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UMBILICAL FOLIATIONS AND TRANSVERSELY HOLOMORPHIC FLOWS

MARCO BRUNELLA & ETIENNE GHYS

1. Introduction

Consider a codimension-1 foliation $\mathscr{F}$ on a closed oriented 3-manifold M equipped with a Riemannian metric g and denote by $\mathscr{N}$ the orthogonal one-dimensional foliation. There is an interesting relationship between the local geometry of the leaves of $\mathscr{F}$ and the transverse structure of $\mathscr{N}$. More precisely, assume for simplicity that $\mathscr{N}$ is oriented by a unit speed flow ϕ^t, denote by π the orthogonal projection of the tangent space TM onto the tangent space $T\mathscr{F}$ and by II_x the second fundamental form at x of the leaf $\mathscr{F}_x$ of $\mathscr{F}$ through the point x. Then, if v is any vector tangent to $\mathscr{F}_x$ at x, one easily checks that:

$$II_x(v) = \frac{d}{dt} g(\pi d\phi^t(v))|_{t=0}.$$

As immediate corollaries, one gets the following:

(1) The leaves of $\mathscr{F}$ are minimal surfaces, i.e., the mean curvature (or the trace of II) vanishes, if and only if the holonomy of $\mathscr{N}$, mapping pieces of leaves of $\mathscr{F}$ to pieces of leaves of $\mathscr{F}$, is area preserving. Equivalently, ϕ^t is volume preserving. Using this remark, D. Sullivan could give a precise description of "taut" codimension-one foliations, i.e., those for which there is a Riemannian metric such that leaves are minimal surfaces. The result can be stated in the following way: a codimension-one foliation on a closed oriented manifold M is *not* taut if and only if there is a compact domain in M whose boundary is a nonempty union of compact leaves and such that the transverse orientation on the boundary points inwards. See [16].

(2) The leaves of $\mathscr{F}$ are totally geodesic, i.e., II vanishes, if and only if the length of the vectors $\pi d\phi^t(v)$ does not depend on t; one says that g is bundle-like for $\mathscr{N}$ or that ϕ^t is a Riemannian flow [3]. This remark made possible the description of all codimension-1 "geodesible" foliations,

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i.e., those for which there is a metric such that leaves are totally geodesic [4], [8]. The list of these foliations is much more restrictive than in case (1).

(3) The leaves of $\mathscr{F}$ are totally umbilical, i.e., for each point x the quadratic form II_x is a multiple of the metric g on the tangent space to $\mathscr{F}_x$, if and only if the holonomy of $\mathscr{N}$ acts conformally on leaves of $\mathscr{F}$. *The purpose of this paper is to classify these umbilical foliations.*

Assume that $\mathscr{F}$ is such an umbilical foliation. On any leaf L of $\mathscr{F}$ the Riemannian metric g defines a conformal structure and, hence, a holomorphic structure (by the existence of the so-called isothermal coordinates). Since the holonomy of $\mathscr{N}$ is conformal from leaves of $\mathscr{F}$ to leaves of $\mathscr{F}$, one deduces that $\mathscr{N}$ is naturally a *transversely holomorphic foliation.* This means that $\mathscr{N}$ is locally defined by submersions onto open sets in $\mathbf{C}$ and that two of these submersions differ (on the intersection of their domains) by a holomorphic map [12].

Conversely, suppose we are given a transversely holomorphic foliation $\mathscr{N}$ on a closed 3-manifold M and assume that $\mathscr{N}$ is transverse to a codimension-one foliation $\mathscr{F}$. The transverse structure of $\mathscr{N}$ induces a conformal structure on every leaf of $\mathscr{F}$. Let g be any Riemannian metric making $\mathscr{F}$ and $\mathscr{N}$ orthogonal and inducing these conformal structures on leaves of $\mathscr{F}$. Then, it is obvious that $\mathscr{F}$ is umbilical for this Riemannian metric g.

Therefore, *it is equivalent to classify umbilical foliations on* 3*-manifolds and transversely holomorphic foliations which are transverse to a codimension-one foliation.*

This approach of the problem was noticed by Y. Carrière jointly with the second author [2].

Let us say that we are basically interested in the qualitative description of umbilical foliations. The expression "umbilical foliation" will have the same meaning as "foliation which is umbilical for some Riemannian metric g".

This paper is organized as follows. In §2 we construct a family of examples and state our main result according to which any umbilical foliation is conjugated to one of these examples. In §3 we establish a very general property of domains of definition of holonomy maps of transversely holomorphic foliations. In §4 we show that our problem reduces indeed to globalizing the holonomy of $\mathscr{N}$. A basic tool is described in §5: the notion of harmonic measure enables us to show that the "distance" between two leaves of $\mathscr{F}$, "measured along $\mathscr{N}$", is a harmonic function on leaves of $\mathscr{F}$, at least if the latter are dense. This leads to a proof of the main result in §6 if the leaves are dense, and in §7 if there is an exceptional minimal set. Finally §§8 and 9 deal with the case where $\mathscr{F}$ has compact leaves: by a surgery technique, one reduces it to §§6 and 7.

We shall always assume that the foliations under consideration are of class C^∞. For simplicity, we also assume that manifolds and foliations are oriented and transversely oriented, and that the ambient manifold M is connected. Finally, since our study constantly switches between the two foliations $\mathscr{F}$ and $\mathscr{N}$, and to avoid confusion, we shall use the expression "1-foliation" to mean "one-dimensional oriented foliation".

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2. Examples and the main result

Example 1. Recall that a Seifert fibration on a closed 3-manifold is a 1-foliation such that all leaves are closed (with finite holonomy). The leaf space of a Seifert fibration is a two-dimensional orbifold and can therefore be equipped with a holomorphic structure (in many ways) [17]. Hence, Seifert fibrations are examples of transversely holomorphic 1-foliations. Many of these (but not all) are transverse to codimension-1 foliations [12] so that we get many examples of umbilical foliations. Note that these Seifert fibrations are also Riemannian foliations in an obvious way, and the umbilical foliations that we construct by this procedure are also geodesible.

Example 2. Let A be an element of $\mathrm{SL}(2, \mathbf{Z})$ with two real distinct positive eigenvalues. Let f_1 and f_2 be the two irrational linear foliations on the torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ by lines parallel to one of the two eigendirections of A. The product $\mathbf{T}^2 \times \mathbf{R}$ can be equipped with two transverse foliations of respective dimensions 1 and 2 and whose leaves are respectively products of leaves of f_1 by points and of f_2 by $\mathbf{R}$. These two foliations are invariant under the diffeomorphism sending $(x, t) \in \mathbf{T}^2 \times \mathbf{R}$ to $(Ax, t+1) \in \mathbf{T}^2 \times \mathbf{R}$, and define two foliations $\mathscr{N}$ and $\mathscr{F}$ on the compact quotient, denoted T_A^3. In this example $\mathscr{N}$ is actually Riemannian, and $\mathscr{F}$ is therefore geodesible (and umbilical). For more details on this example, see [11] or [4]. It is shown in [4] that any codimension-1 foliation on T_A^3 which is transverse to $\mathscr{N}$ is conjugated to $\mathscr{F}$.

Example 3. Let $\mathscr{N}$ be a linear 1-foliation on $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$. It is obviously Riemannian so that any codimension-one foliation transverse to $\mathscr{N}$ is geodesible (and umbilical). The description of these foliations is easy since, by a small perturbation, $\mathscr{N}$ can be changed in a trivial circle bundle over $\mathbf{T}^2$ so that one is back to Example 1.

Example 4. Let ϕ be a Moebius diffeomorphism of the Riemann sphere $\overline{\mathbf{C}}$, of elliptic type, and let $\mathscr{N}$ be the 1-foliation on $\overline{\mathbf{C}} \times \mathbf{S}^1$ obtained by suspending ϕ. It is clearly transversely holomorphic (even Riemannian), so that the foliation by spheres of $\overline{\mathbf{C}} \times \mathbf{S}^1$ is umbilical. By small perturbation, $\mathscr{N}$ can be changed in a Seifert fibration which lifts to a trivial circle bundle in a finite cover of $\overline{\mathbf{C}} \times \mathbf{S}^1$. It follows easily that any codimension-one foliation transverse to $\mathscr{N}$ is conjugated to the foliation by spheres.

Example 5. Let $\mathscr{N}$ be the 1-foliation on $\overline{\mathbf{C}} \times \mathbf{S}^1$ constructed by suspending a parabolic Moebius transformation. It is transversely holomorphic but not Riemannian. A codimension-1 foliation $\mathscr{F}$ transverse to $\mathscr{N}$ cannot have a Reeb component [14] and so must be conjugated to the (geodesible) foliation by spheres [12].

Example 6. We now come to the examples of umbilical but not geodesible foliations. Let λ be a complex number such that $0 < |\lambda| < 1$ and let $\Psi : \mathbf{C} \times [0, +\infty) \to \mathbf{C} \times [0, +\infty)$ be the "homothety" $(z, t) \mapsto (\lambda z, \frac{1}{2}t)$. The 1-foliation by vertical lines of $\mathbf{C} \times [0, +\infty)$ is invariant by Ψ and hence defines a 1-foliation $\mathscr{N}_\lambda$ on the quotient of $\mathbf{C} \times [0, +\infty)\backslash\{(0, 0)\}$ by Ψ, which is diffeomorphic to a solid torus. The 1-foliation $\mathscr{N}_\lambda$ is transverse to the boundary and has a hyperbolic closed leaf in the interior (with "eigenvalue" λ) on which all other leaves accumulate. Clearly, $\mathscr{N}_\lambda$ is transversely holomorphic and, with this structure, the boundary of the solid torus is biholomorphic to the elliptic curve $\mathbf{C}\backslash\{0\}/z \sim \lambda z$. We may obtain transversely holomorphic 1-foliations $\mathscr{N}$ on closed manifolds by glueing two copies of $\mathscr{N}_\lambda$ by a biholomorphism between the boundaries. The resulting manifolds are either lens spaces (possible $\mathbf{S}^3$) or $\mathbf{S}^2 \times \mathbf{S}^1$. The structure of codimension-1 foliations transverse to one of these transversely holomorphic 1-foliations is easily analyzed, thanks to [14]. If the ambient manifold is a lens space, then such a foliation $\mathscr{F}$ must have a Reeb component and each Reeb component must contain a closed leaf of $\mathscr{N}$. It follows that $\mathscr{F}$ is either composed by two Reeb components glued along their common boundary or composed by two Reeb components separated by a product $\mathbf{T}^2 \times [0, 1]$ equipped with a foliation tangent to the boundary and transverse to $\{*\} \times [0, 1]$. If the ambient manifold is $\mathbf{S}^2 \times \mathbf{S}^1$, there is a third possibility: $\mathscr{F}$ is a foliation by spheres (and in this case $\mathscr{N}$ is the suspension of a hyperbolic Moebius transformation of $\overline{\mathbf{C}}$).

Theorem. *Examples* 1 *to* 6 *exhaust the list of transversely holomorphic orientable* 1*-foliations on closed orientable* 3*-manifolds which are transverse to some codimension-one foliation. Equivalently, Examples* 1 *to* 6 *exhaust the list of orientable umbilical codimension-one foliations on closed orientable* 3*-manifolds.*

Note in particular that this strongly restricts the topology of the ambient manifold.

Corollary. *If a closed orientable* 3*-manifold* M *admits an umbilical foliation, then it is diffeomorphic to the total space of a Seifert fibration or of a torus bundle over the circle.*

For each example of transversely holomorphic 1-foliation in the previous list, we give the description of *all* possible transverse foliations. Hence to prove the theorem it will be sufficient to classify transversely holomorphic 1-foliations $\mathcal{N}$ which admit *some* transverse foliation $\mathcal{F}$, instead of classifying directly the pair $(\mathcal{N}, \mathcal{F})$. This remark will be useful in the last two sections.

3. A general property of transversely holomorphic foliations

Let $\mathcal{N}$ be a transversely holomorphic foliation on a closed manifold M, of complex codimension one but of any dimension. A point x in M is said to be *wandering* if there is a small transverse disc D to $\mathcal{N}$ at x such that two distinct points of D belong to different leaves of $\mathcal{N}$. By definition, the space of wandering points is open and saturated by $\mathcal{N}$. The space of wandering leaves is a Riemann surface which might be a priori non-Hausdorff. The following theorem is proved in [10] in the same spirit as Ahlfors' finiteness theorem or Sullivan's theorem on wandering domains for rational maps.

Theorem 3.1. *No Hausdorff points are isolated in the space of wandering leaves of a transversely holomorphic foliation on a compact manifold.*

The following corollaries will be very useful for globalizing the holonomy maps of transversely holomorphic 1-foliations.

Corollary 3.2. *Let* M *be a compact* 3*-manifold with boundary equipped with a transversely holomorphic* 1*-foliation* $\mathcal{N}$ *transverse to the boundary. Assume that at least one leaf of* $\mathcal{N}$ *connects two distinct boundary components of* M. *Then* M *is diffeomorphic to a product* $\Sigma \times [0, 1]$, *and* $\mathcal{N}$ *is conjugated to the foliation by intervals* $\{*\} \times [0, 1]$.

In particular, if $\mathcal{N}$ is a transversely holomorphic 1-foliation on a closed 3-manifold M, and $\Sigma \subset M$ is a surface transverse to $\mathcal{N}$ and intersecting a closed leaf of $\mathcal{N}$, then Σ is a global cross section for $\mathcal{N}$.

Proof. Consider the double $2M$ of M equipped with the double foliation $2\mathcal{N}$ which is obviously transversely holomorphic. The leaf of $2\mathcal{N}$ through a point in $\partial M \subset 2M$ intersects the connected component of ∂M containing it at only one point and is therefore wandering; any connected component of ∂M embeds in the space of wandering leaves. From Theorem 3.1, one deduces that if Σ_1 and Σ_2 are two connected components of ∂M connected by at least one leaf of $\mathcal{N}$, then for all except a finite number of points x of Σ_2, the leaf through x intersects Σ_1. We shall show that, indeed, for *all* points x of Σ_2 the leaf of $\mathcal{N}$ through x intersects Σ_1 and that will prove the corollary.

Let $\{p_1, \cdots, p_n\} \subset \Sigma_2$ be the finite set of points whose leaf does not intersect Σ_1, and choose a point q in Σ_2 different from all p_i. Let $F \subset M$ be the closed set of points whose leaf does not intersect Σ_2. Assume by contradiction that $n \neq 0$. The leaf of $\mathcal{N}$ through p_1 is noncompact; choose a point x in its limit set, necessarily contained in F. Let D be a small disc transverse to $\mathcal{N}$ at x and not intersecting the (compact) leaf through q, and let $K = F \cap D$. One has a well-defined map $\phi : D\backslash K \to \Sigma_2\backslash\{q\}$ sending a point to the (unique) intersection of its leaf with Σ_2.

Of course, ϕ is holomorphic and nonconstant. Also, it is clear that if a sequence z_k of $D\backslash K$ converges to a point of K, then $\phi(z_k)$ can only accumulate in $\{p_1, \cdots, p_n\}$. Choose a nonconstant meromorphic function θ on Σ_2 vanishing on $\{p_1, \cdots, p_n\}$ and having q as its unique pole, and consider $\psi = \theta \circ \phi : D\backslash K \to \mathbf{C}$. If one extends ψ on K by the value 0, we get a continuous function $\psi : D \to \mathbf{C}$ which is holomorphic on the set where it is nonzero. By Radó's theorem [13, p. 255], this implies that ψ is holomorphic.

On the other hand, we know that x is a limit of points of the leaf of p_1 so that there is a sequence of points x_k in $D\backslash K$ converging to x and for which $\phi(x_k) = p_1$ and hence $\psi(x_k) = 0$. This is of course a contradiction to the fact that ψ is holomorphic and nonconstant in D. Thus the corollary is proved. q.e.d.

In the same way we obtain the following.

Corollary 3.3. *Let M be a compact 3-manifold with boundary and corners whose boundary has the form $\Sigma_+ \cup \Sigma_- \cup T$, where Σ_+, Σ_-, and T are nonempty surfaces with boundary, intersecting on their nonempty boundary. Let $\mathcal{N}$ be a transversely holomorphic 1-foliation on M having the following properties:*

(i) *T is diffeomorphic to a disjoint union of cylinders $\mathbf{S}^1 \times [0, 1]$, and $\mathcal{N}$ is tangent to T. The restriction of $\mathcal{N}$ to each cylinder is a trivial 1-foliation by intervals $[0, 1]$.*

(ii) $\mathscr{N}$ *is transverse to* Σ_+ *and* Σ_-, *pointing inwards on* Σ_+ *and outwards on* Σ_-. *Then there is a diffeomorphism from* M *to a product* $\Sigma \times [0, 1]$, *sending* Σ_+, Σ_-, *and* T *to* $\Sigma \times \{0\}$, $\Sigma \times \{1\}$, *and* $\partial\Sigma \times [0, 1]$ *respectively and sending the leaves of* $\mathscr{N}$ *to the intervals* $\{*\} \times [0, 1]$.

Proof. A neighborhood of T is foliated trivially. We can find real analytic simple curves in Σ_+ close to each boundary component of Σ_+. Hence, we can always assume that the boundary of Σ_+ (and Σ_-) consists of real analytic curves. Now, consider the double $2M$ of M along T; it is a compact 3-manifold whose boundary consists of the doubles $2\Sigma_+$ and $2\Sigma_-$ of Σ_+ and Σ_-. Using Schwarz's reflection across the real analytic curves in $\partial\Sigma_+$ and $\partial\Sigma_-$, one sees that the double foliation $2\mathscr{N}$ on $2M$ is transversely holomorphic. The corollary now follows from 3.2 applied to $2\mathscr{N}$.

4. Transversely holomorphic 1-foliations with trivial universal covering

The aim of this section is to show that, in some cases, the main theorem reduces to studying the domains of definition of holonomy maps of $\mathscr{N}$. Remark that the following proposition does not need the existence of a transverse codimension-1 foliation.

Proposition 4.1. *Let* $\mathscr{N}$ *be a transversely holomorphic* 1*-foliation on a closed* 3*-manifold* M. *Assume that the lift* $\widetilde{\mathscr{N}}$ *of* $\mathscr{N}$ *to the universal cover* $\widetilde{M}$ *of* M *is given by the fibers of a global fibration* F *from* $\widetilde{M}$ *to some* (*simply connected*) *surface* S. *Then* $\mathscr{N}$ *is conjugated* (*by a smooth transversely holomorphic diffeomorphism*) *to one of Examples* 1 *to* 6.

Proof. Of course, we can assume that the fibers of F are lines, since otherwise they would be circles and $\mathscr{N}$ would be a Seifert fibration on a lens space.

The transverse holomorphic structure of $\mathscr{N}$ provides a holomorphic structure on S. Therefore, one has three cases to consider.

1. S is the unit disc $\mathbf{D} \subset \mathbf{C}$. The fundamental group Γ of M acts on $\mathbf{D}$ by biholomorphisms and hence by isometries of the Poincaré metric. This defines a transverse invariant (hyperbolic) Riemannian metric for $\mathscr{N}$. The structure of these 1-foliations has been described in [3], [5], [17]: we are in Examples 1 and 2.

2. S is the Riemann sphere $\overline{\mathbf{C}}$. In this case, $\widetilde{M}$ is diffeomorphic to $\overline{\mathbf{C}} \times \mathbf{R}$ and, in particular, has two ends. It follows from [6] that the 1-foliation $\mathscr{N}$ has a global cross section in M so that $\mathscr{N}$ is the suspension of some Moebius automorphism of the Riemann sphere (Examples 4, 5, 6).

3. S is the complex plane $\mathbf{C}$. In this situation, Γ necessarily acts by affine automorphisms of $\mathbf{C}$, but we shall show that it acts by (Euclidean) isometries. One could use the classification of transversely affine flows, given in [9], but we can easily prove it directly. Indeed, as for any 1-foliation on a closed manifold, $\mathcal{N}$ possesses a nontrivial transverse invariant measure which defines a measure μ on $\mathbf{C}$ invariant under Γ. Notice that the only invariant measure of an affine bijection of $\mathbf{C}$ which is not an isometry is a Dirac mass concentrated at its fixed point. Hence we have two possibilities:

(i) Γ has a common fixed point x_0 in $\mathbf{C}$. Since the stabilizer of a point of $\mathbf{C}$ under the action of Γ is the fundamental group of the corresponding leaf of $\mathcal{N}$, this would imply that Γ is infinite cyclic, so that $\widetilde{M}$ is an infinite cyclic covering of a compact manifold and would therefore have two ends. This is a contradiction since $\widetilde{M}$ is diffeomorphic to $\mathbf{C}\times\mathbf{R}$.

(ii) Γ acts by isometries of the complex plane. Once again, this implies that $\mathcal{N}$ admits a transversely invariant Euclidean metric and this has been described in [2]: we are in Examples 1 and 3. q.e.d.

Let us remark that among the examples described in §6 some have a nontrivial universal covering.

5. Harmonic measures and the "distance between leaves"

We shall use the notion of harmonic measure, as introduced by Garnett [7]. Fix a foliation $\mathcal{F}$ on a closed Riemannian manifold M, and denote by $\Delta^{\mathcal{F}}$ the Laplace operator along the leaves, considered as a differential operator on the space of smooth functions on M. A probability measure μ on M is said to be *harmonic* if for every smooth function f on M, the integral $\int(\Delta^{\mathcal{F}} f)\,d\mu$ vanishes. According to [7], such a measure always exists and its support is a closed $\mathcal{F}$-saturated set. In particular, if all leaves are dense, then μ has full support.

Let us analyze the local structure of these measures in the special case where $\mathcal{F}$ is an umbilical foliation on a 3-manifold, whose orthogonal foliation is still denoted by $\mathcal{N}$. Suppose we have an open set U in the ambient manifold M and a diffeomorphism f from a product $\Omega\times(0,1)$ onto U, where Ω is an open set in $\mathbf{C}$, such that:

(i) f maps $\Omega\times\{*\}$ conformally into a leaf of $\mathcal{F}$,

(ii) f maps $\{*\}\times(0,1)$ into a leaf of $\mathcal{N}$.

Since $\mathcal{N}$ is transversely holomorphic, M can be covered by open sets U with these properties. We can disintegrate μ in U via the coordinates given by f, i.e., there exist:

(i) a measure ν on $(0, 1)$, and

(ii) for ν-almost every t in $(0, 1)$ a positive measure λ_t in $\Omega \times \{t\}$, such that for every Borel set B in $\Omega \times (0, 1)$ one has

$$\mu(f(B)) = \int_{(0,1)} \lambda_t(B \cap (\Omega \times \{t\}))\, d\nu(t).$$

Each plaque $\Omega \times \{t\}$ is equipped with a Riemannian metric which is conformal to the metric $|dz|^2$, so that its Laplace operator is a multiple of the classical Laplace operator $\partial^2/\partial z \partial \bar{z}$. If one expresses the condition that μ is harmonic for $\mathcal{F}$ in these local coordinates, one finds that for ν-almost all t the measure λ_t has a density u_t with respect to the area form on $\Omega \times \{t\} \cong \Omega$ which is a nonnegative harmonic function [7].

Fix a point z in Ω and consider the measure θ_z on the arc $f(\{z\} \times (0, 1))$ defined by

$$\theta_z(f(\{z\} \times C)) = \int_C u_t(z)\, d\nu(t),$$

where $C \subset (0, 1)$ is a Borel set. Note that this construction does not depend on the choice of the open set U satisfying the above properties and covering the arc $f(\{z\} \times (0, 1))$.

In summary, *if one chooses a harmonic measure for $\mathcal{F}$, one naturally constructs a measure on any arc which is a piece of leaf of $\mathcal{N}$.* Of course these measures are compatible in the sense that if we restrict one of these to a subarc we get the measure associated to this subarc.

Suppose now that the leaves of $\mathcal{F}$ are dense, so that μ has full support. The above measures θ_z are then without atoms and positive on open sets. We can use this collection of measures to define a topological flow ϕ^t on M, whose orbits are the leaves of $\mathcal{N}$. One simply defines $\phi^t(x)$ for small positive t as the unique point on the leaf of $\mathcal{N}$ through x, in positive direction, such that the small arc joining x and $\phi^t(x)$ in this leaf has measure t. Since the functions u_t are harmonic, we get the following proposition which expresses the fact that some kind of "distance" between leaves of $\mathcal{F}$ is harmonic.

Proposition 5.1. *Let $\mathcal{F}$ be an oriented umbilical foliation on a closed oriented 3-manifold M, and let $\mathcal{N}$ be its orthogonal 1-foliation. Assume that the leaves of $\mathcal{F}$ are dense. Then there is a parametrization of $\mathcal{N}$ by a topological flow ϕ^t with the following property. Let x_1 and x_2 be two points of M on the same leaf of $\mathcal{N}$, let Ω be a small neighborhood of x_1 in the leaf $\mathcal{F}_{x_1}$ of $\mathcal{F}$ through x_1, and let $T : \Omega \to \mathbf{R}$ be a continuous function* *defined near* x_1 *such that* $\phi^{T(x_1)}(x_1) = x_2$ *and that* $\phi^{T(x)}(x)$ *belongs to the leaf* $\mathscr{F}_{x_2}$ *through* x_2. *Then this function* T *is harmonic.*

6. Umbilical foliations with dense leaves

We assume in this section that $\mathscr{F}$ is an umbilical foliation with dense leaves on a closed 3-manifold M, and we still denote its orthogonal 1-foliation by $\mathscr{N}$. We shall use the natural parameter along $\mathscr{N}$ constructed in §5 to show that the foliations $\widetilde{\mathscr{F}}$ and $\widetilde{\mathscr{N}}$, which are the lifts of $\mathscr{F}$ and $\mathscr{N}$ to the universal covering $\widetilde{M}$ of M, are product foliations. Hence, this hypothesis of Proposition 4.1 will be satisfied.

As $\mathscr{F}$ has no Reeb component, none of its leaves is cut by a transverse curve homotopic to zero. In particular, a leaf of $\widetilde{\mathscr{F}}$ and a leaf of $\widetilde{\mathscr{N}}$ intersect at most one point. Leaves of $\widetilde{\mathscr{F}}$ and $\widetilde{\mathscr{N}}$ are closed subsets of $\widetilde{M}$. Note also that all leaves of $\widetilde{\mathscr{F}}$ are planes and that this implies that $\widetilde{M}$ is diffeomorphic to $\mathbf{R}^3$ [12].

Let L_1 and L_2 be two leaves of $\widetilde{\mathscr{F}}$, and consider the open set $\Omega(L_1, L_2)$ of L_1 consisting of those points x in L_1 such that the leaf of $\widetilde{\mathscr{N}}$ through x intersects L_2 at a point denoted $h(L_1, L_2)(x)$.

Lemma 6.1. $\Omega(L_1, L_2)$ *is a simply connected open set in* L_1.

Proof. Let γ_1 be a Jordan curve contained in $\Omega(L_1, L_2)$, bounding a disc D_1 contained in L_1. We shall show that D_1 is contained in $\Omega(L_1, L_2)$. Let γ_2 be the curve $h(L_1, L_2)(\gamma_1)$, and D_2 the disc of L_2 bounded by γ_2. Then D_1, D_2 and the cylinder from γ_1 to γ_2 along $\widetilde{\mathscr{N}}$ determine an embedded (topological) sphere in $\widetilde{M}$ which bounds a ball B. Since all leaves of $\widetilde{\mathscr{N}}$ are closed in $\widetilde{M}$, they intersect B on a compact set. It follows that for every x in D_1 the leaf of $\widetilde{\mathscr{N}}$ through x enters B and has to get out of B at some point of D_2. This shows that D_1 is contained in $\Omega(L_1, L_2)$ and proves the lemma.

Lemma 6.2. *For any two leaves* L_1 *and* L_2 *of* $\widetilde{\mathscr{F}}$, *the open set* $\Omega(L_1, L_2)$ *coincides with* L_1.

Proof. Let L_1 and L_2 be distinct leaves of $\widetilde{\mathscr{F}}$ such that $\Omega = \Omega(L_1, L_2)$ is nonempty. We shall show that Ω coincides with L_1 and this will imply the lemma by connectivity of $\widetilde{M}$.

Using Proposition 5.1, we get a harmonic function $T : \Omega \to \mathbf{R}$. Since leaves are disjoint, this function has constant sign, positive for instance. Let x_0 be a point in the boundary of Ω in L_1. Since the leaf of $\widetilde{\mathscr{N}}$

through x_0 does not cut L_2, it follows that $T(x)$ goes to $+\infty$ as x goes to x_0. This follows from the fact that the topological flow ϕ^t on M lifts to a complete flow on $\widetilde{M}$.

The leaf L_1 is naturally a simply connected Riemann surface, conformally equivalent to the unit disc $\mathbf{D}$ or to the plane $\mathbf{C}$, and Ω is a simply connected open set in L_1. Hence Lemma 6.2 follows from the following sublemma.

Sublemma 6.3. *Let Ω be a nonempty simply connected open set in $\mathbf{D}$ or in $\mathbf{C}$. If there is a positive (super)harmonic function $T : \Omega \to \mathbf{R}^+$ going to $+\infty$ on the boundary $\partial\Omega$, then this boundary is empty, i.e., Ω coincides with $\mathbf{D}$ or $\mathbf{C}$ accordingly.*

Proof. The positive superharmonic function T can be extended to a superharmonic function $\widetilde{T}$ on $\mathbf{D}$ or $\mathbf{C}$ setting $\widetilde{T} = +\infty$ on the complement of Ω. Since Ω is simply connected, its complement is certainly not totally disconnected, and if it were nonempty it would contain a continuum (i.e., a compact connected set with more than one point) and its capacity [15] would be nonzero. This would be a contradiction to the fact that a positive superharmonic function assumes the value $+\infty$ on a set of zero capacity [15, p. 183].

Corollary 6.4. *Let $\mathscr{F}$ be an oriented umbilical foliation with dense leaves on a closed 3-manifold M, and let $\mathscr{N}$ be the orthogonal 1-foliation. Then $\mathscr{F}$ and $\mathscr{N}$ are conjugated to one of Examples 1, 2, 3.*

Proof. By 6.2, we know that any leaf of $\widetilde{\mathscr{N}}$ cuts any leaf of $\widetilde{\mathscr{F}}$ at exactly one point. Hence there is a diffeomorphism from $\widetilde{M}$ to $\mathbf{R}^2 \times \mathbf{R}$ mapping leaves of $\widetilde{\mathscr{F}}$ to $\mathbf{R}^2 \times \{*\}$ and leaves of $\widetilde{\mathscr{N}}$ to $\{*\} \times \mathbf{R}$. The corollary follows from Proposition 4.1, since in Examples 4, 5, 6 the umbilical foliation has compact leaves.

7. Umbilical foliations with an exceptional minimal set

Let $\mathscr{F}$ be a codimension-1 foliation on a closed manifold M, and let $\mathscr{N}$ be a 1-foliation transverse to it. Recall that a minimal set $\mathscr{M}$ of $\mathscr{F}$ can be of three types:

(i) $\mathscr{M} = M$ in case all leaves are dense,

(ii) $\mathscr{M}$ is a compact leaf,

(iii) $\mathscr{M}$ is an exceptional minimal set intersecting transversals on Cantor sets [12].

Umbilical foliations of type (i) have been described in §6. We analyze in this section foliations having a minimal set of type (iii). Note that among

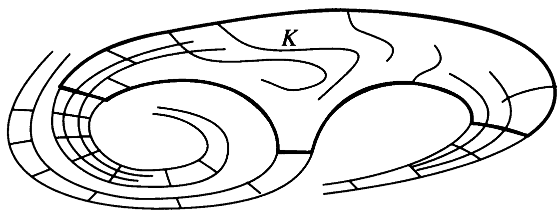

FIGURE 1

the examples described in §2 exceptional minimal sets can only occur in Example 1.

We need the description of open saturated sets given by the *nucleus theorem* that we recall now [12]. Let V be a nonempty connected open set saturated by $\mathcal{F}$. Then there exist:

(i) a (possibly noncompact) manifold $\overline{V}$ with boundary, of the same dimension as M, called the *completion* of M;

(ii) an immersion $i : \overline{V} \to M$ whose restriction to the interior of $\overline{V}$ is a bijection onto V, such that the foliation $\overline{\mathcal{F}} = i^*\mathcal{F}$ is tangent to the boundary of $\overline{V}$, and the foliation $\overline{\mathcal{N}} = i^*\mathcal{N}$ is transverse to the boundary. Moreover, $\overline{\mathcal{N}}$ is "trivial outside a compact set" in the following sense (see Figure 1). There is a compact part K of $\overline{V}$ (the *nucleus*) such that:

(iii) K is a submanifold with boundary and corners, saturated by $\overline{\mathcal{N}}$;

(iv) in the complement of K all leaves of $\overline{\mathcal{N}}$ are compact intervals going from one component of $\partial\overline{V}$ to another one.

Suppose now that $\mathcal{F}$ is an umbilical foliation on a closed 3-manifold M, and let $\mathcal{N}$ be its orthogonal 1-foliation. Suppose $\mathcal{F}$ admits an exceptional minimal set $\mathcal{M} \subset M$. Let V be a connected component of $M \backslash \mathcal{M}$, and let K be a nucleus in $\overline{V}$, as described above. Since $\overline{V}$ is certainly noncompact, the boundary of K in the interior of $\overline{V}$ is nonempty, and we can apply Corollary 3.3 to K. It follows that K is trivially foliated by $\overline{\mathcal{N}}$ so that every leaf of $\overline{\mathcal{N}}$ goes from one component of $\partial\overline{V}$ to another one. Therefore, there is a diffeomorphism from $\overline{V}$ to a product $L \times [0, 1]$ mapping the leaves of $\overline{\mathcal{N}}$ to the arcs $\{*\} \times [0, 1]$. In $\overline{V}$, the foliation $\overline{\mathcal{F}}$ is given by the suspension of a group of diffeomorphisms of $[0, 1]$, so that, in particular, $\overline{\mathcal{F}}$ lifts to a product foliation in the universal covering of $\overline{V}$.

We can now prove the main result of this section.

Proposition 7.1. *Let $\mathcal{F}$ be an oriented umbilical foliation on a closed 3-manifold M, and let $\mathcal{N}$ be its orthogonal 1-foliation. Assume that $\mathcal{F}$*

has an exceptional minimal set $\mathscr{M}$. *Then* $\mathscr{N}$ *and* $\mathscr{F}$ *are conjugated to Example* 1.

Proof. For each connected component V of $M \backslash \mathscr{M}$ we know that $\overline{V}$ can be identified with a product $L \times [0, 1]$. For each x in L, let us collapse all points in $i(\{x\} \times [0, 1])$ to a single point. Doing this construction in each connected component of $M \backslash \mathscr{M}$, one produces a collapsing map $c \colon M \to M'$ onto a space which is clearly a topological 3-manifold homeomorphic to M. Moreover, there are two foliations $\mathscr{F}'$ and $\mathscr{N}'$ on M' such that c is a local homeomorphism when restricted to a leaf of $\mathscr{F}$ and maps a leaf of $\mathscr{N}$ onto a leaf of $\mathscr{N}'$ (by collapsing the arcs in the complement of $\mathscr{M}$). Since $\mathscr{M}$ is an exceptional minimal set and $c(\mathscr{M}) = M'$, all leaves of $\mathscr{F}'$ are dense in M'. Note also that $\mathscr{N}'$ is transversely holomorphic.

Let μ be a harmonic measure on M whose support is $\mathscr{M}$. Then $c_*(\mu)$ is a harmonic measure for $\mathscr{F}'$ with full support (note that, even though $\mathscr{F}'$ is not a smooth foliation, each leaf of $\mathscr{F}'$ is a smooth surface equipped with a conformal structure, so that the definition of harmonic measure of $\mathscr{F}'$ makes sense). The argument of §6 therefore applies to $\mathscr{F}'$: in the universal cover $\widetilde{M}'$ of M' the lifted foliations $\widetilde{\mathscr{F}}'$ and $\widetilde{\mathscr{N}}'$ are product foliations. Moreover, $c : M \to M'$ lifts to a map $\tilde{c} : \widetilde{M} \to \widetilde{M}'$ between universal covers. One can reconstruct $\widetilde{M}$ from $\widetilde{M}'$ by opening a countable collection of leaves L of $\widetilde{\mathscr{F}}'$ and inserting a product $L \times [0, 1]$ foliated as a product.

This shows that there is also a homeomorphism from $\widetilde{M}$ to $\mathbf{R}^2 \times \mathbf{R}$ sending leaves of $\widetilde{\mathscr{F}}$ to $\mathbf{R}^2 \times \{*\}$ and leaves of $\widetilde{\mathscr{N}}$ to $\{*\} \times \mathbf{R}$. Thus the proposition follows from Proposition 4.1.

8. Umbilical foliations with compact leaves: a first case

Let $\mathscr{N}$ be a transversely holomorphic 1-foliation on a closed 3-manifold M, and suppose that $\mathscr{N}$ admits a transverse foliation $\mathscr{F}$. We assume that $\mathscr{F}$ does not contain spherical leaves, otherwise the structure of $\mathscr{N}$ would be evident (a suspension of a Moebius diffeomorphism). A result of [1] allows us to perturb the foliation $\mathscr{F}$ in order to obtain a foliation $\mathscr{F}'$ all of whose compact leaves are tori and their number is finite. The perturbation is in the C^0-topology on plane fields, so that transversality with $\mathscr{N}$ will be preserved. For this reason we will assume that every compact leaf of $\mathscr{F}$ is a torus and that the number of compact leaves is finite.

If $\mathcal{F}$ has dense leaves or admits an exceptional minimal set, then the previous two sections classify $\mathcal{N}$: it is a Riemannian foliation, transversely Euclidean or hyperbolic (Examples 1, 2, 3). In this section and the next one we consider the remaining case, where $\mathcal{F}$ does not have all leaves dense and an exceptional minimal set. It follows that the set of toric leaves is nonempty.

Let M_0 be a connected component of the complement in M of the union of toric leaves, and let $\mathcal{F}_0$, $\mathcal{N}_0$ be the restriction of $\mathcal{F}$, $\mathcal{N}$ to M_0. By the theory of local minimal sets [12] there are two possibilities:

(i) $\mathcal{F}_0$ has a minimal set $\mathcal{M}$ which is exceptional or a proper leaf,

(ii) all the leaves of $\mathcal{F}_0$ are dense in M_0.

We denote the completion of M_0 by $\overline{M}_0$ and the corresponding foliations by $\overline{\mathcal{F}}_0$ and $\overline{\mathcal{N}}_0$; M_0 is a compact manifold whose boundary $\partial \overline{M}_0$ is a nonempty union of tori, $\overline{\mathcal{F}}_0$ is tangent to the boundary, and $\overline{\mathcal{N}}_0$ is transverse.

Proposition 8.1. *If some leaf of $\mathcal{F}_0$ is not dense, then $\overline{M}_0$ is diffeomorphic to either $\overline{\mathbf{D}} \times S^1$ or $\mathbf{T}^2 \times [0, 1]$, and $\overline{\mathcal{N}}_0$ is either a foliation of the type $\mathcal{N}_\lambda$, $0 < |\lambda| < 1$ (Example 6), or a trivial fibration by intervals over $\mathbf{T}^2$.*

Proof. Let $T_1, \cdots, T_k$ be the tori on which the minimal set $\mathcal{M}$ of $\mathcal{F}_0$ accumulates. Because $\mathcal{M}$ is not locally dense, the holonomy of every T_j is cyclic, by Kopell's lemma [12]. For every j let $S_j \subset M_0$ be a torus isotopic and close to T_j, and transverse to $\mathcal{F}_0$ and $\mathcal{N}_0$; let U_j be the open set diffeomorphic to $\mathbf{T}^2 \times (0, 1)$ bounded by S_j and T_j.

Every leaf of $\mathcal{F}_0|_{U_j \cup S_j}$ is topologically an annulus, and holomorphically a punctured closed disc $\overline{\mathbf{D}}^* = \overline{\mathbf{D}} \backslash \{0\}$, so that there exists a diffeomorphism $\phi_j : U_j \cup S_j \to \overline{\mathbf{D}}^* \times S^1$ holomorphic on leaves of $\mathcal{F}_0$, which maps $\mathcal{F}_0|_{U_j \cup S_j}$ to the foliation on $\overline{\mathbf{D}}^* \times \mathbf{S}^1$ whose leaves are $\overline{\mathbf{D}}^* \times \{*\}$. The 1-foliation $\mathcal{N}_0|_{U_j \cup S_j}$ is mapped by ϕ_j to a foliation transverse to the boundary $\partial \overline{\mathbf{D}}^* \times \mathbf{S}^1 \simeq \mathbf{T}^2$ and to every leaf $\overline{\mathbf{D}}^* \times \{*\}$. See Figure 2.

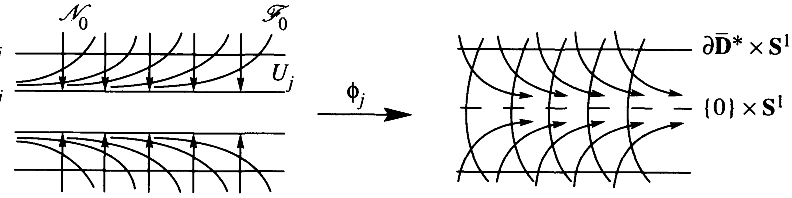

FIGURE 2

The first return map defined by $\phi_{j*}(\mathscr{N}_0|_{U_j \cup S_j})$ on every $\overline{\mathbf{D}}^* \times \{*\}$ is holomorphically conjugated to the linear map $z \mapsto \lambda_j z$, for some $\lambda_j \in \mathbf{D}^*$ (by Riemann extension theorem, Schwarz lemma, and Poincaré linearization theorem). In other words, $\phi_{j*}(\mathscr{N}_0|_{U_j \cup S_j})$ (and hence $\mathscr{N}_0|_{U_j \cup S_j}$) is equivalent to a 1-foliation of the type $\mathscr{N}_\lambda$ with the closed leaf removed.

We now glue these closed leaves to $\overline{M}_0 \backslash \{T_1, \cdots, T_k\}$. Let M_1 be the compact 3-manifold obtained from $\overline{M}_0 \backslash \{T_1, \cdots, T_k\}$ by glueing, for every $j = 1, \cdots, k$, a copy of $\overline{\mathbf{D}} \times \mathbf{S}^1$ via the previous diffeomorphisms $\phi_j : U_j \cup S_j \to \overline{\mathbf{D}}^* \times S^1 \subset \overline{\mathbf{D}} \times S^1$. The complement of the natural inclusion of $\overline{M}_0 \backslash \{T_1, \cdots, T_k\}$ into M_1 is a union of circles $\gamma_1 \cdot \cdots \cdot \gamma_k$. The previous arguments show that on M_1 there are defined a transversely holomorphic 1-foliation $\mathscr{N}_1$ and a foliation $\mathscr{F}_1$ transverse to $\mathscr{N}_1$, such that $\gamma_1, \cdots, \gamma_k$ are closed hyperbolic leaves of $\mathscr{N}_1$, and $(\mathscr{N}_1, \mathscr{F}_1)$ restricted to $M_1 \backslash \{\gamma_1, \cdots, \gamma_k\}$ is equivalent to $(\overline{\mathscr{N}}_0, \overline{\mathscr{F}}_0)$ restricted to $\overline{M}_0 \backslash \{T_1, \cdots, T_k\}$. The compact manifold M_1 can have a boundary, if the tori $\{T_1, \cdots, T_k\}$ do not exhaust the boundary of $\overline{M_0}$; in this case we pass to the double, again denoted by M_1, with foliations again denoted by $\mathscr{N}_1$, $\mathscr{F}_1$.

Let us return to the minimal set $\mathscr{M}$ of $\mathscr{F}_0$. It will correspond to a minimal set $\widehat{\mathscr{M}}$ of $\mathscr{F}_1$, which intersects all the closed hyperbolic leaves $\gamma_1, \cdots, \gamma_k$ (and their doubles, if exist) of $\mathscr{N}_1$. But a transversely holomorphic 1-foliation on a closed 3-manifold with some hyperbolic closed leaf cannot admit a transverse foliation with an exceptional minimal set, by Proposition 7.1. Hence a first conclusion is that $\widehat{\mathscr{M}}$ is a compact leaf, and so $\mathscr{M}$ is a proper leaf.

By Corollary 3.2, $\widehat{\mathscr{M}}$ is a global cross section for $\mathscr{N}_1$, so that $\mathscr{N}_1$ is the suspension of a biholomorphism $f : \widehat{\mathscr{M}} \to \widehat{\mathscr{M}}$. The set $\bigcup_j \gamma_j$ intersects $\widehat{\mathscr{M}}$ in a nonempty set which is contained in the set of hyperbolic periodic points of f. It follows that:

(a) $\widehat{\mathscr{M}}$ is a sphere (and consequently M_1 is diffeomorphic to $\mathbf{S}^2 \times \mathbf{S}^1$, and $\mathscr{F}_1$ is conjugated to the foliation by spheres),

(b) $k \leq 2$, and f is a hyperbolic Moebius diffeomorphism.

It is now easy to return to $\mathscr{F}_0$ and $\mathscr{N}_0$, and to conclude the following:

(i) If $k = 1$, then $(\overline{M}_0, \overline{\mathscr{F}}_0)$ is a Reeb component, and $\overline{\mathscr{N}}_0$ is a foliation of the class $\mathscr{N}_\lambda$, $0 < |\lambda| < 1$.

(ii) If $k = 2$, then $\overline{M}_0$ is diffeomorphic to $\mathbf{T}^2 \times [0, 1]$, and $\overline{\mathscr{N}}_0$ is the obvious fibration by closed intervals over $\mathbf{T}^2$.

9. Umbilical foliations with compact leaves: a second case

We now turn to the case where the leaves of $\mathscr{F}_0$ are dense in M_0.

Proposition 9.1. *If the leaves of $\mathscr{F}_0$ are dense, then $\overline{M}_0$ is diffeomorphic to $\mathbf{T}^2 \times [0, 1]$, and $\overline{\mathscr{N}}_0$ is a fibration by intervals.*

Proof. Let $T_1, \cdots, T_k$ be the (toric) connected components of $\partial \overline{M}_0$. Due to the possible existence of some torus T_j with noncyclic holonomy, we cannot make a "transversely holomorphic surgery" as in the proof of Proposition 8.1, and so we proceed as follows.

First of all, let M_1 be the double of $\overline{M}_0$, and let $\mathscr{F}_1, \mathscr{N}_1$ be the doubles of $\overline{\mathscr{F}}_0, \overline{\mathscr{N}}_0$. We denote again by $T_1, \cdots, T_k$ the toric leaves of $\mathscr{F}_1$, arising from $\partial \overline{M}_0$.

Let U_j be a small tubular neighborhood of T_j in M_1, bounded by two tori transverse to $\mathscr{F}_1$ and $\mathscr{N}_1$. Let $u_j : U_j \to \mathbf{T}^2 \times [-1, 1]$ be a diffeomorphism, with $u_j(T_j) = \mathbf{T}^2 \times \{0\}$, and set $V_j = u_j^{-1}(\mathbf{T}^2 \times (-\frac{1}{2}, \frac{1}{2}))$.

We replace the foliations $\mathscr{N}_1$ and $\mathscr{F}_1$ by two foliations $\widehat{\mathscr{N}_1}$ and $\widehat{\mathscr{F}_1}$, on the same manifold M_1, transverse to each other, with the following properties (see Figure 3):

(a) $\widehat{\mathscr{N}_1} = \mathscr{N}_1$, $\widehat{\mathscr{F}_1} = \mathscr{F}_1$ outside the open set $\bigcup_j U_j$,

(b) in every $\overline{U}_j, \widehat{\mathscr{F}_1}$ is transverse to every torus $u_j^{-1}(\mathbf{T}^2 \times \{t\})$, $t \in [-1, 1]$, so that it is equivalent to a product of a linear foliation on $\mathbf{T}^2$ with the interval $[-1, 1]$.

(c) in every $\overline{U}_j$, $\widehat{\mathscr{N}_1}$ is tangent to T_j, transverse to $u_j^{-1}(\mathbf{T}^2 \times \{t\})$ for $t \neq 0$, and near T_j it is conjugated to the suspension of $h : S^1 \times (-1, 1) \to \mathbf{S}^1 \times (-1, 1)$, $h(\theta, x) = (\theta, \frac{1}{2}x)$.

Remark that $\widehat{\mathscr{N}_1}$ is *not* transversely holomorphic (h cannot be holomorphic!).

The foliation $\widehat{\mathscr{F}_1}$ has dense leaves, so we may consider a harmonic measure μ on M_1 with full support. To construct such a measure, we

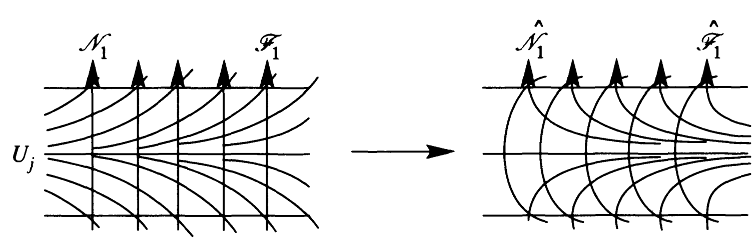

FIGURE 3

put a metric on $T\widehat{\mathscr{F}_1}$, which is Hermitian outside $\bigcup_j U_j$, where $\widehat{\mathscr{N}_1}$ is transversely holomorphic, and the leaves of $\widehat{\mathscr{F}_1}$ are holomorphic. As in §5, this measure can be used to parametrize $\widehat{\mathscr{N}_1}$ with a topological flow $\phi^t: M_1 \to M_1$. However Proposition 5.1 is not valid anymore because $\widehat{\mathscr{N}_1}$ is not transversely holomorphic in all of M_1. To avoid this difficulty, we replace ϕ^t by another flow $\psi^t: M_1 \to M_1$ (still parametrizing $\widehat{\mathscr{N}_1}$) which coincides with ϕ^t outside $\bigcup_j U_j$ and with the property that, for every $t \in \mathbf{R}$, $\psi^t|_{V_j \cap \psi^{-t}(V_j)}$ maps leaves of $\widehat{\mathscr{F}_1}|_{V_j}$ into leaves of $\widehat{\mathscr{F}_1}|_{V_j}$. This is possible thanks to the particular structure of $\widehat{\mathscr{N}_1}$ and $\widehat{\mathscr{F}_1}$ in every neighborhood U_j.

The parametrization of $\widehat{\mathscr{N}_1}$ by ψ^t has the property that the lack of harmonicity of the time needed to go from one leaf to $\widehat{\mathscr{F}_1}$ to another one is concentrated in the (compact) set $\bigcup_j(\overline{U}_j \backslash V_j)$. More precisely, after remarking that every leaf of $\widehat{\mathscr{N}_1}$ intersects $\bigcup_j(\overline{U}_j \backslash V_j)$ in at most two compact intervals, one easily sees that: *there exists a positive constant K such that if x_1, x_2 are two points in the same leaf of $\widehat{\mathscr{N}_1}$, and T is the time needed by ψ^t to go from a neighborhood of x_1 in $(\widehat{\mathscr{F}_1})_{x_1}$ to a neighborhood of x_2 in $(\widehat{\mathscr{F}_1})_{x_2}$, then $|\Delta T| \le K$*. It is important that K does not depend on x_1 and x_2.

We are now in a position very similar to that of §6. Let $\widetilde{M}_1$ be the universal cover of M_1, and $\widetilde{\mathscr{F}_1}$, $\widetilde{\mathscr{N}_1}$ be the lifts of $\widehat{\mathscr{F}_1}$, $\widehat{\mathscr{N}_1}$ ($\widetilde{M}_1$ is diffeomorphic to $\mathbf{R}^3$, $\widetilde{\mathscr{F}_1}$ is a foliation by planes).

Lemma 9.2. *There exists a diffeomorphism $d : \widetilde{M}_1 \to \mathbf{R}^3$ which maps $\widetilde{\mathscr{F}_1}$ to the foliation by horizontal planes, and $\widetilde{\mathscr{N}_1}$ to the foliation by vertical lines.*

Proof. For every $L_1, L_2 \in \widetilde{\mathscr{F}_1}$ let $\Omega(L_1, L_2) \subset L_1$ be as in §6. It is an open simply connected set, and on it there is defined a positive function T, which has bounded Laplacian ($|\Delta T| \le K$), and diverges to $+\infty$ when approaching to the boundary of $\Omega(L_1, L_2)$.

Suppose by contradiction that $\partial\Omega(L_1, L_2)$ is nonempty, and let x be one of its points. Let v be any positive C^2 function defined in a spherical neighborhood O of x in L_1 and satisfying $\Delta v \le -K$. Then the restriction of $T+v$ to $O \cap \Omega(L_1, L_2)$ satisfies the hypotheses of Sublemma 6.3, so we get the desired contradiction, and $\Omega(L_1, L_2)$ is equal to the full leaf L_1. One completes the proof working as in §6. q.e.d.

Let $\widetilde{M}_0$ be the universal cover of M_0, and $\widetilde{\mathscr{F}}_0$, $\widetilde{\mathscr{N}}_0$ be the lifts of $\mathscr{F}_0$, $\mathscr{N}_0$.

Lemma 9.3. *There exists a diffeomorphism* $q : \widetilde{M}_0 \to \mathbf{R}^3$ *which maps* $\widetilde{\mathscr{F}}_0$ *to the foliation by horizontal planes, and* $\widetilde{\mathscr{N}}_0$ *to the foliation by vertical lines.*

Proof. Remark that $(\mathscr{F}_0, \mathscr{N}_0)$ is conjugated to $(\widehat{\mathscr{F}_1}|_{M_0}, \widehat{\mathscr{N}_1}|_{M_0})$ (but $(\overline{\mathscr{F}}_0, \overline{\mathscr{N}}_0)$ is not conjugated to $(\widehat{\mathscr{F}_1}|_{\overline{M}_0}, \widehat{\mathscr{N}_1}|_{\overline{M}_0})$!). The tori $T_1, \cdots, T_k$ in M_1 lift to $\widetilde{M}_1$ to planes saturated by $\widetilde{\mathscr{N}}_1$, and a connected component C of the complement of these planes can be identified with $\widetilde{M}_0$ (such a connected component is diffeomorphic, via the d of the previous lemma, to a product of a domain in the (x, y)-plane, bounded by closed lines, with the z-axis). The universal covering of $(\widehat{\mathscr{F}_1}|_{M_0}, \widehat{\mathscr{N}_1}|_{M_0})$ can be identified with $(\widetilde{\mathscr{F}_1}|_C, \widetilde{\mathscr{N}_1}|_C)$, and the lemma follows from the previous one. q.e.d.

This means (compare §4) that $\mathscr{N}_0$ is transversely hyperbolic or transversely affine, and the same must be true for the double of $\overline{\mathscr{N}}_0$, $\mathscr{N}_1$. But the hyperbolic case cannot occur, because $\mathscr{N}_1$ admits a transverse torus, so that $\mathscr{N}_1$ is transversely affine.

Taking the list in [9] of transversely affine 1-foliations, taking into account that $\mathscr{N}_1$ is a double of some foliation, and recalling that the transverse foliation $\mathscr{F}_1$ has no Reeb component, we obtain that $\mathscr{N}_1$ is a trivial circle fibration over $\mathbf{T}^2$, and finally $\overline{\mathscr{N}}_0$ is a trivial fibration by intervals over $\mathbf{T}^2$. q.e.d.

Propositions 8.1 and 9.1 have the following consequence, which completes the proof of our theorem.

Corollary 9.4. *Let* $\mathscr{N}$ *be a transversely holomorphic 1-foliation on a closed oriented* 3*-manifold* M, *and let* $\mathscr{F}$ *be a foliation transverse to* $\mathscr{N}$, *with a finite number of compact leaves, all of them being tori. If* $\mathscr{F}$ *has not all its leaves dense and does not admit an exceptional minimal set, then* $\mathscr{N}$ *is either a suspension of an automorphism of an elliptic curve* (*Examples* 1 *and* 3) *or belongs to Example* 6.

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University Degli Studi, Bologna
Ecole Normale Supérieure de Lyon