J. DIFFERENTIAL GEOMETRY
85 (2010) 175-185

ON COMPLETE GRADIENT SHRINKING RICCI SOLITONS

Huai-Dong Cao & Detang Zhou

Abstract

In this paper we derive optimal growth estimates on the potential functions of complete noncompact shrinking solitons. Based on this, we prove that a complete noncompact gradient shrinking Ricci soliton has at most Euclidean volume growth. This latter result can be viewed as an analog of the well-known volume comparison theorem of Bishop that a complete noncompact Riemannian manifold with nonnegative Ricci curvature has at most Euclidean volume growth.

1. The results

A complete Riemannian metric g_{ij} on a smooth manifold M^n is called a *gradient shrinking Ricci soliton* if there exists a smooth function f on M^n such that the Ricci tensor R_{ij} of the metric g_{ij} is given by

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some positive constant $\rho > 0$. The function f is called a *potential function* of the shrinking soliton. Note that by scaling g_{ij} one can normalize $\rho = \frac{1}{2}$ so that

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \tag{1.1}$$

Gradient shrinking Ricci solitons play an important role in Hamilton's Ricci flow as they correspond to self-similar solutions, and often arise as Type I singularity models. In this paper, we investigate the asymptotic behavior of potential functions and volume growth rates of complete noncompact gradient shrinking solitons. Our main results are:

Theorem 1.1. *Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton satisfying (1.1). Then the potential function f satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 \le f(x) \le \frac{1}{4}(r(x) + c_2)^2.$$

Received 10/06/2009.

Here $r(x) = d(x_0, x)$ *is the distance function from some fixed point* $x_0 \in M$*, and* c_1 *and* c_2 *are positive constants depending only on* n *and the geometry of* g_{ij} *on the unit ball* $B_{x_0}(1)$*.*

Remark 1.1. In view of the Gaussian shrinker, namely, the flat Euclidean space $(\mathbb{R}^n, g_0)$ with the potential function $|x|^2/4$, the leading term $\frac{1}{4}r^2(x)$ for the lower and upper bounds on f in Theorem 1.1 is optimal. Notice that, by Theorem 1.1, we have a proper, distance-like function on (M^n, g_{ij}, f) defined by $\rho(x) = 2\sqrt{f}$. We also point out that it has been known, by the work of Perelman [**13**], Ni and Wallach [**12**], and Cao, Chen, and Zhu [**3**], that *any 3-dimensional complete noncompact nonflat shrinking gradient soliton is necessarily the round cylinder* $\mathbb{S}^2 \times \mathbb{R}$ *or one of its* $\mathbb{Z}_2$ *quotients*.

Remark 1.2. When the Ricci curvature of (M^n, g_{ij}, f) is assumed to be bounded, Theorem 1.1 was shown by Perelman [**13**]. Also, under the assumption of $Rc \geq 0$, a lower estimate of the form

$$f(x) \geq \frac{1}{8}r^2(x) - c_1'$$

was shown by Ni [**11**] (see also Lemma 2.1 in [**5**]). Moreover, the upper bound in Theorem 1.1 was essentially observed in [**3**], while a rough quadratic lower bound, as pointed out by Carrillo and Ni [**5**] (Section 7, p. 19), could follow from the argument of Fang, Man, and Zhang in [**7**].

Theorem 1.2. *Let* (M^n, g_{ij}, f) *be a complete noncompact gradient shrinking Ricci soliton. Then there exists some positive constant* $C_1 > 0$ *such that*

$$\mathrm{Vol}(B_{x_0}(r)) \leq C_1 r^n$$

for $r > 0$ *sufficiently large.*

Remark 1.3. Feldman, Ilmanen, and Knopf [**8**] constructed a complete noncompact gradient Kähler shrinker on the tautological line bundle $\mathcal{O}(-1)$ of the complex projective space $\mathbb{C}P^{n-1}$ ($n \geq 2$) which has Euclidean volume growth, quadratic curvature decay, and with Ricci curvature changing signs. This example shows that the volume growth rate upper bound in Theorem 1.2 is optimal. Note that, on the other hand, Carrillo and Ni [**5**] showed that *any nonflat gradient shrinking soliton with nonnegative Ricci curvature* $Rc \geq 0$ *must have zero asymptotic volume ratio*, i.e., $\lim_{r\to\infty} \mathrm{Vol}(B_{x_0}(r))/r^n = 0$.

Remark 1.4. There are two well-known theorems on volume growth of geodesic balls for complete noncompact Riemannian manifolds with nonnegative Ricci curvature: a theorem of Yau and Calabi (see [**16**]) asserts that the geodesic balls of such manifolds have at least linear growth, while the classical Bishop volume comparison theorem says (cf. [**14**]) the geodesic balls have at most Euclidean growth. Theorem 1.2

can be viewed as an analog of Bishop's theorem for gradient shrinking solitons. We also like to point out that Xi-Ping Zhu and the first author (see Theorem 3.1 in [**2**]) showed that a complete noncompact gradient shrinking Ricci soliton must have infinite volume. However, it remains an interesting problem to see if a Calabi–Yau-type theorem holds for gradient shrinking solitons, namely, whether every complete noncompact gradient shrinking soliton has at least linear volume growth.

Combining Theorem 1.1 and Theorem 1.2, we also have the following consequence, which was obtained previously in [**10**] and [**15**], respectively.

Corollary 1.1. *Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Then we have*

$$\int_M |u|e^{-f}dV < +\infty$$

for any function u on M with $|u(x)| \le Ae^{\alpha r^2(x)}$, $0 \le \alpha < \frac{1}{4}$ and $A > 0$. In particular, the weighted volume of M is finite,

$$\int_M e^{-f}dV < +\infty.$$

Acknowledgments. We are grateful to Ovidiu Munteanu for pointing out to us the estimate (3.4) on the average scalar curvature which allowed us to remove a certain mild assumption on the scalar curvature in Theorem 1.2. The second author wishes to thank Jiayu Li for some discussions. The first author was partially supported by NSF grants DMS-0354621 and DMS-0506084; the second author was partially supported by CNPq and FAPERJ, Brazil.

2. Asymptotic behavior of the potential function

In this section, we investigate the asymptotic behavior of the potential function of an arbitrary complete noncompact gradient shrinking Ricci solitons and prove Theorem 1.1.

First of all, we need a few useful facts about complete gradient shrinking solitons. The first basic result is due to Hamilton (cf. Theorem 20.1 in [**9**]).

Lemma 2.1. *Let (M^n, g_{ij}, f) be a complete gradient shrinking soliton satisfying (1.1). Then we have*

$$\nabla_i R = 2R_{ij}\nabla_j f,$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant C_0. Here R denotes the scalar curvature of g_{ij}.

As a consequence, by adding the constant C_0 to f, we can assume

$$R+|\nabla f|^2-f=0. \tag{2.1}$$

From now on we will make this normalization on f throughout the paper.

We will also need the following useful result, which is a special case of a more general result on complete ancient solutions due to B.-L. Chen [**6**] (cf. Proposition 5.5 in [**1**]).

Lemma 2.2. *Let (M^n, g_{ij}, f) be a complete shrinking Ricci soliton. Then g_{ij} has nonnegative scalar curvature $R \geq 0$.*

As an immediate consequence of (2.1) and Lemma 2.2, one gets the following result, which was essentially observed by Cao, Chen, and Zhu [**3**] (cf. p. 78–79 in [**3**]).

Lemma 2.3. *Let (M^n, g_{ij}, f) be a complete shrinking Ricci soliton satisfying (1.1) and (2.1). Then*

$$f(x) \leq \frac{1}{4}(r(x)+2\sqrt{f(x_0)})^2, \tag{2.2}$$

$$|\nabla f|(x) \leq \frac{1}{2}r(x)+\sqrt{f(x_0)}, \tag{2.3}$$

and

$$R(x) \leq \frac{1}{4}(r(x)+2\sqrt{f(x_0)})^2. \tag{2.4}$$

Here $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$.

Proof. By Lemma 2.2 and (2.1),

$$0 \leq |\nabla f|^2 \leq f, \quad \text{or} \quad |\nabla \sqrt{f}| \leq \frac{1}{2} \tag{2.5}$$

whenever $f > 0$. Thus $\sqrt{f}$ is an Lipschitz function and

$$|\sqrt{f(x)}-\sqrt{f(x_0)}| \leq \frac{1}{2}r(x).$$

Hence

$$\sqrt{f(x)} \leq \frac{1}{2}r(x)+\sqrt{f(x_0)},$$

or

$$f(x) \leq \frac{1}{4}(r(x)+2\sqrt{f(x_0)})^2.$$

This proves (2.2), from which (2.3) and (2.4) follow immediately when combined with (2.1). q.e.d.

Now (2.2) provides the upper estimate on f in Theorem 1.1. However, proving the lower estimate turns out to be more subtle.

Proposition 2.1. *Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton satisfying the normalization conditions (1.1) and (2.1). Then f satisfies the estimate*

$$f(x) \geq \frac{1}{4}(r(x) - c_1)^2,$$

where c_1 is a positive constant depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$.

Proof. Consider any minimizing normal geodesic $\gamma(s)$, $0 \leq s \leq s_0$ for some arbitrary large $s_0 > 0$, starting from $x_0 = \gamma(0)$. Denote by $X(s) = \dot{\gamma}(s)$ the unit tangent vector along γ. Then, by the second variation of arc length, we have

$$\int_0^{s_0} \phi^2 Rc(X, X) ds \leq (n-1) \int_0^{s_0} |\dot{\phi}(s)|^2 ds \tag{2.6}$$

for every nonnegative function $\phi(s)$ defined on the interval $[0, s_0]$. Now, following Hamilton [**9**], we choose $\phi(s)$ by

$$\phi(s) = \begin{cases} s, & s \in [0,1], \\ 1, & s \in [1, s_0 - 1], \\ s_0 - s, & s \in [s_0 - 1, s_0]. \end{cases}$$

Then

$$\begin{aligned} \int_0^{s_0} Rc(X,X) ds &= \int_0^{s_0} \phi^2 Rc(X,X) ds + \int_0^{s_0} (1-\phi^2) Rc(X,X) ds \\ &\leq (n-1) \int_0^{s_0} |\dot{\phi}(s)|^2 ds + \int_0^{s_0} (1-\phi^2) Rc(X,X) ds \\ &\leq 2(n-1) + \max_{B_{x_0}(1)} |Rc| + \max_{B_{\gamma(s_0)}(1)} |Rc|. \end{aligned}$$

On the other hand, by (1.1), we have

$$\nabla_X \dot{f} = \nabla_X \nabla_X f = \frac{1}{2} - Rc(X,X). \tag{2.7}$$

Integrating (2.7) along γ from 0 to s_0, we get

$$\begin{aligned} \dot{f}(\gamma(s_0)) - \dot{f}(\gamma(0)) &= \frac{1}{2} s_0 - \int_0^{s_0} Rc(X,X) ds \\ &\geq \frac{s_0}{2} - 2(n-1) - \max_{B_{x_0}(1)} |Rc| - \max_{B_{\gamma(s_0)}(1)} |Rc|. \end{aligned}$$

In case g_{ij} has bounded Ricci curvature $|Rc| \leq C$ for some constant $C > 0$, then it would follow that

$$\dot{f}(\gamma(s_0)) \geq \frac{1}{2} s_0 - \dot{f}(\gamma(0)) - 2(n-1) - 2C = \frac{1}{2}(s_0 - c), \tag{2.8}$$

and that

$$f(\gamma(s_0)) \geq \frac{1}{4}(s_0 - c)^2 - f(x_0) - \frac{c^2}{4},$$

proving what we wanted. (Indeed, the above argument was essentially sketched by Perelman (see, p.3 of [**13**]), and a detailed argument was presented in [**4**] (p. 385–386).)

However, since we do not assume any curvature bound in Theorem 1.1, we have to modify the above argument.

First of all, by integrating (2.7) along γ from $s = 1$ to $s = s_0 - 1$ instead and using (2.6) as before, we have

$$\begin{aligned}
\dot{f}(\gamma(s_0-1)) - \dot{f}(\gamma(1)) &= \int_1^{s_0-1} \nabla_X \dot{f}(\gamma(s))ds \\
&= \frac{1}{2}(s_0-2) - \int_1^{s_0-1} Rc(X,X)ds \\
&= \frac{1}{2}(s_0-2) - \int_1^{s_0-1} \phi^2(s)Rc(X,X)ds \\
&\geq \frac{s_0}{2} - 2n + 1 - \max_{B_{x_0}(1)}|Rc| + \int_{s_0-1}^{s_0} \phi^2 Rc(X,X)ds.
\end{aligned}$$

Next, using (2.7) and integration by parts one more time as in [**7**], we obtain

$$\begin{aligned}
\int_{s_0-1}^{s_0} \phi^2 Rc(X,X)ds &= \frac{1}{2}\int_{s_0-1}^{s_0} \phi^2(s)ds - \int_{s_0-1}^{s_0} \phi^2(s)\nabla_X \dot{f}(\gamma(s))ds \\
&= \frac{1}{6} + \dot{f}(\gamma(s_0-1)) - 2\int_{s_0-1}^{s_0} \phi(s)\dot{f}(\gamma(s))ds.
\end{aligned}$$

Therefore,

$$2\int_{s_0-1}^{s_0} \phi(s)\dot{f}(\gamma(s))ds \geq \frac{s_0}{2} - 2n + \frac{7}{6} - \max_{B_{x_0}(1)}|Rc| + \dot{f}(\gamma(1)). \tag{2.9}$$

Furthermore, by (2.5) we have

$$|\dot{f}(\gamma(s))| \leq \sqrt{f(\gamma(s))},$$

and

$$|\sqrt{f(\gamma(s))} - \sqrt{f(\gamma(s_0))}| \leq \frac{1}{2}(s_0 - s) \leq \frac{1}{2},$$

whenever $s_0 - 1 \leq s \leq s_0$. Thus,

$$\max_{s_0-1\leq s\leq s_0} |\dot{f}(\gamma(s))| \leq \sqrt{f(\gamma(s_0))} + \frac{1}{2}. \tag{2.10}$$

Combining (2.9) and (2.10), and noting $2\int_{s_0-1}^{s_0}\phi(s)ds = 1$, we conclude that

$$\sqrt{f(\gamma(s_0))} \geq \frac{1}{2}(s_0 - c_1)$$

for some constant c_1 depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$. This completes the proof of Proposition 2.1 and Theorem 1.1. q.e.d.

3. Volume growth of complete gradient shrinking solitons

In this section, we examine the volume growth of geodesic balls of complete noncompact gradient shrinking Ricci solitons.

Let us define

$$\rho(x) = 2\sqrt{f(x)}.$$

Then, by Theorem 1.1, we have

$$r(x) - c \le \rho(x) \le r(x) + c \tag{3.1}$$

with $c = \max\{c_1, c_2\} > 0$. Also, we have

$$\nabla \rho = \frac{\nabla f}{\sqrt{f}} \quad \text{and} \quad |\nabla \rho| = \frac{|\nabla f|}{\sqrt{f}} \le 1. \tag{3.2}$$

Denote by

$$D(r) = \{x \in M : \rho(x) < r\} \quad \text{and} \quad V(r) = \int_{D(r)} dV.$$

Then, by the co-area formula (cf. [**14**]), we have

$$V(r) = \int_0^r ds \int_{\partial D(s)} \frac{1}{|\nabla \rho|} dA.$$

Hence,

$$V'(r) = \int_{\partial D(r)} \frac{1}{|\nabla \rho|} dA = \frac{r}{2} \int_{\partial D(r)} \frac{1}{|\nabla f|} dA. \tag{3.3}$$

Here we have used (3.2) in deriving the last identity in (3.3).

Lemma 3.1.

$$nV(r) - rV'(r) = 2\int_{D(r)} R dV - 2\int_{\partial D(r)} \frac{R}{|\nabla f|} dV.$$

Proof. Taking the trace in (1.1), we have

$$R + \Delta f = \frac{n}{2}.$$

Thus,

$$\begin{aligned} nV(r) - 2\int_{D(r)} RdV &= 2\int_{D(r)} \Delta f dV \\ &= 2\int_{\partial D(r)} \nabla f \cdot \frac{\nabla \rho}{|\nabla \rho|} \\ &= 2\int_{\partial D(r)} |\nabla f| dV \\ &= 2\int_{\partial D(r)} \frac{f-R}{|\nabla f|} dV \\ &= rV'(r) - 2\int_{\partial D(r)} \frac{R}{|\nabla f|} dV. \end{aligned}$$

q.e.d.

Remark 3.1. As pointed out to us by Ovidiu Munteanu, we have also actually shown that

$$\int_{D(r)} RdV \leq \frac{n}{2} V(r). \tag{3.4}$$

Namely, the average scalar curvature over $D(r)$ is bounded by $n/2$.

Now we are ready to prove **Theorem 1.2**.

Proof. Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Denote

$$\chi(r) = \int_{D(r)} RdV.$$

By the co-area formula, we have

$$\chi(r) = \int_0^r ds \int_{\partial D(s)} \frac{R}{|\nabla \rho|} dA = \frac{1}{2}\int_0^r sds \int_{\partial D(s)} \frac{R}{|\nabla f|} dA.$$

Hence

$$\chi'(r) = \frac{r}{2}\int_{\partial D(r)} \frac{R}{|\nabla f|} dA.$$

Therefore, Lemma 3.1 can be rewritten as

$$nV(r) - rV'(r) = 2\chi(r) - \frac{4}{r}\chi'(r). \tag{3.5}$$

This implies that

$$\begin{aligned} (r^{-n}V(r))' &= 4r^{-n-2} e^{\frac{r^2}{4}} (e^{-\frac{r^2}{4}} \chi(r))' \\ &= 4r^{-n-2}\chi'(r) - 2r^{-n-1}\chi(r). \end{aligned}$$

Integrating the above equation from r_0 to r, we get

$$\begin{aligned} r^{-n}V(r) - r_0^{-n}V(r_0) &= 4r^{-n-2}\chi(r)\Big|_{r_0}^{r} + 4(n+2)\int_{r_0}^{r} r^{-n-3}\chi(r)dr \\ &\quad - 2\int_{r_0}^{r} r^{-n-1}\chi(r)dr \\ &= 4r^{-n-2}\chi(r) - 4r_0^{-n-2}\chi(r_0) \\ &\quad + 2\int_{r_0}^{r} r^{-n-3}\chi(r)(2(n+2) - r^2)dr. \end{aligned}$$

Since $\chi(r)$ is positive and increasing in r, we have, for $r_0 = \sqrt{2(n+2)}$,

$$\begin{aligned} \int_{r_0}^{r} r^{-n-3}\chi(r)(2(n+2) - r^2)dr &\le \chi(r_0)\int_{r_0}^{r} r^{-n-3}(2(n+2) - r^2)dr \\ &= \chi(r_0)\ (-2r^{-n-2} + \frac{1}{n}r^{-n})\Big|_{r_0}^{r}. \end{aligned}$$

Thus,

$$r^{-n}V(r) - r_0^{-n}V(r_0) \le 4r^{-n-2}(\chi(r) - \chi(r_0)) + \frac{2}{n}\chi(r_0)(r^{-n} - r_0^{-n}).$$

It follows that, for $r \ge r_0 = \sqrt{2(n+2)}$,

$$V(r) \le (r_0^{-n}V(r_0))r^n + 4r^{-2}\chi(r). \tag{3.6}$$

On the other hand, by (3.4) we have

$$4r^{-2}\chi(r) \le 2nr^{-2}V(r) \le \frac{1}{2}V(r), \tag{3.7}$$

for r sufficiently large.

Plugging (3.7) into (3.6), we obtain

$$V(r) \le 2r_0^{-n}V(r_0)r^n.$$

Therefore, by (3.1),

$$\mathrm{Vol}(B_{x_0}(r)) \le V(r+c) \le V(r_0)r^n$$

for r large enough. This finishes the proof of Theorem 1.2.

q.e.d.

We conclude with the following volume lower estimate.

Proposition 3.1. *Let (M^n, g_{ij}, f) be a complete noncompact gradient shrinking Ricci soliton. Suppose the average scalar curvature satisfies the upper bound*

$$\frac{1}{V(r)}\int_{D(r)} R dV \le \delta \tag{3.8}$$

for some positive constant $\delta < n/2$ and all sufficiently large r. Then there exists some positive constant $C_2 > 0$ such that

$$\mathrm{Vol}(B_{x_0}(r)) \ge C_2 r^{n-2\delta}$$

for r sufficiently large.

Proof. Combining the assumption (3.8) with Lemma 3.1 and Lemma 2.2, it follows that

$$(n-2\delta)V(r) \leq rV'(r). \tag{3.9}$$

Thus,

$$\int_1^r \frac{V'(s)}{V(s)}ds \geq \int_1^r \frac{n-2\delta}{s}ds.$$

Consequently,

$$V(r) \geq V(1)r^{n-2\delta}.$$

Therefore, in view of (3.1),

$$\mathrm{Vol}(B_{x_0}(r)) \geq V(r-c) \geq 2^{-n}V(1)r^{n-2\delta}$$

for r sufficiently large.

q.e.d.

Remark 3.2. As we mentioned in the introduction, X.-P. Zhu and the first author (see [**2**]) have shown that a complete noncompact gradient shrinking soliton, without any curvature assumption, must have infinite volume. Their proof is, however, more sophisticated, relying on a logarithmic inequality of Carrillo and Ni [**5**] and the Perelman-type noncollapsing argument for complete gradient shrinking solitons.

References

[1] H-D. Cao, *Recent progress on Ricci solitons*, Adv. Lect. Math. (ALM), **11** (2009), 1–38.

[2] H.-D. Cao, *Geometry of complete gradient shrinking solitons*, arXiv:0903.3927v2.

[3] H.-D Cao, B.-L Chen & X.-P Zhu, *Recent developments on Hamilton's Ricci flow*, Surveys in differential geometry, Vol. XII, 47–112, Surv. Differ. Geom., **XII**, Int. Press, Somerville, MA, 2008, MR 2488948, Zbl 1157.53002.

[4] H.-D. Cao & X.P. Zhu, *A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton-Perelman theory of the Ricci flow*, Asian J. Math., **10** (2006), 165–492, MR 2233789, Zbl pre05071705.

[5] J. Carrillo & L. Ni, *Sharp logarithmic Sobolev inequalities on gradient solitons and applications*, arXiv:0806.2417v2.

[6] B.-L Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. **82**(2009), 363–382, MR 2520796, Zbl 1177.53036.

[7] F. Fang, J. Man & Z. Zhang, *Complete gradient shrinking Ricci solitons have finite topological type*, C. R. Math. Acad. Sci. Paris, **346** (2008), 653–656, MR 2423272, Zbl 1144.53083.

[8] M. Feldman, T. Ilmanen & D. Knopf, *Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons*, J. Differential Geom., **65** (2003), 169–209, MR 2058261, Zbl 1069.53036.

[9] R.S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry (Cambridge, MA, 1993), **2**, 7-136, International Press, Combridge, MA, 1995, MR 1375255, Zbl 0867.53030.

[10] F. Morgan, *Manifolds with density*, Notices of the Amer. Math. Soc., **52** (2005), 853–858, MR 2200379, Zbl 1118.53022.

[11] L. Ni, *Ancient solution to Kähler-Ricci flow*, Math. Res. Lett., **12** (2005), 633–653, MR 2189227, Zbl 1087.53061.

[12] L. Ni & N. Wallach, *On a classification of the gradient shrinking solitons*, Math. Res. Lett., **15** (2008), 941–955, MR 2443993, Zbl 1158.53052.

[13] G. Perelmann, *Ricci flow with surgery on three manifolds*, arXiv:math.DG/0303109.

[14] R. Schoen & S.-T Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994, MR 1333601, Zbl 0830.53001.

[15] G. Wei & W. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom., **83** (2009), 377–406.

[16] S.-T Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J., **25** (1976), 659–670, MR 0417452, Zbl 0335.53041 .

Department of Mathematics
Lehigh University
Bethlehem, PA 18015
E-mail address: huc2@lehigh.edu

Instituto de Matemática
Universidade Federal Fluminense
Niterói, RJ 24020, Brazil
E-mail address: zhou@impa.br