isomorphic to X_0. By [**36**], (X_t, ω_t) converges to the compact metric space X_∞ homeomorphic to X_0 in the Gromov-Hausdorff topology. q.e.d.

One corollary of this lemma is the uniqueness of the filling-in for degenerations of Calabi-Yau manifolds.

Corollary 2.3. *Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ and $(\pi'_\Delta : \mathcal{X}' \to \Delta, \mathcal{L}')$ be two degenerations of polarized Calabi-Yau manifolds with Calabi-Yau varieties X_0 and X_0' as the central fibers respectively. If there is a sequence of points $t_k \to 0$ in Δ, and there is a sequence of isomorphism $\psi_k : X_{t_k} \to X'_{t_k}$ such that $\psi_k^*\mathcal{L}|_{X_{t_k}} \cong \mathcal{L}'|_{X'_{t_k}}$, then X_0 is isomorphic to X_0'.*

Proof. If ω_{t_k} and ω'_{t_k} are Ricci-flat Kähler-Einstein metrics representing $c_1(\mathcal{L}|_{X_{t_k}})$ and $c_1(\mathcal{L}'|_{X'_{t_k}})$ respectively, then (X_{t_k}, ω_{t_k}) is isometric to $(X'_{t_k}, \omega'_{t_k})$ as compact metric spaces. By Lemma 2.2, (X_{t_k}, ω_{t_k}) (also

$(X'_{t_k}, \omega'_{t_k}))$ converges to a compact metric space X_∞ homeomorphic to both X_0 and X'_0. We further claim that X_0 is isomorphic to X'_0.

By the proof of Lemma 2.2 and taking some powers of $\mathcal{L}$ and $\mathcal{L}'$, we can assume that there are morphisms $\Psi : \mathcal{X} \to \mathbb{CP}^N$ and $\Psi' : \mathcal{X}' \to \mathbb{CP}^N$ with $\mathcal{L} = \Psi^* \mathcal{O}_{\mathbb{CP}^N}(1)$ and $\mathcal{L}' = \Psi'^* \mathcal{O}_{\mathbb{CP}^N}(1)$ respectively such that $\Psi|_{X_t}$ and $\Psi'|_{X'_t}$ are embeddings for any $t \in \Delta$. Furthermore, there are embeddings $\Phi_{\Sigma_k} : X_{t_k} \to \mathbb{CP}^N$ induced by an orthonormal basis Σ_k of $H^0(X_{t_k}, \mathcal{L}|_{X_{t_k}})$ for each k such that $\Phi_{\Sigma_k}(X_{t_k})$ converges to a Calabi-Yau variety homeomorphic to X_∞ in the Hilbert scheme $\mathcal{H}il^P_N$, denoted still by X_∞. The same arguments as in the proof of Lemma 2.2 show that there are w and $w' \in SL(N+1)$ such that $X_\infty = \sigma(w, \Psi(X_0))$ and $X_\infty = \sigma(w', \Psi'(X'_0))$ where $\sigma : SL(N+1) \times \mathcal{H}il^P_N \to \mathcal{H}il^P_N$ is a natural $SL(N+1)$-action on $\mathcal{H}il^P_N$. Hence X_0 is isomorphic to X'_0. q.e.d.

Remark 2.4. Note that $\mathcal{X}$ may not be birational to $\mathcal{X}'$ in this corollary. If we have a stronger assumption that $\mathcal{X} \backslash X_0$ is isomorphic to $\mathcal{X}' \backslash X'_0$, then the conclusion is a direct consequence of [**2**, Theorem 2.1] and [**30**, Corollary 4.3].

3. Quasi-projective moduli space

In this section, we construct an enlarged moduli space parameterizing certain polarized Calabi-Yau varieties.

3.1. Moduli space for Calabi-Yau manifolds.

In [**45**], Viehweg constructed the coarse moduli space of polarized Calabi-Yau manifolds with a fixed Hilbert polynomial P by using Geometric Invariant Theory (GIT), and it was shown to be a quasi-projective variety. Let's recall the relevant notions and the basic steps of the construction.

The moduli functor $\mathfrak{M}^P$ for polarized Calabi-Yau manifolds with Hilbert polynomial P is a functor from the category of schemes to the category of sets such that $\mathfrak{M}^P(\mathrm{Spec}(\mathbb{C})) = \mathcal{M}^P$ as a set, and for any scheme T, $\mathfrak{M}^P(T) = \{(\pi_T : \mathcal{X} \to T, \mathcal{L})\}/\sim$. Here $\pi_T : \mathcal{X} \to T$ is a flat family of schemes, and $\mathcal{L}$ is a relative ample line bundle on $\mathcal{X}$ such that for any point $t \in T$, $(X_t = \pi_T^{-1}(t), \mathcal{L}|_{X_t}) \in \mathcal{M}^P$. And we say $(\pi_T : \mathcal{X} \to T, \mathcal{L}) \sim (\pi'_T : \mathcal{X}' \to T, \mathcal{L}')$ if there is a T-isomorphism $\tau : \mathcal{X} \to \mathcal{X}'$ and an invertible sheaf $\mathcal{B}$ on T such that $\tau^* \mathcal{L}' \cong \mathcal{L} \otimes \pi_T^* \mathcal{B}$. Theorem 1.13 and Corollary 7.22 of [**45**] assert that there is a quasi-projective scheme $\tilde{\mathcal{M}}^P$ coarsely representing the functor $\mathfrak{M}^P$, i.e. the following hold. There is a natural transformation $\Theta : \mathfrak{M}^P \to \hom(\cdot, \tilde{\mathcal{M}}^P)$ such that $\Theta(\mathrm{Spec}(\mathbb{C})) : \mathfrak{M}^P(\mathrm{Spec}(\mathbb{C})) \to \hom(\mathrm{Spec}(\mathbb{C}), \tilde{\mathcal{M}}^P)$ is bijective, and, for any scheme W and a natural transformation $\Xi : \mathfrak{M}^P \to \hom(\cdot, W)$, there is a unique natural transformation $\Pi : \hom(\cdot, \tilde{\mathcal{M}}^P) \to \hom(\cdot, W)$ such that $\Xi = \Pi \circ \Theta$. This property implies that for any $(\mathcal{X} \to T, \mathcal{L}) \in$

$\mathfrak{M}^P(T)$, there is a unique morphism $T \to \tilde{\mathcal{M}}^P$. We identify $\mathcal{M}^P$ with the set of closed points of $\tilde{\mathcal{M}}^P$ by $\Theta(\mathrm{Spec}(\mathbb{C}))$.

Now we recall the construction of $\tilde{\mathcal{M}}^P$ in [**45**]. Firstly, the functor $\mathfrak{M}^P$ is bounded. More precisely, by Matsusaka's Big Theorem (cf. [**26**]), for any polarized Calabi-Yau manifold (X, L), there is an $m_0 > 0$ depending only on P such that for any $m \geqslant m_0$, L^m is very ample, and $H^i(X, L^m) = \{0\}$, $i > 0$. By choosing a basis Σ of $H^0(X, L^{m_0})$, we have an embedding $\Phi_\Sigma : X \hookrightarrow \mathbb{CP}^N$ such that $L^{m_0} = \Phi_\Sigma^* \mathcal{O}_{\mathbb{CP}^N}(1)$. We regard $\Phi_\Sigma(X)$ as a point in the Hilbert scheme $\mathcal{H}il_N^{P_{m_0}}$ parametrizing the subshemes of $\mathbb{CP}^N$ with Hilbert polynomial P_{m_0}, where $N = h^0(X, L^{m_0}) - 1$. For any other choice Σ', $\Phi_{\Sigma'}(X) = \sigma(u, \Phi_\Sigma(X))$ for a $u \in SL(N+1)$ where $\sigma : SL(N+1) \times \mathcal{H}il_N^{P_{m_0}} \to \mathcal{H}il_N^{P_{m_0}}$ is the $SL(N+1)$-action on $\mathcal{H}il_N^{P_{m_0}}$ induced by the natural $SL(N+1)$-action on $\mathbb{CP}^N$.

Secondly, $\mathfrak{M}^P$ is open (see [**45**, Lemma 1.18]), i.e. for any flat family of polarized varieties $(\pi_Y : \mathcal{X} \to Y, \mathcal{L})$, there is an open subscheme $Y' \subset Y$ such that a morphism $T \to Y$ factors through $T \to Y' \to Y$ if and only if $(\mathcal{X} \times_Y T \to T, \mathfrak{p}^*\mathcal{L}) \in \mathfrak{M}^P(T)$ where $\mathfrak{p} : \mathcal{X} \times_Y T \to \mathcal{X}$ denotes the projection. This is equivalent to that there is an open subscheme $\mathcal{H}_N^o$ of $\mathcal{H}il_N^{P_{m_0}}$ (cf. [**45**, Notions 7.2]) such that a point $p \in \mathcal{H}_N^o$ if and only if $(X_p = \pi_{\mathcal{H}}^{-1}(p), L_p) \in \mathcal{M}^P$ and $L_p^{m_0} \cong \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}$ where $\pi_{\mathcal{H}} : \mathcal{U}_N \to \mathcal{H}il_N^{P_{m_0}}$ is the universal family over the Hilbert scheme $\mathcal{H}il_N^{P_{m_0}}$. The $SL(N+1)$-action σ on $\mathcal{H}il_N^{P_{m_0}}$ induces an $SL(N+1)$-action on $\mathcal{H}_N^o$, denoted still by $\sigma : SL(N+1) \times \mathcal{H}_N^o \to \mathcal{H}_N^o$. The moduli scheme $\tilde{\mathcal{M}}^P$ is constructed by showing that a certain quotient of the $SL(N+1)$-action on $\mathcal{H}_N^o$ exists.

Thirdly, $\mathfrak{M}^P$ is separated (cf. [**45**, Lemma 1.18]), i.e. for any two $(\mathcal{X}_i \to S, \mathcal{L}_i) \in \mathfrak{M}^P(S)$, $i = 1, 2$, any isomorphism of $(\mathcal{X}_1, \mathcal{L}_1)$ onto $(\mathcal{X}_2, \mathcal{L}_2)$ over $S\backslash\{0\}$ extends to a S-isomorphism from $(\mathcal{X}_1, \mathcal{L}_1)$ to $(\mathcal{X}_2, \mathcal{L}_2)$, where $(S, 0)$ is a germ of smooth curve. Thus the $SL(N+1)$-action on $\mathcal{H}_N^o$ is proper and the stabilizers are finite by [**45**, Lemma 7.3]. The separatedness condition implies that if the moduli space exists, i.e. the quotient of the $SL(N+1)$-action σ exists, then it is Hausdorff under the analytic topology.

Finally, $\mathfrak{M}^P$ satisfies further properties, so called the weak positivity and the weak stability, i.e. Assumption 7.19 of [**45**] holds by the proof of Theorem 1.13 in [**45**]. Then the geometric quotient $\mathcal{Q}^o : \mathcal{H}_N^o \to \tilde{\mathcal{M}}^P$ of the $SL(N+1)$-action σ exists (cf. Corollary 3.33 and the proof of Theorem 7.20 in [**45**]). Here the geometric quotient means that $\mathcal{Q}^o$ is a morphism from $\mathcal{H}_N^o$ to a scheme $\tilde{\mathcal{M}}^P$ satisfying the following (cf. [**45**, Definition 3.6]). For any $p \in \mathcal{H}_N^o$ and any $u \in SL(N+1)$, $\mathcal{Q}^o(\sigma(u, p)) = \mathcal{Q}^o(p)$, $\mathcal{O}_{\tilde{\mathcal{M}}^P} = (\mathcal{Q}^o_* \mathcal{O}_{\mathcal{H}_N^o})^{SL(N+1)}$, and, for any two disjoint $SL(N+1)$-invariant closed subscheme W_1 and W_2, we have that $\mathcal{Q}^o(W_1) \bigcap \mathcal{Q}^o(W_2) = \emptyset$, and both of $\mathcal{Q}^o(W_1)$ and $\mathcal{Q}^o(W_2)$ are closed. Furthermore, for any point $p \in \tilde{\mathcal{M}}^P$, the fiber $\mathcal{Q}^{o,-1}(p)$ consists of exactly one $SL(N+1)$-orbit. The existence of $\mathcal{Q}^o$ implies that the $SL(N+1)$-action σ is closed, and the dimension of the stabilizers is constant on connected components. Moreover, $\tilde{\mathcal{M}}^P$ is a quasi-projective variety, and there is an ample sheaf λ on $\tilde{\mathcal{M}}^P$ such that $\mathcal{Q}^{o,*}\lambda = \pi_{\mathcal{H},*}\varpi^{\nu}_{\mathcal{U}_N/\mathcal{H}^o_N}$ for a $\nu \geqslant 1$ by [**45**, Corollary 7.22].

Viehweg also showed in Section 8 of [**45**] that the above construction works for more general moduli functor of certain polarized varieties with semi-ample dualizing sheaf and at worst canonical singularities as long as the boundedness condition, the openness (or the local closedness) condition and the separatedness condition hold. In Section 3.2, we will use the construction in [**45**, Section 8] to obtain an enlarged moduli space of $\mathcal{M}^P$, which also parameterizes certain Calabi-Yau varieties.

We remark that there is an analogue construction of the symplectic reduction (cf. [**28**, Section 8]) to obtain $\mathcal{M}^P$ by using Ricci-flat Kähler-Einstein metrics. If we define a real slice

$$\mathcal{R}^o_N = \bigcup_{p\in\mathcal{H}^o_N} RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \subset \mathcal{H}^o_{N,red}, \tag{3.1}$$

where $RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p})$ is defined by (2.4), then there is a natural $SU(N+1)$-action on $\mathcal{R}^o_N$, and the set theory quotient is $\mathcal{M}^P$. The real slice $\mathcal{R}^o_N$ is an analog of the zero level set of a momentum map in the symplectic reduction.

3.2. Enlarged moduli space. Now we construct the enlarged moduli space. For any polarized Calabi-Yau manifold (X, L) of dimension n with Hilbert polynomial $P = P(\mu) = \chi(X, L^\mu)$, we assume that L is very ample, and $H^i(X, L^\mu) = \{0\}$ for any $i > 0$ and $\mu \geqslant 1$ without loss of generality.

For any $m \geqslant 1$, there is an $N = N(m) > 0$ such that a basis Σ of $H^0(X, L^m)$ induces an embedding $\Phi_\Sigma : X \hookrightarrow \mathbb{CP}^N$. Let $\pi_{\mathcal{H}} : \mathcal{U}_N \to \mathcal{H}ilb^{P_m}_N$ be the universal family over the Hilbert scheme $\mathcal{H}ilb^{P_m}_N$ of the Hilbert polynomial $P_m(\mu) = P(m\mu)$, and $\mathcal{H}^o_N \subset \mathcal{H}ilb^{P_m}_N$ be the open subscheme whose set of closed points parameterizes smooth varieties. The moduli space $\mathcal{M}^P$ is constructed in [**45**] as the geometric quotient $\mathcal{Q}^o : \mathcal{H}^o_N \to \mathcal{M}^P$ under the natural $SL(N+1)$-action σ on $\mathcal{H}^o_N$ as explained above.

Lemma 3.1. *There is an open subscheme $\mathcal{H}_N$ of $\mathcal{H}ilb^{P_m}_N$ such that $\mathcal{H}^o_N \subset \mathcal{H}_N \subset \overline{\mathcal{H}^o_N}$ where $\overline{\mathcal{H}^o_N}$ denotes the Zariski closure of $\mathcal{H}^o_N$ in $\mathcal{H}ilb^{P_m}_N$, and a point $p \in \mathcal{H}_N$ if and only if $X_p = \pi^{-1}_{\mathcal{H}}(p)$ is a Calabi-Yau variety.*

Proof. This result is undoubtedly well-known to experts, and the proof was explained to the author by Chenyang Xu. We use the closed subscheme $\overline{\mathcal{H}_N^o}$ to replace $\mathcal{H}ilb_N^{P_m}$, and try to prove that $\mathcal{H}_N$ is an open subscheme of $\overline{\mathcal{H}_N^o}$.

Note that Calabi-Yau varieties are normal projective varieties, and the normality is an open condition for flat families. Any subset E of $\overline{\mathcal{H}_N^o}$ containing the open subset $\mathcal{H}_N^o$ is constructible (cf. [**16**, Proposition 10.14]), since for any irreducible closed subset Y, if Y is proper, $E\bigcap Y$ is nowhere dense, and otherwise $Y = \overline{\mathcal{H}_N^o}$, which contains $\mathcal{H}_N^o$. The main theorem in [**18**] shows that if $\pi_S : \mathcal{X} \to S$ is a flat morphism from a germ of a variety to a germ of smooth curve $(S, 0)$ whose special fiber $X_0 = \pi_S^{-1}(0)$ has only canonical singularities, then both $\mathcal{X}$ and fibers X_t have only canonical singularities. Thus for any point $p \in \overline{\mathcal{H}_N^o}$, if $X_p = \pi_{\mathcal{H}}^{-1}(p)$ has at worst canonical singularities, then by taking curves passing p and normalizations of curves, there is a neighborhood of p over which fibers of $\pi_{\mathcal{H}}$ have at worst canonical singularities, i.e. having only canonical singularities is an open condition. We denote $\mathcal{W}$ the open subscheme of $\overline{\mathcal{H}_N^o}$ whose set of closed points parameterizes normal varieties with at worst canonical singularities. By [**45**, Lemma 1.19], there is a locally closed subscheme $\mathcal{H}_N$ of $\mathcal{W}$ such that a morphism $T \to \mathcal{W}$ factors through $T \to \mathcal{H}_N$, if and only if $(\mathcal{U}_N \times_{\mathcal{W}} T \to T, \mathfrak{p}^*\varpi_{\mathcal{U}_N/\mathcal{W}}) \sim (\mathcal{U}_N \times_{\mathcal{W}} T \to T, \mathfrak{p}^*\mathcal{O}_{\mathcal{U}_N})$, where $\mathfrak{p} : \mathcal{U}_N \times_{\mathcal{W}} T \to \mathcal{U}_N$ is the projection. Hence a point $p \in \mathcal{H}_N$ if and only if $\varpi_{X_p} \cong \mathcal{O}_{X_p}$, which implies that X_p is a Calabi-Yau variety. Furthermore, $\mathcal{H}_N^o \subset \mathcal{H}_N$, and $\mathcal{H}_N$ is open.

q.e.d.

We define a moduli subfunctor $\mathfrak{M}_m$ of the moduli functor of polarized Gorenstein varieties, i.e. 3) of Examples 1.4 in [**45**], such that

$$\mathfrak{M}_m(\mathrm{Spec}(\mathbb{C})) = \{(X_p = \pi_{\mathcal{H}}^{-1}(p), \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p})|\ p \text{ is a point of } \mathcal{H}_N\}/\sim,$$

where $(X_{p_1}, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_{p_1}}) \sim (X_{p_2}, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_{p_2}})$ if and only if there is an isomorphism $\psi : X_{p_1} \to X_{p_2}$ such that $\mathcal{O}_{\mathbb{CP}^N}(1)|_{X_{p_1}} \cong \psi^*\mathcal{O}_{\mathbb{CP}^N}(1)|_{X_{p_2}}$, which is equivalent to $p_1 = \sigma(u, p_2)$ for an $u \in SL(N+1)$. The functor $\mathfrak{M}_m$ is bounded by the definition, and is open by Lemma 3.1. By [**30**, Corollary 4.3] or the unpublished work [**2**, Theorem 2.1], if $(\mathcal{X}_1 \to S, \mathcal{L}_1)$ and $(\mathcal{X}_2 \to S, \mathcal{L}_2)$ are two flat families of polarized Calabi-Yau varieties over a germ of smooth curve $(S, 0)$, then any isomorphism of these two families over $S\backslash\{0\}$ extends to an isomorphism over S, i.e. the moduli functor $\mathfrak{M}_m$ is also separated.

Now we use the construction in [**45**, Section 8] to prove that $\mathfrak{M}_m$ can be coarsely represented by a quasi-projective variety.

Lemma 3.2. *The coarse moduli space of $\mathfrak{M}_m$ is a quasi-projective variety $\tilde{\mathcal{M}}_m$, which is constructed as a geometric quotient $\mathcal{Q} : \mathcal{H}_N \to \tilde{\mathcal{M}}_m$. There is a positive integer $\nu = \nu(m)$, and an ample line bundle λ_m on $\tilde{\mathcal{M}}_m$ such that $\mathcal{Q}^*\lambda_m = \pi_{\mathcal{H},*}\varpi^{\nu}_{\mathcal{U}_N/\mathcal{H}_N}$. Furthermore, $\tilde{\mathcal{M}}^P$ is an open subscheme of $\tilde{\mathcal{M}}_m$.*

Proof. Note that $\mathfrak{M}_m$ satisfies [**45**, Assumptions 8.22], i.e. $\mathfrak{M}_m$ is bounded, open, separated, and moreover is a moduli functor of varieties with semi-ample dualizing sheaf. Then [**45**, Theorem 8.23] shows that $\mathfrak{M}_m$ can be coarsely represented by a quasi-projective scheme $\tilde{\mathcal{M}}_m$. More precisely, the base changing, local freeness condition, the weak positivity and the weak stability are verified in [**45**, Section 8.6], and then [**45**, Theorem 7.20] shows the existence of the geometric quotient $\mathcal{Q} : \mathcal{H}_N \to \tilde{\mathcal{M}}_m$. Furthermore, there is a positive integer $\nu = \nu(m)$, and an ample line bundle λ_m on $\tilde{\mathcal{M}}_m$ such that $\mathcal{Q}^*\lambda_m = \pi_{\mathcal{H},*}\varpi^{\nu}_{\mathcal{U}_N/\mathcal{H}_N}$ by [**45**, Theorem 8.23, Theorem 7.20 and Corollary 7.22]. Finally, since $\mathcal{H}^o_N \subset \mathcal{H}_N$ is $SL(N+1)$-invariant Zariski open, and $\mathcal{Q}|_{\mathcal{H}^o_N} = \mathcal{Q}^o$, we obtain that $\tilde{\mathcal{M}}^P$ is open in $\tilde{\mathcal{M}}_m$. q.e.d.

Remark 3.3. For a $p \in \mathcal{H}_N$, if X_p is smooth, then $\mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p} = L^m$ where L is an ample line bundle on X_p, and however, if X_p is singular, there may not exist such ample line bundle. Thus some Calabi-Yau variety here could not be embedded to the lower dimensional projective space.

We also have an enlarged real slice of $\mathcal{R}^o_N$. We define

$$\mathcal{R}_N = \bigcup_{p\in\mathcal{H}_N} RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \subset \mathcal{H}_{N,red}, \tag{3.2}$$

where $\mathcal{H}_{N,red}$ is the reduced variety of $\mathcal{H}_N$ with the natural analytic topology, and $RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p})$ is the $SU(N+1)$-orbit induced by the Ricci-flat Kähler-Einstein metric $\omega \in c_1(\mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p})$. By (2.4), $RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \subset O(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \bigcap \mathcal{R}_N$, and by the uniqueness of the Kähler-Einstien metric ω, we obtain

$$RO(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) = O(X_p, \mathcal{O}_{\mathbb{CP}^N}(1)|_{X_p}) \bigcap \mathcal{R}_N,$$

for any $p \in \mathcal{H}_N$. The set theory quotient space

$$\mathcal{M}_m = \mathcal{R}_N/SU(N+1) = \mathcal{H}_{N,red}/SL(N+1) \tag{3.3}$$

with the quotient topology induced by the analytic topology of $\mathcal{H}_{N,red}$, is homeomorphic to the underlying variety of $\tilde{\mathcal{M}}_m$. Note that the reduced Hilbert scheme $\mathcal{H}ilb^{P_m}_{N,red}$ is Hausdorff, and so is the subset $\mathcal{R}_N$. Thus the quotient by a compact Lie group $\mathcal{M}_m = \mathcal{R}_N/SU(N+1)$ is also Hausdorff, which has already been implied by the separatedness of $\mathfrak{M}_m$. For a point $p \in \mathcal{H}_N$, we denote $[X_p] \in \mathcal{M}_m$ the image of p under the quotient map, i.e. $[X_p] = \mathcal{Q}(p)$.

4. Proof of Theorem 1.1 and Theorem 1.3

In this section, we prove Theorem 1.1 and Theorem 1.3. By Matsusaka's Big Theorem, for any polarized Calabi-Yau manifold $(X, L) \in$

$\mathcal{M}^P$, we assume that for any $\mu \geqslant 1$, L^μ is very ample, and $H^i(X, L^\mu) = \{0\}$, $i > 0$ without loss generality.

For any $D > 0$, we define a subset $\mathcal{M}^P(D)$ of $\mathcal{M}^P$ by

$$\mathcal{M}^P(D) =$$

$$\{[X, L] \in \mathcal{M}^P | \text{ Ricci} - \text{flat metric } \omega \in c_1(L) \text{ with } \mathrm{diam}_\omega(X) \leqslant D\}.$$

We have that if $D_1 \leqslant D_2$, then $\mathcal{M}^P(D_1) \subset \mathcal{M}^P(D_2)$, and $\mathcal{M}^P = \bigcup_{D>0} \mathcal{M}^P(D)$. Let's consider an exhaustion

$$\mathcal{M}^P(1) \subset \cdots \subset \mathcal{M}^P(j) \subset \cdots\cdots \subset \mathcal{M}^P = \bigcup_{j\in\mathbb{N}} \mathcal{M}^P(j).$$

Note that for a sequence $[X_k, L_k] \in \mathcal{M}^P(j)$, if (X_k, ω_k) converges to a compact metric space X_∞ in the Gromov-Hausdorff sense, then by Theorem 2.1, there are embeddings $\Phi_k : X_k \hookrightarrow \mathbb{CP}^{N_j}$ for an $N_j > 0$ independent of k such that $L_k^{m_j} \cong \Phi_k^* \mathcal{O}_{\mathbb{CP}^{N_j}}(1)$ for an $m_j > 0$, and $\Phi_k(X_k)$ converges to a Calabi-Yau variety in the Hilbert scheme $\mathcal{H}ilb_{N_j}^{P_{m_j}}$, which is homeomorphic to X_∞, denoted still by X_∞.

Lemma 4.1. *If we denote* $m(l) = \prod_{j=1}^{l} m_j$, *and the sequence* $[X_k, L_k] \in \mathcal{M}^P(l_0)$, *for an* $l_0 \leqslant l$, *i.e.* $\mathrm{diam}_{\omega_k}(X_k) \leqslant l_0$, *then* $[X_\infty] \in \mathcal{M}_{m(l)}$, *where* $\mathcal{M}_{m(l)}$ *is the underlying quasi-projective variety of* $\tilde{\mathcal{M}}_{m(l)}$ *constructed in Lemma* 3.2.

Proof. Note that X_∞ is a Calabi-Yau variety, $\mathcal{O}_{\mathbb{CP}^{N_{l_0}}}(1)|_{X_\infty}$ is very ample, and $(X_\infty, \mathcal{O}_{\mathbb{CP}^{N_{l_0}}}(1)|_{X_\infty})$ represents a point in the Hilbert scheme $\mathcal{H}ilb_{N_{l_0}}^{P_{m_{l_0}}}$. If we denote $m_{l,l_0} = m(l)/m_{l_0}$, then $\mathcal{O}_{\mathbb{CP}^{N_{l_0}}}(m_{l,l_0})|_{X_\infty}$ is very ample and without any higher cohomology (cf. [**45**, Corollary 2.36]). The Hilbert polynomial of $(X_\infty, \mathcal{O}_{\mathbb{CP}^{N_{l_0}}}(m_{l,l_0})|_{X_\infty})$ is $P_{m(l)}(k) = P_{m_{l_0}}(m_{l,l_0} \cdot k)$. Thus there is an embedding $\Psi_\infty : X_\infty \to \mathbb{CP}^{N_{m(l)}}$ such that $\mathcal{O}_{\mathbb{CP}^{N_{l_0}}}(m_{l,l_0})|_{X_\infty} = \Psi_\infty^* \mathcal{O}_{\mathbb{CP}^{N_{m(l)}}}(1)$, where $N_{m(l)} = P_{m(l)}(1) - 1$. We have $\Psi_\infty(X_\infty) \in \mathcal{H}_{N_{m(l)}} \subset \mathcal{H}ilb_{N_{m(l)}}^{P_{m(l)}}$. Note that $[\Psi_\infty(X_\infty)] = \mathcal{Q}_l(\Psi_\infty(X_\infty))$, where $\mathcal{Q}_l : \mathcal{H}_{N_{m(l)}} \to \tilde{\mathcal{M}}_{m(l)}$ is the quotient map in Lemma 3.2. We obtain the conclusion by identifying $\Psi_\infty(X_\infty)$ with X_∞. q.e.d.

Lemma 4.2. *There is a continuous inclusion $i_l : \mathcal{M}_{m(l)} \hookrightarrow \mathcal{M}_{m(l+1)}$.*

Proof. For any l, let $\pi_{\mathcal{H}} : \mathcal{U}_{N_{m(l)}} \to \mathcal{H}ilb^{P_{m(l)}}_{N_{m(l)}}$ be the universal family. The line bundle $\mathcal{L}_{m(l)} = \Upsilon^* \mathcal{O}_{\mathbb{CP}^{N_{m(l)}}}(1)$ is a relative very ample where Υ is the composition of the embedding and the projection $\mathcal{U}_{N_{m(l)}} \hookrightarrow \mathbb{CP}^{N_{m(l)}} \times \mathcal{H}ilb^{P_{m(l)}}_{N_{m(l)}} \to \mathbb{CP}^{N_{m(l)}}$. For a point $p \in \mathcal{H}_{N_{m(l)}}$, if $X_p = \pi^{-1}_{\mathcal{H}}(p)$, then $\mathcal{L}^{m_{l+1}}_{m(l)}|_{X_p}$ has no higher cohomology, and $(X_p, \mathcal{L}^{m_{l+1}}_{m(l)}|_{X_p})$ has the Hilbert polynomial $P_{m(l+1)}(k) = P_{m(l)}(m_{l+1} \cdot k)$. Note that $N_{m(l+1)} = P_{m(l+1)}(1) - 1$. Thus we have embeddings $\Theta : \pi^{-1}_{\mathcal{H}}(\mathcal{H}_{N_{m(l)}}) \hookrightarrow \mathbb{P}(\pi_{\mathcal{H}*}(\mathcal{L}^{m_{l+1}}_{m(l)})) \cong \mathbb{CP}^{N_{m(l+1)}} \times \mathcal{H}_{N_{m(l)}}$ with $\mathcal{L}^{m_{l+1}}_{m(l)} = \Theta^* \mathcal{L}_{m(l+1)}$. For two choices of embeddings Θ and Θ', there is a section $u \in \mathcal{O}_{\mathcal{H}_{N_{m(l)}}}(SL(N_{m(l+1)}+1) \times \mathcal{H}_{N_{m(l)}})$ such that $\Theta'(X_p) = \sigma(u(p), \Theta(X_p))$ under the $SL(N_{m(l+1)}+1)$-action σ on $\mathcal{H}ilb^{P_{m(l+1)}}_{N_{m(l+1)}}$. By the universal property of Hilbert scheme, we obtain a morphism $\mathfrak{I}_l : \mathcal{H}_{N_{m(l)}} \to \mathcal{H}ilb^{P_{m(l+1)}}_{N_{m(l+1)}}$ such that $\pi^{-1}_{\mathcal{H}}(\mathcal{H}_{N_{m(l)}}) = \mathcal{U}_{N_{m(l+1)}} \times_{\mathcal{H}ilb^{P_{m(l+1)}}_{N_{m(l+1)}}} \mathcal{H}_{N_{m(l)}}$, and furthermore, $\mathfrak{I}_l(\mathcal{H}_{N_{m(l)}}) \subset \mathcal{H}_{N_{m(l+1)}}$. We also denote $\mathfrak{I}_l$ the corresponding morphism $\pi^{-1}_{\mathcal{H}}(\mathcal{H}_{N_{m(l)}}) \to \mathcal{U}_{N_{m(l+1)}}$ without any confusion. If p and $p' \in \mathcal{H}_{N_{m(l)}}$ satisfy $p = \sigma(w, p')$ for a $w \in SL(N_{m(l)}+1)$, then $(X_p, \mathcal{L}^{m_{l+1}}_{m(l)}|_{X_p}) \sim (X_{p'}, \mathcal{L}^{m_{l+1}}_{m(l)}|_{X_{p'}})$. Hence $\mathfrak{I}_l$ is equivariant under the $SL(N_{m(l)}+1)$ and $SL(N_{m(l+1)}+1)$ actions. We obtain a map $i_l : \mathcal{M}_{m(l)} \to \mathcal{M}_{m(l+1)}$ by taking the quotients, which is continuous under the analytic topology.

If $i_l([X_1]) = i_l([X_2])$, for two $[X_1]$ and $[X_2] \in \mathcal{M}_{m(l)}$, then there is an isomorphism $\tilde{\psi} : \mathfrak{I}_l(X_1) \to \mathfrak{I}_l(X_2)$ with $\tilde{\psi}^* \mathcal{L}_{m(l+1)}|_{\mathfrak{I}_l(X_2)} \cong \mathcal{L}_{m(l+1)}|_{\mathfrak{I}_l(X_1)}$. Thus we obtain an isomorphism $\psi : X_1 \to X_2$ with $\psi^* \mathcal{L}^{m(l+1)}_{m(l)}|_{X_2} \cong \mathcal{L}^{m(l+1)}_{m(l)}|_{X_1}$. Hence $\psi^* \mathcal{L}_{m(l)}|_{X_2} \cong \mathcal{L}_{m(l)}|_{X_1}$, and $[X_1] = [X_2]$, i.e. i_l is injective. q.e.d.

Proof of Theorem 1.1 . We define

$$\overline{\mathcal{M}}^P = \bigcup_{l \in \mathbb{N}} \mathcal{M}_{m(l)} \tag{4.1}$$

by using the inclusions i_l. Note that $\mathcal{M}^P$ is an open dense subset of each $\mathcal{M}_{m(l)}$ by Lemma 3.2 and thus of $\overline{\mathcal{M}}^P$.

We extend the map $\mathcal{CY} : \mathcal{M}^P \to \mathcal{M}et$ to a map $\overline{\mathcal{CY}} : \overline{\mathcal{M}}^P \to \overline{\mathcal{CY}(\mathcal{M}^P)}$ by the following. For any $x \in \mathcal{M}_{m(l)} \subset \overline{\mathcal{M}}^P$, let ω_l be the Ricci-flat Kähler-Einstein metric on X_p representing $c_1(\mathcal{O}_{\mathbb{CP}^{N_{m(l)}}}|_{X_p})$, where $p \in \mathcal{H}_{N_{m(l)}} \subset \mathcal{H}ilb^{P_{m(l)}}_{N_{m(l)}}$, $\mathcal{Q}_{m(l)}(p) = x$, and $X_p = \pi^{-1}_{\mathcal{H}}(p)$ from the construction in Section 3.2. For a normalized curve $f : \Delta \to \mathcal{H}_{N_{m(l)}}$ with $f(\Delta^*) \subset \mathcal{H}^o_{N_{m(l)}}$, we have a degeneration of polarized Calabi-Yau manifolds $(\mathcal{U}_{N_{m(l)}} \times_{\mathcal{H}_{N_{m(l)}}} \Delta \to \Delta, \mathfrak{p}^*\mathcal{O}_{\mathbb{CP}^{N_{m(l)}}}(1)|_{\mathcal{U}_{N_{m(l)}}})$ with central fiber X_p, where $\mathfrak{p}$ is the projection to the first factor. By Lemma 2.2, ω_l induces a compact metric space structure on X_p.

We define

$$\overline{\mathcal{CY}}(x) = \Big(X_p, \frac{1}{m(l)}\omega_l\Big).$$

If we consider $i_l(x) \in \mathcal{M}_{m(l+1)}$, then X_p is isomorphic to $\mathfrak{I}_l(X_p)$, $(X_p, \frac{1}{m(l)}\omega_l)$ is isometric to $(\mathfrak{I}_l(X_p), \frac{1}{m(l+1)}\omega_{l+1})$ as metric spaces by $i_l(x) = \mathcal{Q}_{m(l+1)}(\mathfrak{I}_l(X_p))$, and $\mathfrak{I}_l^*\mathcal{O}_{\mathbb{CP}^{N_{m(l+1)}}}(1)|_{\mathfrak{I}_l(X_p)} \cong \mathcal{O}_{\mathbb{CP}^{N_{m(l)}}}(m_{l+1})|_{X_p}$. Thus $\overline{\mathcal{CY}}$ is well-defined. If X_p is smooth, there is an ample line bundle L_p such that $L_p^{m(l)} \cong \mathcal{O}_{\mathbb{CP}^{N_{m(l)}}}(1)|_{X_p}$ and $[X_p, L_p] \in \mathcal{M}^P$. Hence $\frac{1}{m(l)}\omega_l \in c_1(L_p)$ and $\overline{\mathcal{CY}}|_{\mathcal{M}^P} = \mathcal{CY}$.

For any compact metric space $(Y, d_Y) \in \overline{\mathcal{CY}(\mathcal{M}^P)}$, we have a sequence $[X_k, L_k] \in \mathcal{M}^P(l_1)$, for an $l_1 > \operatorname{diam}_{d_Y}(Y)$, such that (X_k, ω_k) converges to (Y, d_Y) in the Gromov-Hausdorff sense, where $\omega_k \in c_1(L_k)$ is the Ricci-flat Kähler-Einstein metric. By Lemma 4.1, there is a Calabi-Yau variety X_∞ homeomorphic to Y and satisfying that X_∞ can be embedded in $\mathbb{CP}^{N_{m(l_1)}}$ and $[X_\infty] \in \mathcal{M}_{m(l_1)}$. Furthermore, the metric structure d_Y is induced by the singular Ricci-flat Kähler-Einstein metric $\omega \in \frac{1}{m(l_1)}c_1(\mathcal{O}_{\mathbb{CP}^{N_{m(l_1)}}}(1)|_{X_\infty})$ by Theorem 2.1, which implies that $\overline{\mathcal{CY}}([X_\infty]) = (X_\infty, \omega)$, i.e. $\overline{\mathcal{CY}}$ is surjective. We obtain i), ii) and iii).

Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds satisfying the condition in iv). We assume that $\mathcal{L}$ is relative very ample, and $[X_t, \mathcal{L}|_{X_t}] \in \mathcal{M}^P$, $t \in \Delta^*$. By (2.5), there is an $l_2 > 0$ such that $\operatorname{diam}_{\omega_t}(X_t) \leqslant l_2$ for $t \in \Delta^*$, where ω_t is the unique Ricci-flat Kähler-Einstein metric representing $c_1(\mathcal{L}|_{X_t})$. By the above construction, we have a $m(l_2) > 0$ such that we have a morphism $\tilde{\Psi} : \mathcal{X} \hookrightarrow \Delta \times \mathbb{CP}^{N_{m(l_2)}} \to \mathbb{CP}^{N_{m(l_2)}}$ with $\mathcal{L}^{m(l_2)}|_{X_t} \cong \tilde{\Psi}^*\mathcal{O}_{\mathbb{CP}^{N_{m(l_2)}}}(1)|_{X_t}$. There is a unique morphism $\rho : \Delta \to \mathcal{M}_{m(l_2)}$ such that $\rho(t) = [\tilde{\Psi}(X_t)]$ by the conclusion of $\mathcal{M}_{m(l_2)}$ coarsely representing the functor $\mathfrak{M}_{m(l_2)}$ in Lemma 3.2. The Gromov-Hausdorff convergence in iv) is a consequence of Lemma 2.2. q.e.d.

Proof of Theorem 1.3. For any point $x \in \overline{\mathcal{M}}^P \backslash \mathcal{M}^P$, we assume that $x \in \mathcal{M}_{m(l)} \subset \overline{\mathcal{M}}^P$ for an $m(l) > 0$. Let $p \in \mathcal{H}_{N_{m(l)}} \subset \mathcal{H}ilb^{P_{m(l)}}_{N_{m(l)}}$, $\mathcal{Q}_{m(l)}(p) = x$, and $X_p = \pi_{\mathcal{H}}^{-1}(p)$, where $\pi_{\mathcal{H}} : \mathcal{U}_{N_{m(l)}} \to \mathcal{H}_{N_{m(l)}}$ is the universal family. Let $\tau : \Delta \to \mathcal{H}_{N_{m(l)}}$ be a morphism such that $\tau(0) = p$ and $\tau(\Delta^*) \subset \mathcal{H}^o_{N_{m(l)}}$, and $\mathcal{X} = \mathcal{U}_{N_{m(l)}} \times_{\mathcal{H}_{N_{m(l)}}} \Delta \to \Delta$ be the degeneration of Calabi-Yau manifolds. Since the central fiber X_p is a Calabi-Yau variety, [**46**, Proposition 2.3] shows that the Weil-Petersson distance between the interior Δ^* and p is finite. Hence we obtain i) by composing the quotient map $\mathcal{Q}_{m(l)}$.

Let $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds. If we assume that the Weil-Petersson distance between Δ^* and 0 is finite, then after a finite base change $(\pi_\Delta : \mathcal{X} \to \Delta, \mathcal{L})$ is birational to a new family $(\pi'_\Delta : \mathcal{X}' \to \Delta, \mathcal{L}')$ such that $(\mathcal{X}\backslash X_0, \mathcal{L}) \cong (\mathcal{X}'\backslash X'_0, \mathcal{L}')$, and X'_0 is a Calabi-Yau variety by [**44**, Theorem 1.2]. We obtain ii) by the same argument as in the proof of iv) in Theorem 1.1. q.e.d.

Remark 4.3. Note that $\overline{\mathcal{M}}^P$ parameterizes certain Calabi-Yau varieties, which are proven to be K-stable by [**29**]. Hence Theorem 1.1 gives an evidence to the conjecture of the existence of K-moduli spaces (cf. [**30**, Conjecture 3.1]).

5. A remark for compactifications

Finally, we remark that there actually is a natural Gromov-Hausdorff compactification of $\overline{\mathcal{M}}^P$. If we define the normalized Calabi-Yau map

$$\mathcal{NCY} : \mathcal{M}^P \to \mathcal{M}et, \quad \text{by} \quad [X, L] \mapsto (X, \mathrm{diam}_\omega^{-2}(X)\omega),$$

where $\omega \in c_1(L)$ is the unique Ricci-flat Kähler-Einstein metric, then the Gromov's precompactness theorem (cf. [**10**, **34**]) asserts that the closure $\overline{\mathcal{NCY}(\mathcal{M}^P)}$ of $\mathcal{NCY}(\mathcal{M}^P)$ in $\mathcal{M}et$ is compact. Moreover, the map

$$\overline{\mathcal{CY}(\mathcal{M}^P)} \to \overline{\mathcal{NCY}(\mathcal{M}^P)}, \quad (Y, d_Y) \mapsto (Y, \mathrm{diam}_{d_Y}^{-1}(Y)d_Y)$$

is injective and continuous. However, because of the collapsing phenomenon, the algebro-geometric structure of $\overline{\mathcal{NCY}(\mathcal{M}^P)}$ is unclear, and it is not a compactification in the usual algebraic geometry sense. The Gromov-Hausdorff compactification is studied for the moduli spaces of compact Riemann surfaces and Abelian varieties in a recent preprint [**32**].

Let $(\mathcal{X} \to \Delta, \mathcal{L})$ be a degeneration of polarized Calabi-Yau manifolds of dimension n such that the diameter of the Ricci-flat Kähler metric $\omega_t \in c_1(\mathcal{L}|_{X_t})$ tends to infinite when $t \to 0$, i.e. $\mathrm{diam}_{\omega_t}(X_t) \to \infty$. Since $\mathrm{Vol}_{\omega_t}(X_t) = \frac{1}{n!}c_1^n(\mathcal{L}|_{X_t}) \equiv \text{const.}$, (X_t, ω_t) must collapse (cf. [**1**]), i.e. for metric 1-balls $B_{\omega_t}(1)$, $\mathrm{Vol}_{\omega_t}(B_{\omega_t}(1)) \to 0$ when $t \to 0$. If $0 \in \Delta$ is a large complex limit point (cf. [**12**]), a refined version of the Strominger-Yau-Zaslow (SYZ) conjecture (cf. [**39**]) due to Gross, Wilson, Kontsevich and Soibelman (cf. [**15**, **23**, **24**]) says that $\mathrm{diam}_{\omega_t}(X_t) \sim \sqrt{-\log|t|}$, and $(X_t, \mathrm{diam}_{\omega_t}^{-2}(X_t)\omega_t)$ converges to a compact metric space (B, d_B) in the Gromov-Hausdorff sense. If $h^{i,0}(X_t) = 0$, $1 \leqslant i < n$, then B is homeomorphic to S^n. Furthermore, there is an open subset $B_0 \subset B$

with $\operatorname{codim}_{\mathbb{R}} B\backslash B_0 \geqslant 2$, B_0 admits a real affine structure, and the metric d_B is induced by a Monge-Ampère metric g_B on B_0, i.e. under affine coordinates $x_1, \cdots, x_n$, there is a potential function ϕ such that

$$g_B = \sum_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} dx_i dx_j, \quad \text{and } \det\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j}\right) = 1.$$

This conjecture was verified by Gross and Wilson for fibred K3 surfaces with only type I_1 singular fibers in [**15**], and was studied for higher dimensional HyperKähler manifolds in [**13**, **14**]. In [**24**], it was further conjectured that the Gromov-Hausdorff limit B is homeomorphic to the Calabi-Yau skeleton of the Berkovich analytic space associated to $\mathcal{X}\times_\Delta \Delta^*$ by taking some base change if necessary, which gives an algebro-geometric description of B. If we grant this version of SYZ conjecture, we will have a nice algebro-geometric structure for the compactification at least for some one dimensional moduli space $\mathcal{M}^P$.

Example 5.1. A simple concrete example for Theorem 1.1 and Theorem 1.3 is the mirror Calabi-Yau 3-fold of the quintic 3-fold constructed in [**5**] (cf. Section 18 in [**12**]), i.e. X_t is the crepant resolution of the quotient

$$Y_s = \left\{[z_0, \cdots, z_4] \in \mathbb{CP}^4 | z_0^5 + \cdots + z_4^5 + s z_0 \cdots z_4 = 0\right\} / (\mathbb{Z}_5^5/\mathbb{Z}_5)$$

of the quintic by $\mathbb{Z}_5^5/\mathbb{Z}_5$, where $s^5 = t \in \mathbb{C}$. By choosing a polarization, $\mathcal{M}^P = \mathbb{C}\backslash\{1\}$, and 0 is an orbifold point of $\mathcal{M}^P$. When $t = 1$, X_1 is a Calabi-Yau variety with finite ordinary double points, and however, $t = \infty$ is a large complex limit point, which implies that $t = \infty$ is the cusp end of $\mathcal{M}^P$ and has infinite Weil-Petersson distance. Thus $\overline{\mathcal{M}}^P = \mathbb{C}$. Again, if we grant the refined version of SYZ conjecture, the point $t = \infty$ corresponds to the S^3 with a Monge-Ampère metric on an open dense subset, and consequently, $\overline{\mathcal{M}}^P$ has a natural Gromov-Hausdorff compactification $\mathbb{CP}^1$ with a continuous surjection $\overline{\mathcal{NCY}}$: $\mathbb{CP}^1 \to \overline{\mathcal{NCY}(\mathcal{M}^P)}$ extending $\mathcal{NCY}$.

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