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CONVEX DECOMPOSITIONS OF REAL PROJECTIVE SURFACES. I: π-ANNULI AND CONVEXITY

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Dedicated to the memory of Sookja Lee

Abstract

A real projective surface is a surface with a flat real projective structure. A π-annulus is an easy-to-construct real projective annulus with geodesic boundary. Let Σ be an orientable compact real projective surface with convex boundary and negative Euler characteristic. We prove that there is a π-annulus with a projective map to Σ whenever Σ is not convex.

The real projective plane $\mathbf{RP}^2$ is the quotient space of $\mathbf{R}^3 - \{O\}$ for the origin O under the equivalence relation determined by

$$\mathbf{x} \sim \mathbf{y} \quad \text{if and only if} \quad \mathbf{x} = s\mathbf{y},$$

where $s \in \mathbf{R} - \{0\}$ and $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$. The action of the general linear group $\mathrm{GL}(3, \mathbf{R})$ induces a transitive action of the projective general linear group $\mathrm{PGL}(3, \mathbf{R})$ on $\mathbf{RP}^2$. Felix Klein's Erlangen program states that real projective geometry is the study of properties of $\mathbf{RP}^2$ invariant under the action of $\mathrm{PGL}(3, \mathbf{R})$ (Goldman [10]). Given a differentiable surface, an atlas of charts to $\mathbf{RP}^2$ such that transition functions are restrictions of elements of $\mathrm{PGL}(3, \mathbf{R})$ induces real projective geometric properties locally and consistently on the surface from $\mathbf{RP}^2$. A maximal element of the collection of such atlases is said to be a *real projective structure.* (We omit the word *real* from the words *real projective* from now on since the projective structures that we work with are real projective structures.) A differentiable surface with a projective structure is said to be a *projective surface*, and an immersion from a projective surface to projective surface preserving projective structures is said to be a *projective* map. Let $\mathbf{RP}^2$ have the obvious projective structure (with a single coordinate chart). $\mathrm{PGL}(3, \mathbf{R})$ consists of projective automorphisms of $\mathbf{RP}^2$. The standard unit sphere

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$\mathbf{S}^2$ may be viewed as the quotient space of $\mathbf{R}^3 - \{O\}$ by the equivalence relation determined by

$$\mathbf{x} \sim \mathbf{y} \quad \text{if and only if} \quad \mathbf{x} = s\mathbf{y},$$

where $s \in \mathbf{R}^+$ and $\mathbf{x}$, $\mathbf{y} \in \mathbf{R}^3$ hold. Let $\mathbf{S}^2$ have the projective structure induced from its double covering map to $\mathbf{RP}^2$; the sphere $\mathbf{S}^2$ is a projective surface. The transformations induced from elements of $\mathrm{GL}(3, \mathbf{R})$ from the projective automorphism group $\mathrm{Aut}(\mathbf{S}^2)$, a Lie group isomorphic to the group $\mathrm{SL}_{\pm}(3, \mathbf{R})$ of linear automorphisms of $\mathbf{R}^3$ with determinant ± 1. Let $\mathbf{RP}^1$ be a one-dimensional subspace of $\mathbf{RP}^2$; let $\mathbf{RP}^2 - \mathbf{RP}^1$ be given the projective structure restricted from $\mathbf{RP}^2$. The complement $\mathbf{RP}^2 - \mathbf{RP}^1$ has a unique natural affine structure, such that affine automorphisms of $\mathbf{RP}^2 - \mathbf{RP}^1$ are precisely the restrictions of projective automorphisms of $\mathbf{RP}^2$ preserving $\mathbf{RP}^1$. Moreover, we give the restricted projective structure for every subsurface of $\mathbf{RP}^2$ or $\mathbf{S}^2$. Thus, every subsurface of $\mathbf{RP}^2$ or $\mathbf{S}^2$ is a projective surface. Furthermore, the quotient of such a subsurface under the properly discontinuous and free action of a projective automorphism group of the subsurface inherits a projective structure. The projective torus that is the quotient projective surface of $\mathbf{RP}^2 - \mathbf{RP}^1$ under the action of the group of projective automorphisms corresponding to affine translations by integer-valued vectors is an example. Also the Klein model of the hyperbolic plane provides more examples of quotient projective surfaces. Let $\mathrm{SO}(1, 2)$ be the group of linear automorphisms of $\mathbf{R}^3$ that leave invariant the quadratic form $-x_1^2 + x_2^2 + x_3^2$. This can be identified with a subgroup of $\mathrm{PGL}(3, \mathbf{R})$ in a standard manner. Each hyperbolic surface (in the Riemannian geometry sense) can be realized as a *hyperbolic projective surface*, i.e., the quotient projective surface of the interior of the standard disk in $\mathbf{RP}^2$ by the action of a discrete and torsion free subgroup of $\mathrm{SO}(1, 2)$. For more details, refer to Goldman [9] and Thurston [22].

A *projective homeomorphism* is a homeomorphism that is a projective map. The boundary of a projective surface is said to be convex if each boundary point has a neighborhood projectively homeomorphic to a subsurface of $\mathbf{RP}^2 - \mathbf{RP}^1$ convex under the affine geometry. (The projective surfaces that we consider are projective surfaces with convex boundary.) The main objects of our study are closed projective surfaces as in many older papers (see Goldman [9], [10], [11], Koszul [15], [16], and Kuiper [17]). However techniques of this paper apply without any difficulties to compact projective surfaces with convex boundary. Furthermore, we need to consider compact projective surfaces with geodesic boundary for purposes in [5]. Therefore, we study closed projective surfaces and compact projective surfaces with convex boundary.

A compact projective surface with nonpositive Euler characteristic is said to be *convex* if the projective surface is projectively homeomorphic to a quotient projective surface of a domain in $\mathbf{RP}^2 - \mathbf{RP}^1$ convex under the affine geometry. Such convex projective surfaces form a relatively plentiful and well-understood class of compact projective surfaces. Closed hyperbolic projective surfaces are examples. Koszul [16] showed that small projective deformations of hyperbolic projective surfaces are in the class also; more recently, Goldman [11] produced one-parameter families of deformed projective surfaces from projective triangle group constructions. Many properties of the projective surfaces in the class are made known by a line of research dating back to 1930s. Let Ω be the domain in $\mathbf{RP}^2 - \mathbf{RP}^1$ of which a closed projective surface of the class is a quotient projective surface. Benzécri [1], Kac and Vinberg [13], and Kuiper [17] proved that Ω is strictly convex, that $\operatorname{bd}\Omega$ either is a conic or does not belong to C^2, and that each closed surface in the projective surface is freely homotopic to a unique closed geodesic. Recently, Goldman [11] proved a surprising result that the deformation space of convex projective structures on a closed surface with negative Euler characteristic χ is homeomorphic to a cell of dimension -8χ. (For consequences of Goldman's result, this paper, and the next paper [5], see [6] and [11].)

Another class of well-understood compact projective surfaces consists of tori, Klein bottles, annuli, and Möbius bands with projective structures. Nagano and Yagi [20] studied projective structures on tori and Klein bottles. Goldman [8] and Sullivan and Thurston [21] produced and classified many examples of projective annuli, and Möbius bands with geodesic boundary.

Simple examples of these are π-annuli. A *great disk* is a closed hemisphere of $\mathbf{S}^2$. A *lune* is a closed disk that is the closure of a component of $\mathbf{S}^2 - l_1 - l_2$ for two distinct great circles l_1 and l_2. A lune is a proper subset of a great disk and is bounded by two geodesic segments connecting a given antipodal pair of points. Let A be a great disk or a lune, let α be a segment in $\operatorname{bd} A$ ending at two antipodal points, and let x be a point of the manifold interior α^o. The quotient projective surface of the domain $A^o \cup (\alpha^o - \{x\})$ under the properly discontinuous and free action of $\langle \vartheta \rangle$ where $\vartheta \in \operatorname{Aut}(\mathbf{S}^2)$ is said to be a *π-annulus*. The article [5] contains detailed constructions of π-annuli and their classifications. In particular,

π-annuli are shown to be compact annuli. The articles of Goldman [8], [9], [11] present related constructions and projective surfaces.

Furthermore, projective surfaces of the above two classes can be combined to produce compact projective surfaces. Given a projective surface M and subsurfaces $M_1, \cdots, M_n$ of M, we say that M is the *sum* of $M_1, \cdots, M_n$ if $M = M_1 \cup \cdots \cup M_n$, and $M_i \cap M_j$ is the union of mutually disjoint simple closed geodesics or is the empty set for every two integers i and j such that $1 \leq i < j \leq n$. Sullivan and Thurston [21] observed that many projective surfaces can be constructed by summing convex projective surfaces and projective annuli with geodesic boundary in a process known as grafting; they and Goldman [9] produced many examples of compact projective surfaces by summation (see also Maskit [18], Hejhal [12]).

A natural question that Thurston and Goldman asked around 1977 is whether such sums yield all compact projective surfaces with negative Euler characteristic. In this series of papers, we give a positive answer, the *admissible decomposition theorem* [5]. By the above method it is easy to construct a closed nonconvex projective surface by grafting a projective annulus including a π-annulus into a closed convex projective surface. Loosely speaking, we cut open a closed convex projective surface along a simple closed geodesic and past by projective maps the resulting boundary components with the appropriate boundary components of a nonconvex annulus with geodesic boundary (with matching holonomy), say obtained as in Sullivan and Thurston [21]. In [5], we show a converse result that an arbitrary compact nonconvex projective surface with negative Euler characteristic includes a π-annulus, which is sufficient to prove the admissible decomposition theorem.

Main Theorem. *Suppose that Σ is an orientable compact projective surface with convex boundary and negative Euler characteristic, and that Σ is not convex. Then there is a π-annulus Λ with a projective map $\Phi: \Lambda \to \Sigma$.*

Given a projective surface M, its universal cover $\widetilde{M}$ has a uniquely induced projective structure. Every projective map defined on an open subset of $\widetilde{M}$ to $\mathbf{S}^2$ extends to a global projective map from $\widetilde{M}$. A projective structure on M therefore has an $\mathbf{S}^2$-development pair $(\mathbf{dev}, h)$ where $\mathbf{dev}: \widetilde{M} \to \mathbf{S}^2$ is an immersion, said to be a developing map, where $h: \pi_1(M) \to \mathrm{Aut}(\mathbf{S})^2)$ is a homomorphism, said to be a holonomy homomorphism. Every $\mathbf{S}^2$-development pair $(\mathbf{dev}', h'(\cdot))$ equals $(\psi \circ \mathbf{dev}, \psi \circ h(\cdot) \circ \psi^{-1})$ where $\psi \in \mathrm{Aut}(\mathbf{S}^2)$. (Note that M is developed into $\mathbf{S}^2$ instead of $\mathbf{RP}^2$.)

The sphere $\mathbf{S}^2$ has the distance metric $\mathbf{d}$ induced from the standard Riemannian metric μ of constant curvature 1, such that arcs are geodesic if and only if they are geodesic in the Riemannian sense (up to parameter changes), i.e., represented by arcs in great circles. The developing map **dev** of M induces the Riemannian metric on $\widetilde{M}$, denoted by μ again, and μ induces a distance metric on $\widetilde{M}$, denoted by $\mathbf{d}$ again. $\mathbf{d}$ and μ are said to be the *spherical metric* and the *spherical Riemannian metric* of $\widetilde{M}$ *associated with* **dev** respectively. Let $\check{M}$ denote the metric completion of $\widetilde{M}$; let $\widetilde{M}_\infty$ denote $\check{M} - \widetilde{M}$. The sets $\widetilde{M}$ and $\widetilde{M}_\infty$ are a dense open subset and a closed subset of $\check{M}$ respectively. $\check{M}$ and $\widetilde{M}_\infty$ have well-defined topology (independently of the choice of the $\mathbf{S}^2$-development pair). We call $\check{M}$ the *projective completion* of M, and call $\widetilde{M}_\infty$ the *ideal boundary* of $\widetilde{M}$, and its elements *ideal points*. **dev** extends uniquely to a distance decreasing map from $\check{M}$, and each deck transformation ϑ extends uniquely to a self-homeomorphism of $\check{M}$, preserving $\widetilde{M}$ and $\widetilde{M}_\infty$. By abuse of terminology, these extended maps are said to be a *developing map* and a *deck transformation* respectively and are denoted by **dev** and ϑ respectively. A *developing image* of a subset of $\check{M}$ is the image of the subset under a given developing map. (See [14].)

To prove the main theorem, we obtain a geometric object, a crescent, and refine it into a π-annulus. §1 provides the foundations for geometry, and an appendix discusses convergence of convex subsets of a projective completion. The second part, §§2, 3, and 4, discusses geometric objects, i.e., crescents, concave subsets, and great disks for compact projective surfaces. Let $\check{\Sigma}$ be a projective completion of Σ. The last part, consisting of §§5, 6, and 7, presents the proof, which consists of obtaining three geometric objects sequentially:

(I) a crescent in $\check{\Sigma}$,

(II) a simplistic crescent in $\check{\Sigma}$, and

(III) a π-annulus with a projective map to Σ.

In §1, convexity is defined, convex subsets are classified, and their intersection properties are explored. In the appendix, we discuss a criterion for the convergence of sequences of convex disks in the projective completions.

Crescents are discussed in §2. In particular, the *transversal intersection property* is crucial.

A *great disk* or a *lune* in $\check{M}$ is a subset such that the developing map of M restricted to it is an imbedding onto a great disk or a lune respectively. A subset $\mathscr{R}$ of $\check{M}$ is said to be a *crescent* if $\mathscr{R}$ is a great disk or a lune,

$\mathscr{R}^o \subset \widetilde{M}$, $\delta\mathscr{R} \cap \widetilde{M} \neq \varnothing$, and $\delta\mathscr{R} \cap \widetilde{M}_\infty$ includes a great line, i.e., a line connecting an antipodal pair of points. An example of a crescent is the projective completion of a π-annulus. Let B be a π-annulus that is the quotient projective surface of $(L^o \cup \alpha^o) - \{x\}$ where L is a lune, α is an edge of L, and x is a point of α^o. The universal cover $\widetilde{B}$ can be identified with $(L^o \cup \alpha^o) - \{x\}$. Then the projective completion of B can be identified with L. Since $L^o \subset \widetilde{B}$, $\delta L \cap \widetilde{B} = \alpha^o - \{x\}$, and $\delta L \cap (L - \widetilde{B})$ includes the other edge of L, it follows that L is a crescent. (For more examples, see §2 and Figure 1.)

In §3 we generalize the notion of crescents to obtain sets that are invariant under deck transformations. A nonempty connected subset A of $\widetilde{M}$ is said to be *concave* if A satisfies the following (a), (b), and (c):

(a) Each component of $\operatorname{bd} A \cap \widetilde{M}$ is a maximal line in $\widetilde{M}$ and is an open line.

(b) Each component of $\operatorname{bd} A \cap \widetilde{M}$ is a subset of $\delta\mathscr{R}$ for a crescent or a great disk $\mathscr{R}$ included in A.

(b′) Each component of $\operatorname{bd} A \cap \widetilde{M}$ is a subset of $\delta\mathscr{R}$ for a crescent $\mathscr{R}$ included in A.

(c) If $\operatorname{int}(A \cap \widetilde{M}) \cap \vartheta(A) \neq \varnothing$ for a deck transformation ϑ, then $A = \vartheta(A)$.

In condition (b), $\mathscr{R}$ may be a great disk that is not a crescent. If A satisfies (a), (b′), and (c), then A is *properly concave*. A line in $\widetilde{M}$ is *simplistic* if it corresponds to a non-null-homotopic simple closed curve in M, and a crescent $\mathscr{R}$ is *simplistic* if a component of $\delta\mathscr{R} \cap \widetilde{M}$ is simplistic. In a sum projective surface S constructed by Sullivan and Thurston and Goldman, a properly concave set is realized as a connected portion of the completion $\check{S}$ corresponding to attached annuli. In §6, examples of concave sets will be obtained by taking a union of crescents.

Concavity Lemma. *Suppose that A is a concave subset of $\widetilde{M}$ and that α is a component of* $\operatorname{bd} A \cap \widetilde{M}$. *Then α is a simplistic. Furthermore, if A is properly concave, then A includes a simplistic crescent $\mathscr{R}$ such that $\alpha \subset \delta\mathscr{R}$.*

In §4, we show by the concavity lemma that if $\widetilde{M}$ includes a great disk, then M is a great disk or is projectively homeomorphic to $\mathbf{RP}^2$ or $\mathbf{S}^2$. This implies that if $\chi(M) \leq 0$, then geodesics in $\widetilde{M}$ are imbeddings into $\widetilde{M}$.

We begin the proof of the main theorem in §5, where we obtain a crescent in $\check{\Sigma}$ completing the first step (I). First, find a triangle with an edge ν intersecting an ideal point. Choose a sequence of points of ν converging to a ideal point. Pull back these points by deck transformations to a compact neighborhood of a fundamental domain of the universal cover $\tilde{\Sigma}$ along with the triangle. The techniques of the appendix show that an appropriate sequence of the image triangles does not degenerate and, hence, a subsequence converges to a crescent.

In §6, we refine the previously obtained crescent to a simplistic crescent (II). The methods depend on whether $\check{\Sigma}$ includes a great-disk crescent or not. Suppose that $\check{\Sigma}$ includes a great-disk crescent. Since the crescent itself is properly concave, the concavity lemma implies that the crescent is simplistic. Suppose that $\check{\Sigma}$ includes no great-disk crescent. We define an equivalence relation on the collection of crescents. Two overlapping crescents are considered equivalent, and this generates the equivalence relation. Given a crescent $\mathscr{R}$, the union $\Lambda(\mathscr{R})$ of crescents equivalent to $\mathscr{R}$ is properly concave. The concavity lemma yields that $\Lambda(\mathscr{R})$ includes a simplistic crescent.

Last, §7 completes step (III) and the proof of the main theorem. We show that a simplistic crescent is almost the universal cover of a π-annulus.

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1. Introductory material on projective surfaces

We discuss notation in §1.1, list examples of projective surfaces in §1.2, and in §1.3 we define convex subsets, show that a simply convex subset is a subset of an ellipsoid, introduce the notion of dimension, and classify convex sets according to dimension. In §1.4, given a projective surface, we introduce our notion of convexity for the subsets of the projective completion of the surface. The definition differs from those in the older papers (see Benzécri [1], Goldman [9], [10], [11], and Kuiper [17]), and has better properties. In relevant situations, a developing map restricted to convex subsets are imbeddings. Also, the properties stated in Theorem 1.7 do not hold for the older definitions. Since we cannot prove that the closure of a convex subset is convex, we introduce the notion of tame subsets. The closure of a tame subset is tame, and the developing map restricted to a tame subset is an imbedding. We classify tame subsets according to dimension and show that the preliminary notion of convexity given in the introduction agrees with the definition given here.

Next, we compare the intersection of two subsurfaces of the universal cover and the intersection of their developing images, using this, explore the intersection properties of convex subsets in §1.7, and then discuss the dipping intersection property in §1.9.

Lastly, we introduce the notion of tiny disks, which we often use to understand local phenomena, and, further, exhibit a compatibility relation of the spherical metric and a complete metric on the universal cover restricted to a tiny disk.

1.1. Let X be a metric space with distance function η, let A be a compact subset of X, and let $x \in X$. We define the distance between x and A by

$$\eta(x, A) = \eta(A, x) = \inf\{\eta(x, y) | y \in A\}.$$

When A and B are compact subsets of X, the Hausdorff distance between A and B is given by

$$\eta^H(A, B) = \sup\{\eta(x, B), \eta(A, y) | x \in A, \ y \in B\}.$$

It is well known that if X is compact, then the collection of compact subsets of X forms a compact metric space with distance function η^H.

An η-ball of radius r, where $r \geq 0$, around x is the subset

$$B_\eta(x, r) = \{y \in X | \eta(x, y) \leq r\}.$$

Let M be a projective surface. In this paper, given a subset A of a projective completion $\check{M}$ or the universal cover $\widetilde{M}$, let $\operatorname{bd} A$ and $\operatorname{int} A$ respectively denote the topological boundary and the topological interior of A with respect to the topology of $\check{M}$. In this paper, an *imbedding* is a topological imbedding. Let A be a submanifold of $\widetilde{M}$ or an imbedded manifold in $\check{M}$. We agree that δA and A^o denote the boundary and the interior of A as a manifold respectively. In general, $\operatorname{bd} A$ and $\operatorname{int} A$ differ from δA and A^o respectively.

A geodesic is a continuous map $\alpha: I \to M$ where I is an interval of $\mathbf{R}$ that is locally geodesic with respect to the charts of M. A subset of $\mathbf{S}^2$ is a *line* if it is the image of an injective geodesic and is not homeomorphic to a circle. A *segment* is the image of an injective geodesic defined on a closed interval of $\mathbf{R}$. A segment is an imbedded arc and is a line. Lastly, disks or lunes that we consider are compact unless we say otherwise.

1.2. Let us list some examples of projective surfaces (see Figure 1), which are simple in the sense that their developing maps are inclusion maps. To obtain more complicated examples, add these to a hyperbolic projective surface. However, the resulting surfaces are too complicated to be pursued here.

First, let us introduce a *homogeneous coordinate system* on $\mathbf{S}^2$. Given a vector (x_1, x_2, x_3) in $\mathbf{R}^3 - \{O\}$, let $[x_1, x_2, x_3]$ denote the semi-infinite, open line starting at O and passing through (x_1, x_2, x_3). Each point of $\mathbf{S}^2$ equals a unique open line $[x_1, x_2, x_3]$. Let $e_1 = [1, 0, 0]$, let $e_2 = [0, 1, 0]$, and let $e_3 = [0, 0, 1]$. Given mutually distinct points x, y, and z, we use following notations. $-x$: the antipodal point of x; $\overline{xyz}$: the shortest segment with endpoints x and z and passing through y; let $\overline{xy}$: the shortest segment with endpoints x and y whenever $x \neq -y$. Let B_2 denote the lune bounded by segments $\overline{e_2 e_1 - e_2}$ and $\overline{e_2 e_3 - e_2}$; let B_3 denote the lune bounded by $\overline{e_3 e_1 - e_3}$ and $\overline{e_3 e_2 - e_3}$. Let H_A be the open great disk defined by $x_1 > 0$. Let $T_{\lambda_1, \lambda_2, \lambda_3}$ denote the projective automorphism of $\mathbf{S}^2$ corresponding to the diagonal matrix with eigenvalues λ_1, λ_2, and λ_3 with respective eigenvectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ (see Figure 1, next page).

Let Σ_1 be the quotient projective surface of $(B_2 \cap B_3 \cap H_A) - \{e_1\}$ by the action of $\langle T_{3,1,1/3} \rangle$, i.e., the subgroup generated by $T_{3,1,1/3}$. Let Σ_2 be the quotient projective surface of $(B_2 \cap H_A) - \{e_1\}$ by the action of $\langle T_{3,1,1/3} \rangle$. Let Σ_3 be the quotient projective surface of $H_A \cup \overline{e_2 e_3}^o \cup \overline{e_2 e_3}^o$ by the action of

$$\left\langle \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1/4 \end{bmatrix} \right\rangle .$$

Let Σ_4 be the quotient projective surface of $((B_2 \cup B_3) - \{e_1\}) \cap H_A$ by the action of $\langle T_{4,1/2,1/2} \rangle$. Finally, let Σ_5 be the quotient projective surface of $H_A - \{e_1\}$ by the action of $\langle T_{4,1/2,1/2} \rangle$.

The projective surface Σ_1 is a compact convex projective annulus. Σ_2 and Σ_3 are π-annuli, Σ_4 is the union of two π-annuli that overlap on a compact convex projective annulus, and Σ_5 is the sum of two π-annuli.

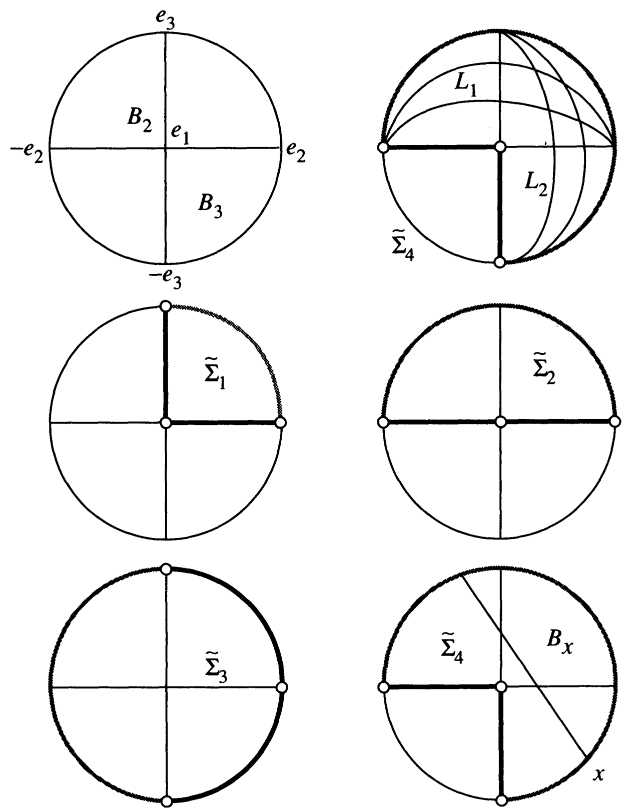

FIGURE 1

For each $i = 1, 2, 3, 4$, we may identify the universal cover $\tilde{\Sigma}_i$ of Σ_i with the domain of which Σ_i is a quotient, and identify a developing map with the inclusion map. Each point x of $\mathrm{Cl}(H_A) - \{e_1\}$ has coordinates (ρ, θ), $0 < \rho \leq \pi/2$, $0 \leq \theta < 2\pi$, where ρ denotes the **d**-distance from e_1 to x, and θ denotes the oriented μ-angle between $\overline{e_1 x}$ and $\overline{e_1 e_2}$. (Recall from the introduction the standard Riemannian metric μ of curvature 1 on $\mathbf{S}^2$ and the associated distance metric $\mathbf{d}$.) The universal cover $\tilde{\Sigma}_5$ of Σ_5 may be identified with $\{(\rho, \theta) | 0 < \rho < \pi/2, \theta \in \mathbf{R}\}$. A developing map sends the element of the form (ρ, θ), $0 < \rho < \pi/2$, $\theta \in \mathbf{R}$, to (ρ, θ') where $0 \leq \theta' < 2\pi$ and $\theta' = \theta \pmod{2\pi}$. This is an infinite-cycle covering map.

By identification by the (extended) developing map of Σ_1, we have

$$\check{\Sigma}_1 = B_2 \cap B_3 \quad \text{and} \quad \Sigma_{1,\infty} = \{e_1\} \cup \overline{e_2 e_3}.$$

Similarly, we have

$$\check{\Sigma}_2 = B_2, \qquad \tilde{\Sigma}_{2,\infty} = \{e_1\} \cup \overline{e_2e_3 - e_2};$$
$$\check{\Sigma}_3 = \mathrm{Cl}(H_A), \qquad \tilde{\Sigma}_{3,\infty} = \{e_2\} \cup \overline{e_3 - e_2 - e_3};$$
$$\check{\Sigma}_4 = B_2 \cup B_3, \qquad \tilde{\Sigma}_{4,\infty} = \{e_1\} \cup \overline{-e_2e_2 - e_3}.$$

The completion $\check{\Sigma}_5$ can be identified with the quotient space of $\{(\rho, \theta)|0 \leq \rho \leq \pi/2, \theta \in \mathbf{R}\}$ by the equivalence relation given by

$$(\rho_1, \theta_1) \sim (\rho_2, \theta_2) \quad \text{if and only if} \quad \rho_1 = \rho_2 = 0$$

where $\theta_1, \theta_2 \in \mathbf{R}$. The developing map sends the equivalence class $[\rho, \theta]$ of (ρ, θ), $\rho > 0$, $\theta \in \mathbf{R}$, to (ρ, θ') where $0 \leq \theta' < 2\pi$ and $\theta' = \theta \pmod{2\pi}$; the developing map sends $[0, 0]$ to e_1.

1.3. Because of antipodes, the set of lines in $\mathbf{S}^2$ divides into three projectively invariant classes: the first is the set of great lines, the second is the set of simply convex lines, and the third is the set of nonconvex lines. A *great line* is a line whose endpoints are antipodal, a *simply convex line* is a line that lies properly in an open great line, and a *nonconvex line* is a line that includes a great segment properly. We may characterize these lines as follows: a line is great if and only if its $\mathbf{d}$-length equals π, a line is simply convex if and only if its $\mathbf{d}$-length is less than π, and a line is nonconvex if and only if its $\mathbf{d}$-length is greater than π.

Great lines and simply convex lines are called *convex lines*. A subset A of $\mathbf{S}^2$ is said to be *convex* if given every two points of A, the subset A includes a convex segment connecting two points. For convenience, the empty set is not considered convex. A great circle and a lune are convex. A component of $\mathbf{S}^2 - \mathbf{S}^1$ for a great circle $\mathbf{S}^1$ is an open great disk, which is convex, and is projectively homeomorphic to $\mathbf{RP}^2 - \mathbf{RP}^1$ where $\mathbf{RP}^1$ is a one-dimensional subspace of $\mathbf{RP}^2$. Similarly to the case of $\mathbf{RP}^2 - \mathbf{RP}^1$, the open great disk has a unique natural affine structure. A subset of the open great disk is convex if and only if it is convex under the affine geometry of the open great disk. The universal covering domain of a π-annulus is not convex.

A convex subset A of $\mathbf{S}^2$ is said to be *simply convex* if there is a positive constant c such that the $\mathbf{d}$-length of every segment in A is less than $\pi - c$. An *ellipsoid* is a component of the subset of $\mathbf{S}^2$ of points corresponding to nonzero vectors in $\mathbf{R}^3$ with nonnegative values under a given nondegenerate quadratic form with index of inertia 1. An ellipsoid is a disk: a subset is an ellipsoid if and only if it is the standard unit disk with respect to an affine coordinate system of an open great disk including it. An ellipsoid is simply convex, and so is $B_2 \cap B_3$ (see §1.2). A dense convex subset of a great disk or a lune is not simply convex.

Moreover, a simply convex subset is a subset of an ellipsoid and hence is a precompact subset of an open great disk. Let A be a simply convex subset of $\mathbf{S}^2$; let x be the point in the complement of A such that $\mathbf{d}(x, A)$ achieves the maximum value of $\{\mathbf{d}(y, A) | y \in \mathbf{S}^2\}$. The simple convexity yields that $\mathbf{d}(x, A) > \pi/2$ and hence that a $\mathbf{d}$-ball whose radius is less than π, is an ellipsoid including A.

In $\mathbf{S}^2$, there are three types of great spheres: a *great sphere of dimension zero*, or the set consisting of two antipodal points; a *great sphere of dimension one*, or a great circle; and a *great sphere of dimension two*, or $\mathbf{S}^2$ itself. Each convex subset A is a subset of a unique great sphere of dimension i for $i = 0, 1, 2$ such that no great sphere of lower dimension includes A; here, i is said to be the *dimension* of A. A convex subset of dimension zero is a set of consisting of a single point; a convex subset of dimension one is a great circle, a great line, or a simply convex line; and a compact convex subset of dimension two equals $\mathbf{S}^2$, a great disk, a lune, or a simply convex subset. Since a compact simply convex subset is a compact subset of an open great disk convex under the affine geometry, it is a disk when its dimension is two. Hence, a compact convex subset of dimension two is a disk whenever it is not identical with $\mathbf{S}^2$. The closure of a convex subset is convex, the closure of a simply convex subset is simply convex, and a dense convex subset of $\mathbf{S}^2$ equals $\mathbf{S}^2$. Therefore, a convex subset of dimension two equals $\mathbf{S}^2$ or a dense convex subset of a great disk, a lune, or a simply convex disk.

Our notion of convexity is similar to the affine convexity (see [2]). Let A be a convex subset of $\mathbf{S}^2$. Then $\operatorname{int} A$ is a convex subset whenever $\operatorname{int} A \neq \emptyset$. Also $\mathrm{Cl}(A)$ is convex. Suppose that the dimension of A is two and A is not identical with $\mathbf{S}^2$. Then $\operatorname{int} A$ is an open disk, $\mathrm{Cl}(A)$ is a compact disk, $\operatorname{int}\mathrm{Cl}(A) = \operatorname{int} A$, and $\operatorname{bd} A$ is homeomorphic to a circle. (These hold also if A is compact and if $\operatorname{int} A$, $\operatorname{int}\mathrm{Cl}(A)$, and $\operatorname{bd} A$ are replaced by A^o, $\mathrm{Cl}(A)^o$, and δA respectively.) If x is a point not belonging to $\operatorname{int} A$, then $\mathbf{S}^2$ includes a great disk H such that $x \in H$ and $A \cap H^o = \emptyset$. Suppose that A is compact, and α is an imbedded geodesic satisfying $\alpha \cap A^o \neq \emptyset$. Let $\beta = \alpha \cap A$. The subset β is a connected line such that $\beta^o \subset A^o$. If furthermore α is a great circle, then $\delta\beta \subset \delta A$.

However, the intersection properties of our notion of convexity differ slightly from the properties of affine convexity. Let A and B be two convex subsets of $\mathbf{S}^2$. Then $A \cap B$ is either the pair of antipodal points or a convex subset or an empty set. Hence, if $A \cap B$ contains at least three points or the pair of nonantipodal points, then $A \cap B$ is convex.

1.4. Let M be an arbitrary projective surface, and let $\widetilde{M}$ be its universal cover. Let $(\mathbf{dev}, h)$ be its $\mathbf{S}^2$-development pair, and let $\check{M}$ and $\widetilde{M}_\infty$ be the projective completion and the ideal boundary of $\widetilde{M}$ respectively; let $\mathbf{d}$ be the spherical metric associated with $\mathbf{dev}$. The notion of convexity for subsets of $\mathbf{S}^2$ extends to a notion for subsets of $\check{M}$. We first generalize the notion of lines. A continuous map $\alpha: I \to \check{M}$, where I is an interval of $\mathbf{R}$, is a geodesic if $\mathbf{dev} \circ \alpha$ is a geodesic. A subset of $\check{M}$ is a *line* if it is the image of an injective geodesic and is not homeomorphic to a circle; the subset is a *segment* if it is the image of an injective geodesic defined on a closed interval. Define the spherical length or the $\mathbf{d}$-length of a line α to be the $\mathbf{d}$-length of the geodesic corresponding to $\mathbf{dev}|\alpha$. The subset $\{[\pi/2, \theta] | \theta \in \mathbf{R}\}$ of $\check{\Sigma}_5$ is a line whose spherical length is infinite. If the $\mathbf{d}$-length of α equals π, then α is said to be *great*; if the $\mathbf{d}$-length of α is less than π, then α is said to be *simply convex*. In $\check{\Sigma}_4$, $\overline{e_2 e_3 - e_2}$, $\overline{e_2 e_3}^o$, and $\overline{-e_2 e_3 - e_3}^o$ are a great segment, a simply convex line, and a line respectively (see §1.2).

Great lines and simply convex lines are again said to be *convex*. ($\mathbf{dev}$ restricted to a convex line corresponds to an injective geodesic onto a convex line in $\mathbf{S}^2$.) A subset A of $\check{M}$ is *convex* if given points x and y of A, A includes a convex segment containing x and y. A convex subset A of $\check{M}$ is *simply convex* if there is a positive constant c such that the $\mathbf{d}$-length of each segment in A is less than $\pi - c$. A lune is convex but not simply convex. The line of $\check{\Sigma}_5$, $\{[\pi/2, \theta] | \theta \in \mathbf{R}\}$, is not convex.

Unfortunately, we cannot prove that the closure of a convex subset is convex at the moment. If A is a convex subset of $\widetilde{M}$, then $\mathbf{dev}|A$ is an isometry with respect to $\mathbf{d}$ on $\widetilde{M}$ and $\mathbf{d}$ on $\mathbf{S}^2$. Since $\mathbf{d}$ on $\check{M}$ is complete, $\mathbf{dev}|\,\mathrm{Cl}(A)$ for the closure $\mathrm{Cl}(A)$ of A in $\check{M}$ is an imbedding onto $\mathrm{Cl}(\mathbf{dev}(A))$. If A is a convex subset of a convex compact subset B of $\check{M}$, then $\mathbf{dev}|B$ is an imbedding onto $\mathbf{dev}(B)$. We say that A is a *tame* subset if A is a convex subset of $\widetilde{M}$ or a convex subset of a convex compact subset of $\check{M}$. It follows that if A is tame, then $\mathbf{dev}|A$ is an imbedding onto $\mathbf{dev}(A)$. Moreover if A is a tame, then $\mathbf{dev}|\,\mathrm{Cl}(A)$ is an imbedding onto a convex subset $\mathrm{Cl}(\mathbf{dev}(A))$. Therefore, $\mathrm{Cl}(A)$ is a convex subset of $\check{M}$ and is tame. (In $\check{\Sigma}_4$, the subsets $B_2^o \cap B_3^o$ and $\overline{e_2 e_3 - e_2}^o$ are tame subsets.)

Tame subsets may be classified according to their developing images. For example, if A is compact, and $\mathbf{dev}(A)$ is a lune, then A is said to be a *lune*. If $\mathbf{dev}(A)$ is identical with $\mathbf{S}^2$, then A is said to be a *great sphere*. The *dimension* of A is the dimension of $\mathbf{dev}(A)$. A tame subset of dimension zero is the set of a point. A tame subset of dimension one is a great circle or a convex line, which may be a great line or a simply convex line. A compact tame subset of dimension two is a great sphere, a great disk, a lune, or a simply convex disk. Since a dense convex subset of $\mathbf{S}^2$ equals $\mathbf{S}^2$, a tame subset of dimension two is either a great sphere or a dense convex subset of a great disk, a lune, or a simply convex disk.

In this paper, convex disks or lunes that we consider are tame and are such that their manifold interiors are subsets of $\widetilde{M}$ unless we say otherwise. (Recall also from §1.1 that disks and lunes that we consider are compact unless we say otherwise.)

1.5. We say that M is *convex* (resp. *simply convex*) if $\widetilde{M}$ is convex (resp. simply convex). For example, hyperbolic projective surfaces are simply convex. Σ_1 is simply convex, π-annuli are not convex, and Σ_4 and Σ_5 are not convex. The following lemma shows that this definition is consistent with the preliminary definition given in the introduction. (Note that the open great disk G in the following lemma may be replaced by $\mathbf{RP}^2 - \mathbf{RP}^1$ for a one-dimensional subspace $\mathbf{RP}^1$.)

Lemma. *Suppose that M is compact. Then M is convex if and only if one of the following holds*:

(1) $\chi(M) > 0$, *and M is projectively homeomorphic to* $\mathbf{RP}^2$, $\mathbf{S}^2$, *or a convex disk in* $\mathbf{S}^2$.

(2) $\chi(M) = 0$, *and M is projectively homeomorphic to a quotient projective surface of a convex domain in an open great disk G.*

(3) $\chi(M) < 0$, *and M is projectively homeomorphic to a quotient projective surface of a bounded convex domain in an open great disk G; thus, M is simply convex.*

Proof. Suppose that M is convex. $\widetilde{M}$ is thus tame, and $\mathbf{dev}|\widetilde{M}$ is an imbedding onto $\mathbf{dev}(\widetilde{M})$. Let Γ be the image of the holonomy homomorphism h associated with $\mathbf{dev}$. There is a commutative diagram

$$\begin{array}{ccc}
\widetilde{M} & \xrightarrow{\mathbf{dev}} & \mathbf{dev}(\widetilde{M}) \\
{\scriptstyle \mathbf{p}}\downarrow & & {\scriptstyle q}\downarrow \\
M & \xrightarrow{\mathbf{dev}'} & \mathbf{dev}(\widetilde{M})/\Gamma
\end{array}$$

where $\mathbf{p}$ denotes the universal covering map, q the quotient map, and $\mathbf{dev}'$ the induced map. It is easy to see that $\mathbf{dev}'$ is a projective homeomorphism.

Since $\mathbf{dev}(\widetilde{M})$ is a convex subset of $\mathbf{S}^2$, it follows that $\mathbf{dev}(\widetilde{M})$ is a great sphere or that $\mathrm{Cl}(\mathbf{dev}(\widetilde{M}))$ is a great disk, a lune, or a simply convex disk (see §1.3).

If $\mathbf{dev}(\widetilde{M})$ is a great sphere, then $\mathbf{dev}$ is a homeomorphism to $\mathbf{S}^2$. Thus, M is as in (1).

Suppose that $\mathbf{dev}(\widetilde{M})$ is not a great sphere. If $\chi(M) > 0$, then $\widetilde{M}$ is compact. Hence, $\mathbf{dev}(\widetilde{M})$ is a compact convex disk, $\mathbf{p}$ is a trivial covering map, and M is as in (1).

We will now assume that $\chi(M) \leq 0$. Suppose that $\mathrm{Cl}(\mathbf{dev}(\widetilde{M}))$ is a great disk. Then $\mathrm{Cl}(\mathbf{dev}(\widetilde{M})) = H$ for a great disk H. Let $\alpha = \mathbf{dev}(\widetilde{M}) \cap \delta H$. We claim that $\alpha = \varnothing$. Since $\mathbf{dev}(\widetilde{M}^o) = H^o$, it follows that $\alpha = \mathbf{dev}(\delta \widetilde{M})$; thus, α is a boundaryless one-dimensional manifold in δH or the empty set. Since $\mathbf{dev}(\widetilde{M})$ is a convex subset of H, α is convex or empty. Thus α is either δH itself, a great line, a simply convex line, or the empty set. Since $\chi(M) \leq 0$, α does not equal δH. Suppose that α is a great line or a simply convex line. The unique boundary component of $\mathbf{dev}(\widetilde{M})$ equals α. Since $\mathbf{dev}(\widetilde{M})$ is Γ-invariant, so is α. Since $\mathbf{dev}(\widetilde{M}) = \alpha \cup H^o$, $\mathbf{dev}(\widetilde{M})/\Gamma$ is a compact surface with unique boundary component α/Γ. Since the action of Γ on α is properly discontinuous and free, $\mathbf{dev}(\widetilde{M})/\Gamma$ is homotopy equivalent to α/Γ. However, no surface with connected boundary is homotopy equivalent to its unique boundary component. This is a contradiction. Therefore, $\alpha = \varnothing$, and $\mathbf{dev}(\widetilde{M}) = H^o$.

Since H^o has an affine structure, and elements of Γ preserve H^o and hence restrict to affine automorphisms of H^o, it follows that M has an affine structure. By Milnor [19], $\chi(M) = 0$. Since $\mathbf{dev}'$ is a projective homeomorphism, M is as in (2).

Suppose that $\mathrm{Cl}(\mathbf{dev}(\widetilde{M}))$ is a lune. Then the vertices of the lune is not a subset of $\mathbf{dev}(\widetilde{M})$ since they are Γ-invariant. Thus, $\mathbf{dev}(\widetilde{M})$ is a subset of an open hemisphere. Also, similarly to Lemma 2.5 [11], $\chi(M) = 0$. Thus, M is as in (2).

Suppose that $\mathrm{Cl}(\mathbf{dev}(\widetilde{M}))$ is a simply convex disk. Then, clearly, $\mathrm{Cl}(\mathbf{dev}(\widetilde{M}))$ is a bounded convex subset of an open hemisphere, and $\widetilde{M}$ and M are simply convex; M is as in (2) or (3).

The converse part follows from the obvious fact that quotient surfaces in (1), (2), and (3) are convex.

1.6. Now we study intersection properties of convex subsets of $\widetilde{M}$ beginning with the following lemma.

Lemma. *Let A and B be open subsurfaces of $\widetilde{M}$. Suppose that $A \cap B \neq \varnothing$,.* $\mathbf{dev}|A$ *and* $\mathbf{dev}|B$ *are imbeddings, and* $\mathbf{dev}(A) \cap \mathbf{dev}(B)$ *is* *a path-connected subset of* $\mathbf{S}^2$. *Then* $\mathbf{dev}|A \cap B$ *is an imbedding onto* $\mathbf{dev}(A) \cup \mathbf{dev}(B)$. *Moreover,* $\mathbf{dev}|A \cup B$ *is an imbedding onto* $\mathbf{dev}(A) \cup \mathbf{dev}(B)$.

Proof. The lemma is a consequence of the following: Given an arbitrary path in $\mathbf{dev}(\widetilde{M})$ and an initial point in $\widetilde{M}$, there exists at most one lift to $\widetilde{M}$. (Compare to Proposition 1.3.1 [3].)

1.7. Suppose that F_1 and F_2 are convex disks in $\check{M}$. Since F_1^o and F_2^o are dense subsets of F_1 and F_2, respectively, the following statements are equivalent:

$$F_1^o \cap F_2 \neq \varnothing; \qquad F_1 \cap F_2^o \neq \varnothing; \qquad F_1^o \cap F_2^o \neq \varnothing.$$

Similarly, these are also equivalent to the following equivalent statements:

$$\operatorname{int}(F_1 \cap \widetilde{M}) \cap F_2 \neq \varnothing; \qquad \operatorname{int}(F_2 \cap \widetilde{M}) \cap F_1 \neq \varnothing;$$
$$\operatorname{int} F_2 \cap F_2 \neq \varnothing; \qquad \operatorname{int} F_2 \cap F_1 \neq \varnothing.$$

We say that F_1 and F_2 *overlap* if the above statements hold, and say that a subset of $\check{M}$ is *backed* if it is a subset of a convex disk in $\check{M}$.

Intersection Theorem. *Suppose that* F_1 *and* F_2 *overlap, and suppose that* α *is a backed convex segment or a great circle such that* $\alpha \cap F_1^o \neq \varnothing$. *Then the following hold*:

(1) $\mathbf{dev}|F_1 \cap F_2$ *is an imbedding onto* $\mathbf{dev}(F_1) \cap \mathbf{dev}(F_2)$.

(2) $\mathbf{dev}|F_1 \cup F_2$ *is an imbedding onto* $\mathbf{dev}(F_1) \cup \mathbf{dev}(F_2)$.

(3) $F_1 \cap F_2$ *is a convex disk.*

(4) $F_1 \cup F_2$ *is a disk.*

(5) $F_1^o \cap F_2^o = (F_1 \cap F_2)^o$, *and* $F_1^o \cup F_2^o = (F_1 \cup F_2)^o$. *These are nonempty subsets.*

(6) $F_1 \cap \alpha$ *is a convex subsegment* β *such that* $\beta^o \subset F_1^o$.

Proof. (1) Since $\mathbf{dev}(F_1) \cap \mathbf{dev}(F_2)$ is convex (see §1.3) and has a nonempty interior, it is a disk. Since $\mathbf{dev}(F_1)$ and $\mathbf{dev}(F_2)$ are disks as well,

$$\mathbf{dev}(F_1) \cap \mathbf{dev}(F_2) = \operatorname{Cl}(\mathbf{dev}(F_1)^o \cap \mathbf{dev}(F_2)^o).$$

By Lemma 1.6, $\mathbf{dev}|F_1^o \cap F_2^o$ is an imbedding onto $\mathbf{dev}(F_1)^o \cap \mathbf{dev}(F_2)^o$. Hence, $F_1^o \cap F_2^o$ is a convex open disk, and its closure K, a convex disk, is a subset of $F_1 \cap F_2$. Since

$$\mathbf{dev}(K) = \mathbf{dev}(F_1) \cap \mathbf{dev}(F_2),$$

and $\mathbf{dev}|F_1$ and $\mathbf{dev}|F_2$ are injective, we have

$$K = F_1 \cap F_2 \quad \text{and} \quad \mathbf{dev}(F_1 \cap F_2) = \mathbf{dev}(F_1) \cap \mathbf{dev}(F_2).$$

This implies (1), and (2), (3), (4), and (5) easily follow from (1).

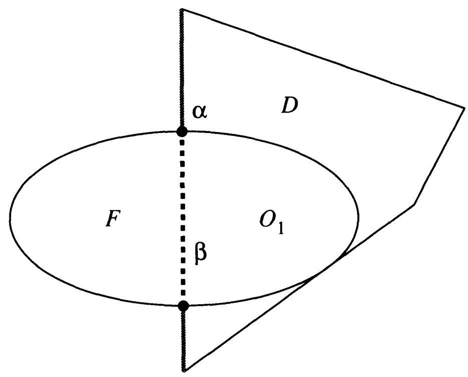

FIGURE 2. A DIPPING INTERSECTION.

(6) Since α is backed, $\check{M}$ includes a convex disk K such that $\alpha \subset K$. Since $\mathbf{dev}|F_1 \cup K$ is an imbedding onto $\mathbf{dev}(F_1) \cup \mathbf{dev}(K)$ by (1), we have (6) (see §1.3).

1.8. Example. Given $\theta_1 \in \mathbf{R}$, let B_{θ_1} be the subset of $\check{\Sigma}_5$ given by

$$B_{\theta_1} = \{[\rho, \theta] | 0 \leq \rho \leq \pi/2, \ \theta_1 \leq \theta \leq \theta_1 + \pi\}.$$

B_0 and $B_{-\pi/2}$ are tame subsets imbedding onto B_2 and B_3 respectively under the developing map of Σ_5. The above theorem applies here. $B_{-\pi}$ and B_0 intersect at $\{[\rho, 0] | 0 \leq \rho \leq \pi/2\}$; however, the intersection of the developing images of $B_{-\pi}$ and B_0 is $\overline{e_2 e_1 - e_2}$, which differs from the developing image of $B_{-\pi} \cap B_0$. Moreover, the developing map restricted to $B_{-\pi} \cup B_0$ is not injective (see §1.2).

1.9. We now discuss a crucial technical corollary of Theorem 1.7. Let D be a convex disk in $\check{M}$ such that δD includes a convex segment or a great circle α. We say that a convex disk F *is dipped into* (D, α) if it has the following properties (see Figure 2):

(a) F and D overlap.

(b) $F \cap \alpha$ is a segment β such that $\delta\beta \subset \delta F$, $\beta^o \subset F^o$.

(c) $F - \beta$ has two convex components O_1 and O_2 such that

$$\mathrm{Cl}(O_1) = O_1 \cup \beta = F - O_2, \qquad \mathrm{Cl}(O_2) = O_2 \cup \beta = F - O_1.$$

(d) $F \cap D$ is identical with $\mathrm{Cl}(O_1)$ or $\mathrm{Cl}(O_2)$.

(A useful heuristic idea is that of a slice of bread dipped into a bowl of milk.) We say that F is *dipped into* (D, α) *nicely* if the following statements hold:

(i) F is dipped into (D, α).

(ii) $F \cap D^o$ is identical with O_1 or O_2.

(iii) $\delta(F \cap D) = \beta \cup \xi$ for a compact arc ξ in $\mathrm{bd}\, F$ where $\beta \cap \xi = \delta\beta$. (So, in this case, $\delta\beta \subset \mathrm{bd}\, F$.)

Corollary. *Suppose that F and D overlap, and that $F^o \cap (\delta D - \alpha^o) = \varnothing$. Then F is dipped into (D, α). Assume the following two equivalent conditions*:

(1) $F^o \cap \alpha \neq \varnothing$.

(2) $F \not\subset D$.

If $F \cap (\delta D - \alpha^o) = \varnothing$ furthermore, then F is dipped into (D, α) nicely.

Proof. Each of the properties (a), (b), (c), and (d) is proved below. (a) is a part of the premise.

As a consequence of (a) and Theorem 1.7, the proofs of (b), (c), and (d) are implied by their proofs under the assumption that D and F are subsets of $\mathbf{S}^2$. So, assume that D, $F \subset \mathbf{S}^2$.

(b) By Theorem 1.7(6), $F \cap \alpha$ is a convex segment. Let it be denoted by β. Since $F^o \cap \delta\alpha = \varnothing$ by the premise, $\delta\beta \subset \delta F$. Since $F^o \cap \beta \neq \varnothing$, it follows that $\beta^o \subset F^o$.

(c) Suppose that $\mathbf{S}^1$ is the great circle including α. Then $\mathbf{S}^1 \cap F = \beta$. Let H_1 and H_2 be components of $\mathbf{S}^2 - \mathbf{S}^1$. Since $F^o \cap \mathbf{S}^1 \neq \varnothing$, we have $H_1 \cap F \neq \varnothing$ and $H_2 \cap F \neq \varnothing$. Let $O_1 = H_1 \cap F$, and $O_2 = H_2 \cap F$. Then (c) follows.

(d) Since $F^o \cap D^o \neq \varnothing$, a point x belongs to it. Since x does not belong to β, the point x belongs to O_1 or O_2. Let us assume without loss of generality that $x \in O_1$ and hence that $x \in O_1^o$. Since we have $O_1^o \cap \delta D = \varnothing$, $O_1^o \cap \mathrm{bd}\, D = \varnothing$, and $\mathrm{Cl}(O_1) \subset D$, it follows that $\mathrm{Cl}(O_1) \subset F \cap D$. Since $D \cap H_1 \neq \varnothing$ and $\alpha \subset \mathbf{S}^1$, we have $D \subset \mathrm{Cl}(H_1)$ and $O_2 \cap D = \varnothing$; thus, $\mathrm{Cl}(O_1) = F \cap D$.

Let us prove the second part of the corollary. Assume that $\mathrm{Cl}(O_1) = F \cap D$ without loss of generality. Since $F \cap \delta D = \beta$,

$$F \cap D^o = \mathrm{Cl}(O_1) - \delta D = (O_1 \cup \beta) - \delta D = O_1.$$

Since $\mathrm{Cl}(O_1)$ is a disk, $\delta\,\mathrm{Cl}(O_1)$ is homeomorphic to a circle. Let $\xi' = \delta\,\mathrm{Cl}(O_1) - \beta$. Since β is a segment, the closure ξ of ξ' is a compact arc. We have

$$\delta\,\mathrm{Cl}(O_1) = \beta \cup \xi \quad \text{and} \quad \beta \cap \xi = \delta\beta.$$

Since $\delta D \cap F = \beta$, it follows that $\xi' \subset D^o$. Thus, $\xi \subset \mathrm{bd}\, F$.

1.10. Example. B_0 is dipped into $(B_{-\pi/2}, \nu)$ in Example 1.8 where

$$\nu = \{[\rho, \theta] | 0 \leq \rho \leq \pi/2,\ \theta = -\pi/2,\ \pi/2\}.$$

Let d' be the spherical metric associated with the developing map of $\check{\Sigma}_5$. The d'-ball $B([\pi/4, \pi/2], \pi/8)$ is dipped into $(B_{-\pi/2}, \nu)$ nicely. The projective complection $\check{\Sigma}_4$ includes the lune L_1 bounded by $\overline{e_2[1/2, 0, 1/2] - e_2}$ and $\overline{e_2[1/4, 0, 1/2] - e_2}$, and includes the second lune L_2 bounded by $\overline{e_3[1/2, 1/2, 0] - e_3}$ and $\overline{e_3[1/4, 1/2, 0] - e_3}$. The lune L_1 is not dipped into (L_2, ν) for any edge ν of L_2 (see §1.2 and Figure 1).

1.11. In addition to the metric $\mathbf{d}$ associated with $\mathbf{dev}$, we need a complete Riemannian metric and the associated distance metric on M. The reader may choose them. The complete metrics induce a complete Riemannian metric and a complete distance metric on $\widetilde{M}$. We call these an *original Riemannian metric* and an *original metric* of $\widetilde{M}$ respectively.

Let d be the original metric. For each point x of $\widetilde{M}$, every neighborhood of x includes a compact neighborhood that is a simply convex disk in $\widetilde{M}$. Let us call such a neighborhood a *tiny disk* of x. This is a bounded subset of $\widetilde{M}$ with respect to d.

We exhibit a compatibility relation between $\mathbf{d}$ and d holding on a tiny disk. Let ν and μ be the original Riemannian metric of $\widetilde{M}$ and the spherical Riemannian metric of $\widetilde{M}$ associated with $\mathbf{dev}$ respectively. Let B be a tiny disk. Because B is compact, there exists a positive constant c depending on B such that

$$c^{-1}\nu(\mathbf{x}, \mathbf{y}) \leq \mu(\mathbf{x}, \mathbf{y}) \leq c\nu(\mathbf{x}, \mathbf{y})$$

for vectors $\mathbf{x}$ and $\mathbf{y}$ at every point of B. Therefore,

$$c^{-1}d(x, y) \leq \mathbf{d}(x, y) \leq cd(x, y),$$

where x, $y \in B$. The constant c depends only on B and the choice of the original metric d and the spherical metric $\mathbf{d}$.

2. Crescents

Professor Goldman and I would loosely describe a crescent as a half-plane. Indeed, a crescent is projectively "equivalent" to a half-plane if the boundary is ignored. For example, in $\check{\Sigma}_4$, the lune B_x bounded by $\overline{x[1/2, 0, 1/2] - x}$ and $\overline{xe_3 - x}$ for a point x of $\overline{e_2 - e_3}^o$ is a crescent, which is a half-plane. As in the case of a half-plane, one side of a crescent lies in the ideal boundary, and the other side has finite points. Moreover, the intersection properties of two overlapping crescents are projectively "equivalent" to the intersection properties of two overlapping half-planes (see Theorem 2.6). Because of these properties similar to those of half-planes, the existence of crescents in the projective completion of a projective surface imposes a very strong condition on the global geometry of the projective surface.

In this section, we discuss properties of crescents in $\check{M}$ when M is compact, after discussing some elementary properties, show that overlapping great-disk crescents are identical if $\chi(M) < 0$, and finally present the transversal intersection property.

2.1. Recall the definition of crescents given in the introduction. A subset $\mathscr{R}$ of $\check{M}$ is a *crescent* if $\mathscr{R}$ is a great disk or a lune, $\delta\mathscr{R} \cap \widetilde{M} \neq \emptyset$, and $\delta\mathscr{R} \cap \widetilde{M}_\infty$ includes a great line (compare Chapter 2 of Benzécri [1]). (Note the condition $\mathscr{R}^o \subset \widetilde{M}$ hidden by a convention made in §1.4.) Suppose that $\mathscr{R}$ is a crescent. Let $\alpha_{\mathscr{R}}$ denote the unique maximal element of the set of open lines included in $\delta\mathscr{R} \cap \widetilde{M}_\infty$ and including a great line in $\delta\mathscr{R} \cap \widetilde{M}_\infty$. Let $\nu_{\mathscr{R}}$ denote $\delta\mathscr{R} - \alpha_{\mathscr{R}}$ Recall also that $\mathscr{R}$ is *simplistic* if a component of $\delta\mathscr{R} \cap \widetilde{M}$ is a simplistic line, i.e., a line corresponding to a simple closed curve in M.

Consider B_2 in $\check{\Sigma}_4$. This is a crescent such that

$$\nu_{B_2} = \overline{e_2e_1 - e_2} \quad \text{and} \quad \alpha_{B_2} = \overline{e_2e_3 - e_2}^o$$

hold. B_2 is a simplistic crescent with simplistic lines $\overline{e_1e_2}^o$ and $\overline{e_1 - e_2}^o$. The previously mentioned B_x is the lune in $\check{\Sigma}_4$ bounded by $\overline{x[1/2, 0, 1/2] - x}^o$ and $\overline{xe_3 - x}^o$ where $x \in \overline{e_2 - e_3}^o$. The subset B_x is a crescent but is not a simplistic crescent. We have

$$\nu_{B_x} = \overline{x[1/2, 0, 1/2] - x} \quad \text{and} \quad \alpha_{B_x} = \overline{xe_3 - x}^o.$$

An example of a crescent that is also a great disk is $\check{\Sigma}_3$, which is simplistic as well. L_1 and L_2 in $\check{\Sigma}_4$ are not crescents (see §1.2 and Example 1.10).

2.2. Let $\mathscr{R}$ be a crescent. If $\mathscr{R}$ is a lune, then $\nu_{\mathscr{R}}$ is a great segment, and $\alpha_{\mathscr{R}}$ is a great line. If $\mathscr{R}$ is a great disk, then $\nu_{\mathscr{R}}$ is a convex segment, and $\alpha_{\mathscr{R}}$ is a line whose $\mathbf{d}$-length is greater than or equal to π. For each deck transformation ϑ, the subset $\vartheta(\mathscr{R})$ is also a crescent such that

$$\nu_{\vartheta(\mathscr{R})} = \vartheta(\nu_{\mathscr{R}}) \quad \text{and} \quad \alpha_{\vartheta(\mathscr{R})} = \vartheta(\alpha_{\mathscr{R}}).$$

2.3. The subset $\mathscr{R}$ is a disk, the finite edge $\nu_{\mathscr{R}}$ is a segment, and $\mathscr{R} - \nu_{\mathscr{R}}$ is a noncompact disk with boundary $\alpha_{\mathscr{R}}$. Moreover, $\mathscr{R} - \nu_{\mathscr{R}}$ is an open subset of $\check{M}$. Let $x \in \mathscr{R} - \nu_{\mathscr{R}}$; let $y \in \check{M} - \mathscr{R}$. Then $\mathbf{d}(x, \nu_{\mathscr{R}}) > c$ for a positive constant c. Since every path on $\widetilde{M}$ connecting a point of $\mathscr{R}^o$ and a point of $\widetilde{M} - \mathscr{R}$ passes through $\nu_{\mathscr{R}}$, it follows that $\mathbf{d}(x, y) > c/2$. Hence, $\mathscr{R} - \nu_{\mathscr{R}}$ is an open subset of $\check{M}$.

2.4. Suppose that $\check{M}$ includes a convex disk F overlapping a crescent $\mathscr{R}$. Since $\mathrm{Cl}(\alpha_{\mathscr{R}}) \cap F^o = \emptyset$, by Corollary 1.9, F either is a subset of $\mathscr{R}$ or is dipped into $(\mathscr{R}, \nu_{\mathscr{R}})$. If, furthermore, F is a subset of $\widetilde{M}$, then F either is a subset of $\mathscr{R}$ or is dipped into $(\mathscr{R}, \nu_{\mathscr{R}})$ nicely. In particular, this is true if F is a tiny disk.

What happens when two crescents intersect? In general, their intersection may be complicated (see Example 1.8). Here, we study overlapping crescents.

Let us start with the simplest case. Suppose that $\mathscr{R}_1$ and $\mathscr{R}_2$ are crescents. If $\mathscr{R}_2 \subset \mathscr{R}_1$, then $\mathscr{R}_2$ is bounded by $\nu_{\mathscr{R}_2}$ and a subset $\alpha_{\mathscr{R}_2}$ of $\alpha_{\mathscr{R}_1}$. Moreover, $\mathscr{R}_2 \subset \mathscr{R}_1$ and $\nu^o_{\mathscr{R}_2} \subset \mathscr{R}_1$ if and only if $\mathscr{R}_2$ is a proper subset of $\mathscr{R}_1$.

Next, suppose that $\mathscr{R}_1$ and $\mathscr{R}_2$ are arbitrary overlapping great-disk crescents. We claim that if $\chi(M) < 0$, then $\mathscr{R}_1 \neq \mathscr{R}_2$. Suppose not. By Theorem 1.7, $\mathbf{dev}|\mathscr{R}_1 \cup \mathscr{R}_2$ is an imbedding onto $\mathbf{dev}(\mathscr{R}_1) \cup \mathbf{dev}(\mathscr{R}_2)$. The following statements hold:

- $\mathbf{dev}(\mathscr{R}_1) \neq \mathbf{dev}(\mathscr{R}_2)$.
- $\mathscr{R}_1 \cup \mathscr{R}_2$ is a disk.
- $\delta(\mathscr{R}_1 \cup \mathscr{R}_2) = \mathrm{Cl}(\alpha_{\mathscr{R}_1}) \cup \mathrm{Cl}(\alpha_{\mathscr{R}_2})$.

We deduce two consequences. First, since $\mathbf{dev}(\mathscr{R}_1) \neq \mathbf{dev}(\mathscr{R}_2)$, it follows that $\mathscr{R}_1^o \cup \mathscr{R}_2^o$ is not convex. Second, since each point of $\alpha_{\mathscr{R}_1} \cup \alpha_{\mathscr{R}_2}$ belongs to a convex neighborhood $\mathscr{R}_1$ or $\mathscr{R}_2$, by Lemma 2.5, $\mathscr{R}_1^o \cup \mathscr{R}_2^o = \widetilde{M}$, and $\widetilde{M}$ is convex. This is a contradiction. Thus, if $\chi(M) < 0$, then $\mathscr{R}_1 \neq \mathscr{R}_2$ holds.

2.5. Lemma. *Suppose that $\chi(M) < 0$ and that F is a disk in $\check{M}$ such that $F^o \subset \widetilde{M}$ and $\delta F \subset \widetilde{M}_\infty$. Suppose that there is a finite subset L of δF such that F includes a relative convex-disk neighborhood of each point of $\delta F - L$. Then $\widetilde{M}$ equals F^o and is convex.*

Proof. Since $\mathrm{bd}\, F^o \subset \delta F$, we deduce that $\mathrm{bd}\, F^o \cap \widetilde{M} = \emptyset$, $F^o = \widetilde{M}$, and $F = \check{M}$. Let L' be the subset of elements of L that do not have relative convex-disk neighborhoods in F. Suppose that $L' \neq \emptyset$. Since F and δF are invariant under deck transformations of $\check{M}$, so is L'. Let ϑ be a deck transformation. Since M is orientable and δF is a closed curve, $\langle\vartheta\rangle$ acts on L' and on the set of components of $\delta F - L'$ cyclically. If one of the actions of $\langle\vartheta\rangle$ is not trivial, then the Brouwer fixed point theorem implies that ϑ fixes a point in F^o. This is absurd. Since each point of L' is fixed by every deck transformation, Lemma 2.5 [11] implies the contradiction that $\chi(M) = 0$, and $L' = \emptyset$, and F and $\widetilde{M}$ are convex (see Thurston [22]).

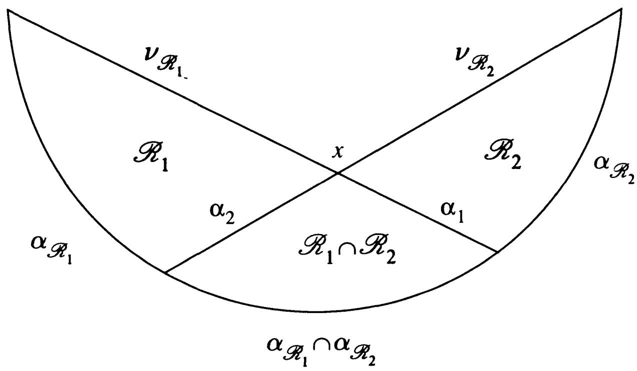

FIGURE 3. A TRANSVERSAL INTERSECTION

2.6. The most important case, yielding the transversal intersection property, is presented now. Two convex segments *intersect transversally* at a point x if x is a point of intersection, and their developing images intersect at $\mathbf{dev}(x)$ transversally. Suppose that $\mathscr{R}_1$ and $\mathscr{R}_2$ in $\check{M}$ are overlapping lune crescents. $\mathscr{R}_1$ and $\mathscr{R}_2$ intersect *transversally* if the following properties hold ($i = 1$, $j = 2$; or $i = 2$, $j = 1$):

(a) There exists a unique point x of intersection of $\nu_{\mathscr{R}_1}$ and $\nu_{\mathscr{R}_2}$. Furthermore, x is the transversal point of intersection and belongs to $\nu^o_{\mathscr{R}_1} \cap \nu^o_{\mathscr{R}_2}$.

(b) Let $\alpha_i = \nu_{\mathscr{R}_i} \cap \mathscr{R}_j$. The subset α_i is a convex segment connecting x to a point in $\alpha_{\mathscr{R}_j}$, is the closure of a component of $\nu_{\mathscr{R}_i} - \{x\}$, and satisfies $\alpha^o_i \subset \mathscr{R}^o_j$.

(c) $\mathscr{R}_1 \cap \mathscr{R}_2$ is the closure of a component of $\mathscr{R}_j - \alpha_i$.

(d) Both $\alpha_{\mathscr{R}_1} \cap \alpha_{\mathscr{R}_2}$ and $\alpha_{\mathscr{R}_1} \cup \alpha_{\mathscr{R}_2}$ are open lines in $\widetilde{M}_\infty$.
(See Figure 3.)

The crescents B_0 and $B_{-\pi/2}$ in $\check{\Sigma}_5$ (see Example 1.8) form an example of two transversally intersecting crescents. We do not want to have an intersection like that of L_1 and L_2 in $\check{\Sigma}_4$ (see Example 1.10).

Transversal Intersection Theorem. *Suppose that $\chi(M) < 0$ and $\widetilde{M}$ is not convex. Suppose that $\mathscr{R}_1$ and $\mathscr{R}_2$ are overlapping lune crescents. Then either $\mathscr{R}_1$ and $\mathscr{R}_2$ intersect transversally or $\mathscr{R}_1 \subset \mathscr{R}_2$ or $\mathscr{R}_2 \subset \mathscr{R}_1$.*

Proof. Assume that we have $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$, and also that $\mathscr{R}_1 \not\subset \mathscr{R}_2$ and $\mathscr{R}_2 \not\subset \mathscr{R}_1$. By Corollary 1.9, $\mathscr{R}_j$ is dipped into $(\mathscr{R}_i, \nu_{\mathscr{R}_i})$. Hence, the following statements hold:

(i) $\nu_{\mathscr{R}_i} \cap \mathscr{R}_j$ is a closed connected segment α_i such that $\delta\alpha_i \subset \delta\mathscr{R}_j$, $\alpha^o_i \subset \mathscr{R}^o_j$.

(ii) $\mathscr{R}_i \cap \mathscr{R}_j$ is the convex disk that is the closure of a component of $\mathscr{R}_j - \alpha_i$.

Since α_i is a segment with endpoints in $\delta\mathscr{R}_j$, one of the following properties holds:

(α) Both endpoints of α_i are in $\alpha_{\mathscr{R}_j}$.

(β) Both points are in $\nu_{\mathscr{R}_j}$.

(γ) One of them is in $\nu_{\mathscr{R}_j}$, and the other is in $\alpha_{\mathscr{R}_j}$.

Clearly (α) is impossible since $\alpha_i \not\subset \alpha_{\mathscr{R}_j}$. In case ($\beta$), we have $\alpha_i = \nu_{\mathscr{R}_i}$, $\delta\nu_{\mathscr{R}_i} \subset \delta\nu_{\mathscr{R}_j}$, and $\nu^o_{\mathscr{R}_i} \subset \mathscr{R}^o_j$. (ii) yields two possibilities: $\mathscr{R}_i \cap \mathscr{R}_j$ is the disk in $\mathscr{R}_j$ bounded by $\nu_{\mathscr{R}_i}$ and $\nu_{\mathscr{R}_j}$, or by $\nu_{\mathscr{R}_i}$ and $\alpha_{\mathscr{R}_j}$. Since $\mathscr{R}_i \not\subset \mathscr{R}_j$, the former holds; thus, $\mathscr{R}_i \cup \mathscr{R}_j$ is a disk bounded by segments $\mathrm{Cl}(\alpha_{\mathscr{R}_i})$ and $\mathrm{Cl}(\alpha_{\mathscr{R}_j})$. Lemma 2.5 now contradicts the premise that $\widetilde{M}$ is not convex. The case (γ) implies a transversal intersection. Since $\alpha_i \not\subset \mathrm{Cl}(\alpha_{\mathscr{R}_j})$, the endpoint of α_i in $\nu_{\mathscr{R}_j}$ is in $\nu^o_{\mathscr{R}_j}$. This and the relation

$$\nu_{\mathscr{R}_i} \cap \nu_{\mathscr{R}_j} = \nu_{\mathscr{R}_1} \cap \alpha_2 = \alpha_1 \cap \nu_{\mathscr{R}_2} = \alpha_1 \cap \alpha_2$$

imply (a). Condition (b) follows from (i), (ii), and (a); conditions (c) and (d) follow from (ii).

3. Concavity

A concave subset of $\check{M}$ may be thought of as the complement of a convex set in a great disk with geodesic boundary components satisfying an invariance property. In general, this picture is oversimplified since not all concave subsets correspond to subsets of great disks. The notion of concave subsets is a generalization of the notion of crescents. A result of §2.4 implies that a great-disk crescent and its overlapping image under a deck transformation are identical. Since this is not true for a lune crescent in general, we need to obtain an invariant subset from a collection of crescents, which turns out to be concave (see §6).

We will prove the concavity lemma here since this can be done independently of many other results and we need the lemma for the proof of Theorem 4.1. To do this, we recall the dipping intersection property (see §2.4), and prove two claims, from which the concavity lemma follows immediately.

3.1. A *maximal line* in $\widetilde{M}$ is a line that is not a proper subset of any line or an imbedded closed geodesic in $\widetilde{M}$. A line l is maximal in $\widetilde{M}$

and is open if and only if the d-length of each component of l removed a point of l^o is infinite. A connected subset A of $\check{M}$ is said to be *concave* if A satisfies (a), (b), and (c).

(a) Each component of $\mathrm{bd}\, A \cap \widetilde{M}$ is a maximal line in $\widetilde{M}$ and is an open line.

(b) Each component of $\mathrm{bd}\, A \cap \widetilde{M}$ is a subset of $\delta\mathscr{R}$ for a crescent or a great disk $\mathscr{R}$ included in A.

(b′) Each component of $\mathrm{bd}\, A \cap \widetilde{M}$ is a subset of $\delta\mathscr{R}$ for a crescent $\mathscr{R}$ included in A.

(c) If $\mathrm{int}(A \cap \widetilde{M}) \cap \vartheta(A) \neq \emptyset$ for a deck transformation ϑ, then $A = \vartheta(A)$.

If A satisfies (a), (b′), and (c), then A is said to be *properly concave*. An example of a concave subset is $\check{\Sigma}_3$ in itself or B_2 in $\check{\Sigma}_4$. Also $\check{\Sigma}_5$ is concave in itself; B_x in $\check{\Sigma}_4$ for x in $\overline{e_2 - e_3}^o$ is not concave (see §1.2 and §2). L_1 and L_2 in $\check{\Sigma}_4$ are not concave (see Example 1.10).

We prove that each component of $\mathrm{bd}\, A \cap \widetilde{M}$ for a concave subset A is simplistic, and that each component of $\mathrm{bd}\, A \cap \widetilde{M}$ is a subset of $\delta\mathscr{R}$ for a simplistic crescent $\mathscr{R}$ included in A if A is a properly concave subset. We shall divide the proof into several steps as follows.

3.2. The first step is a recollection of a result of §2.4. Let A be a concave subset of $\check{M}$; let α be a component of $\mathrm{bd}\, A \cap \widetilde{M}$. Let $x \in \widetilde{M}$; let $B(x)$ be a tiny disk of x. Suppose that $\mathrm{int}\, B(x) \cap \alpha \neq \emptyset$. The line α is a subset of $\delta\mathscr{R}$ for a crescent or a great disk $\mathscr{R}$ included in A. It follows that $\mathscr{R}^o$ includes either $B(x)^o$ or a component C_α of $B(x) - \alpha$. Since $\mathrm{int}(A \cap \widetilde{M}) \supset \mathscr{R}^o$, $\mathrm{int}(A \cap \widetilde{M}) \supset B(x)^o$, or $\supset C_\alpha$.

3.3. The second step is to prove the claim that given a deck transformation ϑ and a component α of $\mathrm{bd}\, A \cap \widetilde{M}$ for a concave set A, either $\alpha = \vartheta(\alpha)$ or $\alpha \cap \vartheta(\alpha) = \emptyset$ holds. We prove the equivalent statement that if $\alpha \cap \vartheta(\alpha) \neq \emptyset$ for a deck transformation ϑ, then $\alpha = \vartheta(\alpha)$. Let $x \in \alpha \cap \vartheta(\alpha)$. If α and $\vartheta(\alpha)$ are tangent at x, then $\alpha = \vartheta(\alpha)$. Suppose that α and $\vartheta(\alpha)$ are transversal at x. Let $B(x)$ and C_α be determined as above. Since $\vartheta(\alpha) \cap B(x)^o \neq \emptyset$ or $\vartheta(\alpha) \cap C_\alpha \neq \emptyset$, it follows that $\mathrm{int}(A \cap \widetilde{M}) \cap \vartheta(A) \neq \emptyset$. Thus, $A = \vartheta(A)$. Since α and $\vartheta(\alpha)$ are components of $\mathrm{bd}\, A \cap \widetilde{M}$, we have $\alpha = \vartheta(\alpha)$. This is a contradiction.

3.4. We need

Lemma. *Suppose that* α *is an imbedded geodesic in* $\widetilde{M}$ *such that* $\alpha^o \cap \delta\widetilde{M} \neq \emptyset$. *Then* $\alpha \subset \delta\widetilde{M}$.

Proof. It is straightforward to prove.

The following is a consequence of the above lemma:

If α is a maximal line in $\widetilde{M}$ and is an open line, and $\alpha^o \cap \delta\widetilde{M} \neq \emptyset$, then α is identical with a component of $\delta\widetilde{M}$.

3.5. The third step is to prove the claim that given a concave subset A, the number of distinct images of a component α of $\mathrm{bd}\, A \cap \widetilde{M}$ intersecting a compact subset K of $\widetilde{M}$ is finite.

Suppose not. Then K contains a point x that is a cluster point of $\bigcup_{\vartheta \in \pi_1(M)} \vartheta(\alpha)$ but is not in it. By §3.4, the images of α intersecting a common component of δM are identical. Let $B(x)$ be a tiny disk of x such that $B(x) \cap \delta\widetilde{M}$ is a connected arc; consequently, no more than one image of α may intersect $B(x) \cap \delta\widetilde{M}$. Hence, we may extract a sequence $\{x_i\}$ converging to x where $x_i \in \vartheta_i(\alpha) \cap B(x)^o$, $\vartheta_i \in \pi_1(M)$, $i = 1, 2, 3, \cdots$, and $\vartheta_i(\alpha) \neq \vartheta_j(\alpha)$ whenever $i \neq j$, $i, j = 1, 2, 3, \ldots$. Let $\mathscr{R}$ be a crescent such that $\alpha \subset \delta\mathscr{R}$ and $\mathscr{R} \subset A$. For each i, let $\mathbf{x}_i$ be the boundary orientation vector at x_i of $\vartheta_i(\mathscr{R})$. We may assume without loss of generality that $\{\mathbf{x}_i\}$ converges. For each i, let $B(x)(i, L)$ and $B(x)(i, R)$ be the left and right components of $B(x) - \vartheta_i(\alpha)$ respectively. We have for each i

$$B(x)(i, L) \subset \vartheta_i(\mathscr{R}^o) \cap B(x) \subset \mathrm{int}(\vartheta_i(A) \cap \widetilde{M}) \cap B(x).$$

Since the previous claim (§3.3) implies that $\vartheta_i(\alpha) \cap \vartheta_j(\alpha) = \emptyset$ for $i \neq j$, $i, j = 1, 2, 3, \ldots$, there are some positive integers k and l, sufficiently large, satisfying $\vartheta_k(\alpha) \cap B(x) \subset B(x)(l, L)$ (see Figure 4). Since $B(x)(l, L) \subset \mathrm{int}(\vartheta_l(A) \cap \widetilde{M})$, $\vartheta_k(A) = \vartheta_l(A)$; this implies the absurdity $\vartheta_k(\alpha) \cap B(x) \subset \mathrm{int}(\vartheta_k(A) \cap \widetilde{M})$.

3.6. Now, the proof of the concavity lemma follows: Let α be a component of $\mathrm{bd}\, A \cap \widetilde{M}$ for a concave subset A of $\check{M}$; let $\mathbf{p}\colon \widetilde{M} \to M$ be the

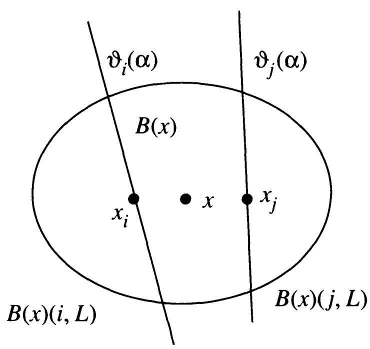

FIGURE 4. $\vartheta_i(\alpha)$, $B(x)(i, L)$, AND SO ON.

universal covering map. By the first claim, $\mathbf{p}|\alpha$ is either injective or an infinite cyclic covering map onto a simple closed curve. Since the second claim implies that $\mathbf{p}(\alpha)$ is compact, α is simplistic. Hence the lemma follows.

4. The great-disk theorem

A great disk in $\widetilde{M}$ is a disk such that **dev** restricted to it is an imbedding onto a hemisphere of $\mathbf{S}^2$. As an application of the concavity lemma, we will prove the following theorem.

4.1. Great-Disk Theorem. *Suppose that M is compact and that $\widetilde{M}$ includes a great disk. Then M is projectively homeomorphic to* (1) *a great disk,* (2) $\mathbf{RP}^2$, *or* (3) $\mathbf{S}^2$.

Remark. The theorem easily generalizes to an n-dimensional version for n great than 2 with an almost identical proof: Let M be an n-dimensional projective manifold with convex boundary; replace the term *great disk* with the term *n-dimensional great ball*, and replace $\mathbf{RP}^2$ and $\mathbf{S}^2$ with $\mathbf{RP}^n$ and $\mathbf{S}^n$ respectively.

4.2. Let us begin the proof. Suppose that $H \cap \delta\widetilde{M} \neq \varnothing$. Since $\delta H \subset \delta\widetilde{M}$ by Lemma 3.4, $H = \widetilde{M}$. The universal covering map $\mathbf{p}\colon \widetilde{M} \to M$ is the trivial covering map and a projective map. Hence (1) follows.

Suppose that $H \subset \widetilde{M}^o$. Let p be the center of $\mathbf{dev}(H)$; recall that for $\varepsilon \geq 0$,

$$B(-p, \varepsilon) = \{q \in \mathbf{S}^2 | \mathbf{d}(q, -p) \leq \varepsilon\}.$$

Let us consider the collection of open disks that include H and such that **dev** restricted to each of them is an imbedding onto the complement of $B(-p, \varepsilon)$ where $0 \leq \varepsilon < \pi/2$. Clearly, the collection is not empty, and, by Lemma 1.6, contains a unique maximal element P, where $\mathbf{dev}|P$ is an imbedding onto $\mathbf{S}^2 - B(-p, \varepsilon)$ for some ε $(0 \leq \varepsilon < \pi/2)$.

Suppose that $\varepsilon = 0$. The condition that $P = \widetilde{M}$ shows that the holonomy group action must leave $\mathbf{dev}(P)$, $\{-p\}$, and $\{p\}$ invariant, so that deck transformations fix the point $(\mathbf{dev}|P)^{-1}(p)$. Since this is absurd, P is a proper subset of $\widetilde{M}$. Let B be an open tiny disk neighborhood of a boundary point of P. By Lemma 1.6, $P \cup B$ is a great sphere, and, hence, that $\widetilde{M}$ is a great sphere. (2) and (3) follow.

4.3. We will not need the following results to go on. Let A be a connected open subset of $\widetilde{M}$. A developing map $\mathbf{dev}|A$ of A induces a metric μ' on A such that $\mu' = \mu|A$, where μ is the spherical Riemannian metric of $\widetilde{M}$ induced by $\mathbf{dev}$. Let $\mathbf{d}'$ be the distance metric on A induced from μ'.

Suppose that $\mathbf{dev}|A$ is an imbedding with a positive constant c such that if $x, y \in A$, then

$$c^{-1}\mathbf{d}'(x, y) \leq \mathbf{d}(\mathbf{dev}(x), \mathbf{dev}(y)) \leq c\mathbf{d}'(x, y).$$

Thus we claim that $\mathbf{dev}|\operatorname{Cl}(A)$ is an imbedding onto $\operatorname{Cl}(\mathbf{dev}(A))$. Let us prove this. The inequalities imply that $\mathbf{dev}|A$ extends to a continuous map $\mathbf{dev}'\colon \check{A} \to \mathbf{S}^2$ that is an imbedding onto $\operatorname{Cl}(\mathbf{dev}(A))$. Since $\check{M}$ is complete, and the inclusion map $\imath\colon A \to \widetilde{M}$ is distance decreasing with respect to $\mathbf{d}'$ and $\mathbf{d}$, $\check{\imath}$ extends to a continuous map $\check{\imath}\colon \check{A} \to \operatorname{Cl}(A)$. Since $\check{A}$ is compact, and A is dense in $\operatorname{Cl}(A)$, $\check{\imath}$ is a map onto $\operatorname{Cl}(A)$. From $\mathbf{dev}' = \mathbf{dev} \circ \check{\imath}$, it follows that $\check{\imath}$ is injective and therefore an imbedding onto $\operatorname{Cl}(A)$. Hence the claim is proved.

4.4. Suppose that $\mathbf{dev}|A$ is an imbedding onto $\mathbf{S}^2 - B(-p, r)$, $0 < r < \pi/2$. Since the above inequalities hold for a positive c, $\mathbf{dev}|\operatorname{Cl}(A)$ is an imbedding onto $\operatorname{Cl}(\mathbf{dev}(A))$, which equals $\mathbf{S}^2 - B(x, r)^o$.

4.5. The proof of Theorem 4.1 is completed by showing that $\varepsilon > 0$ is absurd. By lifting to a finite cover M_1 of M, if necessary, so that its universal covering map $\mathbf{p}_1\colon \widetilde{M} \to M_1$ imbeds H, we may assume without loss of generality that $\mathbf{p}|H$ is an imbedding; that is, H does not intersect any of its images under deck transformations of $\check{M}$.

Let $F = \operatorname{Cl}(P)$. By §4.4, F is a disk such that $\mathbf{dev}|F$ is an imbedding onto the complement of the open disk $B(-p, \varepsilon)^o$, $0 < \varepsilon < \pi/2$. Since P is maximal, $\delta F \cap (\widetilde{M}_\infty \cup \delta\widetilde{M}) \neq \emptyset$. By Lemma 3.4, $\delta F \cap \widetilde{M}_\infty \neq \emptyset$. Hence, F includes a great disk H_1 such that $\delta H_1 \cap \widetilde{M}_\infty = \{q\}$ and $H_1 - \{q\} \subset F^o$ hold for a point q contained in $\widetilde{M}_\infty \cap \delta F$. Let $\alpha = \delta H_1 - \{q\}$ (see Figure 5, next page).

4.6. Lemma. *H_1 is concave.*

Proof. Since H_1 satisfies the concavity conditions (a) and (b), we need to show that it satisfies condition (c). First, we slide H in F. Choose an imbedding $\imath\colon H \times [0, 1] \to F \times [0, 1] \subset \check{M} \times [0, 1]$ having the properties

$$\imath(H \times \{0\}) = H \times \{0\}, \qquad \imath(H \times \{1\}) = H_1 \times \{1\},$$

when $t \in [0, 1]$, $\imath|H \times \{t\}$ is an imbedding onto a great disk $H(t) \times \{t\}$, and $H(t) \subset F^o$ whenever $0 \leq t < 1$. Clearly, $H(t)^o \subset F^o$ whenever $0 \leq t \leq 1$. Thus there is a continuous function $f\colon \check{M} \times [0, 1] \to \mathbf{R}$

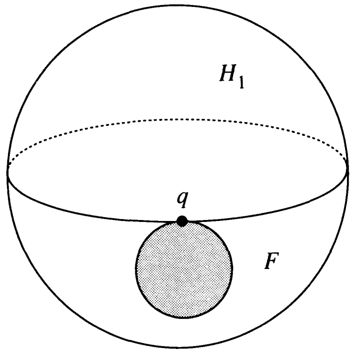

FIGURE 5. F AND H_1.

satisfying

$$f|\imath(H^o \times [0, 1]) > 1, \qquad f|\imath(\delta H \times [0, 1]) \equiv 1,$$

and

$$f|(\check{M} \times [0, 1] - \imath(H \times [0, 1])) < 1.$$

Suppose that $\mathbf{p}|H(t_0)$ is not an imbedding for some t_0 such that $0 < t_0 < 1$; that is, $H(t_0)$ intersects $\vartheta(H(t_0))$ for a deck transformation ϑ. So, $\vartheta(H(t_0))$ includes a point x such that $f(x, t_0) \geq 1$. Hence the function $f': [0, 1] \to \mathbf{R}$ defined by

$$f'(t) = \sup\{f \circ (\vartheta \times \mathrm{Id}) \circ \imath(x, t) | x \in H\}, \qquad 0 \leq t \leq 1,$$

is continuous. Since $f'(0) < 1$ and $f'(t_0) \geq 1$, it follows that $f'(t_1) = 1$ for a real number t_1 such that $0 < t_1 < 1$. Therefore $H(t_1)$ intersects $\vartheta(H(t_1))$ tangentially at a common boundary point. For being imbedded closed geodesics of the spherical metric μ on $\widetilde{M}$, both $\delta H(t_1)$ and $\delta\vartheta(H(t_1))$ are identical. Thus, $H(t_1) \cup \vartheta(H(t_1))$ is a great sphere in $\widetilde{M}$ including P. This yields the contradiction that P is not maximal. Hence $\mathbf{p}|H(t)$ is an imbedding as long as $t \in [0, 1)$.

Suppose that $\mathrm{int}(H_1 \cap \widetilde{M}) \cap \vartheta(H_1) \neq \varnothing$ for a deck transformation ϑ. Then $H_1^o \cap \vartheta(H_1) \neq \varnothing$, and therefore the great disk H contains a point x such that $f \circ (\vartheta \times \mathrm{Id}) \circ \imath(x, 1) > 1$. Since $(\vartheta \times \mathrm{Id})$ is continuous, the interval $[0, 1)$ contains a real number t satisfying $f \circ (\vartheta \times \mathrm{Id}) \circ \imath(x, t) > 1$; that is, $H(t)^o \cap \vartheta(H(t)) \neq \varnothing$, which contradicts the above claim. Hence, H_1 is concave.

4.7. By the concavity lemma and Lemma 4.6, α is simplistic; thus, an infinite cyclic subgroup generated by a deck transformation acts freely on

α. However, this is contradicted by the geometry of α that $\alpha = \mathbf{S}^1 - \{q\}$ where $\mathbf{S}^1$ is a great circle in $\check{M}$. For an arbitrary action on α of a subgroup of the deck transformation group, since q is a fixed point of the action, the unique point of α corresponding to $-\mathbf{dev}(q)$ under $\mathbf{dev}$ is a fixed point also. Hence the proof of Theorem 4.1 is completed.

4.8. Let us state the consequences of the Theorem 4.1. Suppose that $\chi(M) \leq 0$. This condition implies that each geodesic in $\widetilde{M}$ does not self-intersect, so that at most one geodesic connects two given points of $\widetilde{M}$ if their developing images are not antipodal or identical. By these two facts, a geodesic in $\widetilde{M}$ imbeds onto a lines in $\widetilde{M}$. (Thus, lines in $\widetilde{M}$ are topologically imbedded whenever $\chi(M) \leq 0$.)

Let α be a line in $\widetilde{M}$. A priori, $\mathrm{Cl}(\alpha)$ may be very complicated. However, if the $\mathbf{d}$-length of α is finite and is not $2n\pi$ for every positive integer n, then $\mathrm{Cl}(\alpha)$ is a segment such that $\mathrm{Cl}(\alpha) - \alpha^o$ is the set of two points. If the $\mathbf{d}$-length of α is $2n\pi$ for a positive integer n, then $\mathrm{Cl}(\alpha)$ is as stated above or is a closed curve such that $\mathrm{Cl}(\alpha) - \alpha^o$ is the set of a point. Consequently, in these cases, there can be at most two *endpoints*, i.e., the elements of $\mathrm{Cl}(\alpha) - \alpha^o$.

5. The existence of crescents

Let Σ be a compact nonconvex projective surface with negative Euler characteristic; let $(\mathbf{dev}, h)$ be an $\mathbf{S}^2$-development pair of Σ. Let $\tilde{\Sigma}$ be the universal cover of Σ, and let $\check{\Sigma}$ be the projective completion of Σ; let $\tilde{\Sigma}_\infty$ be the ideal boundary of $\tilde{\Sigma}$. Let μ and $\mathbf{d}$ be the spherical Riemannian metric and the spherical metric associated with $\mathbf{dev}$ respectively; let d be an original metric on $\tilde{\Sigma}$. In this section, we obtain a crescent in $\check{\Sigma}$, completing the first step (I).

Recall that a closed nonconvex projective surface Σ' can be constructed by grafting a projective annulus including a π-annulus into a closed convex projective surface. Call the π-annulus Λ'. Then a component of $(\mathbf{p}')^{-1}(\Lambda')$ for the universal covering map $\mathbf{p}': \tilde{\Sigma}' \to \Sigma'$ equals $B^o \cup (\alpha^o - \{x\})$ for a lune or a great disk B in a projective completion $\check{\Sigma}'$ of Σ', and an edge α of B, and a point x of α^o. The closure of B^o, i.e., B, is a crescent. Hence, for each nonconvex surface constructed this way, step (I) can be carried out.

Let us sketch the process of the first step (I). The author is indebted to Peter Doyle in some parts of this intuitive description. The nonconvexity implies that $\check{\Sigma}$ includes a triangle R intersecting ideal points at the interior of an edge ν and only at ν^o. Choose a maximal line α in $\nu \cap \tilde{\Sigma}$ that starts from an endpoint of ν. Let us imagine that we live in $\tilde{\Sigma}$ and that we are walking toward an ideal point along α. Let $\xi = \delta R - \nu^o$. We find that the arc ξ gets further and further away from us, but the visual angle of ξ is always π. Thus, it never looks small. (This corresponds to Lemma 5.4.) We claim that a sequence of these images of ξ tends to a subset of $\tilde{\Sigma}_\infty$, giving rise to an arc in $\tilde{\Sigma}_\infty$, and that the corresponding sequence of images of R tends to a crescent. More precisely, we do as follows: We choose a monotone sequence on α converging to an ideal point and choose a compact neighborhood K of a fundamental domain of $\tilde{\Sigma}$. Now, we pull back points of the sequence by deck transformations to points inside K along with the triangle, extract the convergent sequences $\{p_i\}$ and $\{R_i\}$ respectively from the image points and image triangles, and employ techniques of the appendix and Lemma 5.4 and Lemma 5.5 to show that $\{R_i\}$ converges to a crescent.

An example illustrating this is as follows: Let R be a triangle in $\check{\Sigma}_4$ such that only one of its edges ν intersects e_1 in ν^o where $R - \{e_1\} \subset \tilde{\Sigma}_4$. Choose a point p of $\nu^o - \{e_1\}$. Then apply $T^i_{4,1/2,1/2}$, $i = 1, 2, 3, \ldots$, to p to produce a monotone sequence $\{p_i\}$ converging to e_1. Now, apply $T^{-i}_{4,1/2,1/2}$ to p_i and R for each i. It follows that $\{T^{-i}_{4,1/2,1/2}(R)\}$ converges to a crescent (see §1.2).

This is a generalization of Fried's work on similarity manifolds [7]. It was not clear that attempting to generalize his work was a good idea. Initially, I believed that any suitable sequence of the image triangles degenerates and abandoned this approach quickly. But on the second time that I thought about this, I realized that the subsequence never degenerates because of the fact stated in Lemma 5.4.

5.1. We begin the proof. A *triangle* is a simply convex disk in a projective completion bounded by three segments. The criterion for convexity can be stated as an extension problem. Let R be a triangle in $\mathbf{S}^2$; let $R' = R - \nu^o$ for an edge ν of R. Then a projective surface M is convex if and only if every projective map $\eta: R' \to M$ extends to a projective map $\eta': R \to M$. Fried [7] was first to use this in his classification of similarity manifolds. Recently, Carrière [3] used this to study affine manifolds of *discompacitè* one.

Since Σ is a nonconvex projective surface, by the above criterion, $\check{\Sigma}$ includes a triangle R with edges ν, κ_1, and κ_2 such that $R \cap \tilde{\Sigma}_\infty = \nu^o \cap \tilde{\Sigma}_\infty \neq \emptyset$. Let v_1, v_2, and v_3 be the vertices of R such that v_1, v_2, and v_3 are opposite to κ_1, κ_2, and ν respectively. Let ξ be the

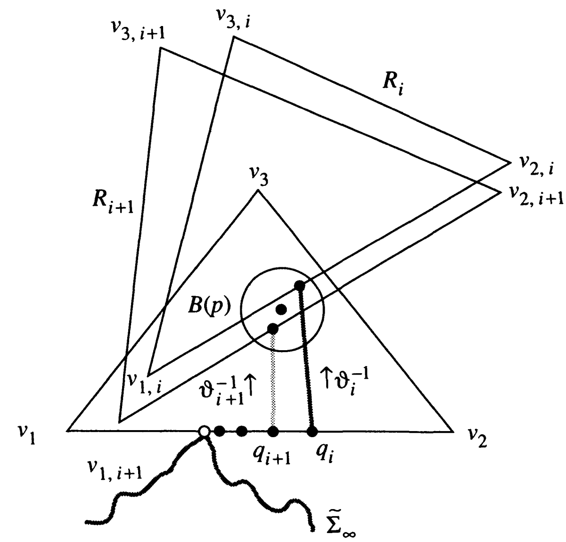

FIGURE 6. PULLBACK SEQUENCES

compact arc $\kappa_1 \cup \kappa_2$, which together with ν bounds R.

5.2. *We make choices.* Let α be the component of $\nu \cap \tilde{\Sigma}$ containing a point of κ_1. The subset α is a half-open line with an endpoint in $\tilde{\Sigma}_\infty$. Let $\delta^+\alpha$ denote the endpoint, which is the unique point of the set $\mathrm{Cl}(\alpha) - \alpha$. Let us choose an arbitrary monotone sequence $\{q_i\}$ on α converging to $\delta^+\alpha$ with respect to $\mathbf{d}$. Let us choose a fundamental domain in $\tilde{\Sigma}$, and let K be a compact neighborhood of the fundamental domain such that $K \subset \tilde{\Sigma}$ holds. For each positive integer i, we choose a deck transformation ϑ_i and a point p_i of K such that $\vartheta_i(p_i) = q_i$.

For convenience of notation, let $v_{j,i}$, $\kappa_{k,i}$, ξ_i, ν_i, and R_i, $i = 1, 2, 3, \ldots$, $j = 1, 2, 3$, $k = 1, 2$, be the images under ϑ_i^{-1} of v_j, κ_k, ξ, ν, and R, respectively. Let $\mathbf{n}_i$ be the outer-normal vector to ν_i at p_i with respect to the spherical Riemannian metric μ for each i.

Since $p_i \in K$ for each positive integer i, the compact subset K includes a point p that is an accumulation point of $\{p_i\}$. By choosing subsequences, if necessary, we may assume without loss of generality that $\{p_i\}$, $\{\mathbf{dev}(v_{j,i})\}$, $\{\mathbf{dev}(\kappa_{k,i})\}$, $\{\mathbf{dev}(\nu_i)\}$, $\{\mathbf{dev}(R_i)\}$, and $\{\mathbf{n}_i\}$, for $j = 1, 2, 3$ and $k = 1, 2$, converge to points p, $v_{j,\infty}$, compact subsets $\kappa_{k,\infty}$, ν_∞, R_∞, and a vector in $UT_p(\tilde{\Sigma})$ respectively. We have thus chosen our sequences $\{p_i\}$, $\{v_{j,i}\}$, $\{\kappa_{k,i}\}$, $\{\nu_i\}$, and $\{R_i\}$ for $j = 1, 2, 3$ and $k = 1, 2$ (see Figure 6).

5.3. *We will now obtain a crescent.* First, we prove the nondegeneracy of $\{R_i\}$: Let $B(p)$ be a tiny disk of p. By Lemma 5.4, there is a positive integer N such that $B(p) \cap \xi_i = \varnothing$ whenever $i > N$. By Lemma 3 of the appendix, there is an integer N_1 greater than N such that R_i includes a common open disk whenever $i > N_1$. Since $\mathbf{dev}(R_i)$ includes the developing image of the common open disk whenever $i > N_1$, the subset R_∞ is a convex disk.

Second, we show that R_∞ is a triangle, a lune, or a great disk. Part 2 of the appendix shows that each of $\kappa_{1,\infty}$, $\kappa_{2,\infty}$, and ν_∞ is a segment or a set consisting of a point and that $\delta R_\infty = \kappa_{1,\infty} \cup \kappa_{2,\infty} \cup \nu_\infty$ holds. Given the collection of segments among $\kappa_{1,\infty}$, $\kappa_{2,\infty}$, and ν_∞, their manifold interiors can be shown to be mutually disjoint. Therefore, R_∞ is a triangle, a lune, or a great disk.

Additionally, $\{\mathbf{dev}(\xi_i)\}$ converges to a compact arc ξ_∞, which equals $\kappa_{1,\infty} \cup \kappa_{2,\infty}$. Hence, R_∞ is a disk bounded by compact arcs ξ_∞ and ν_∞.

Lastly, we show that R_∞ corresponds to a crescent: By the convergence theorem in Part 4 of the appendix, $\check{\Sigma}$ includes a convex disk R^u, a segment ν^u in R^u, and a compact arc ξ^u in R^u such that $\mathbf{dev}$ restricted to them are imbeddings to R_∞, ν_∞, and ξ_∞ respectively. Let us observe

- $\xi^u \subset \tilde{\Sigma}_\infty$ by Lemma 5.4 and Theorem 4 of the appendix.
- $\nu^u \cap \tilde{\Sigma} \neq \varnothing$ since $\tilde{\Sigma}$ is not convex.
- ξ^u and ν^u bound R.

These facts and Lemma 5.5 imply that R^u is a crescent. Thus the first step (I) is complete.

5.4. Lemma. *We have $d(p, \xi_i) \to \infty$ as $i \to \infty$, and, equivalently, there is a positive integer N for each compact subset G of $\tilde{\Sigma}$ such that $G \cap \xi_i = \varnothing$ whenever $i > N$.*

Proof. This follows from the facts that ξ_i is a compact arc in $\tilde{\Sigma}$, and ϑ_1^{-1} is an isometry of $\tilde{\Sigma}$ with respect to d.

5.5. Lemma. *ν_∞ is a great segment, and R_∞ is a lune or a great disk.*

Proof. Recall that $B(p)$ is a tiny disk of p. Let $c = d(p, \operatorname{bd} B(p))$. There is a positive integer N such that $d(p, p_i) < c/2$ whenever $i > N$. Hence, $d(p_i, \operatorname{bd} B(p)) > c/2$ whenever $i > N$. Since α is a semi-infinite with respect to d, it properly includes another semi-infinite, half-open, line α' with respect to d. There is a positive integer N_1 such that for each positive integer i greater than N_1, the line α' includes a closed interval ω_i that contains q_i and such that the d-length of the path on ω_i from q_i to each endpoint of ω_i is $c/2$. Hence $\{\omega_i\}$ satisfies the following properties:

(i) Since $\{q_i\}$ is monotone, it follows that $\mathbf{d}$-length$(\omega_i) \to 0$ as $i \to \infty$.

(ii) Since $\omega_i \subset \alpha'$ whenever $i > N_1$, there is a positive constant c' such that $\inf\{\mathbf{d}(x, \delta\nu) | x \in \omega_i\} > c'$ whenever $i > N_1$.

(iii) Since $\vartheta_i^{-1}(\omega_i) \subset B(p)$ whenever $i > N$, N_1, there is a positive constant C (see §1.11) such that $\mathbf{d}$-length$(\vartheta_i^{-1}(\omega_i)) \geq C^{-1}$ whenever $i > N$, N_1.

The above properties imply easily that ν_∞ is a great segment, and, hence, R_∞ is a lune or a great disk.

6. Obtaining a simplistic crescent

Now, a simplistic crescent in $\check{\Sigma}$ will be found from the crescent obtained in the previous section. This will complete the second step (II). Recall from the introduction the two possible situations. The first case is elementary. The second case is the one where every crescent is a lune crescent. We define special subsets $\Lambda(\mathscr{R})$, $\delta_\infty\Lambda(\mathscr{R})$, and $\Lambda_1(\mathscr{R})$ of $\check{\Sigma}$ given a crescent $\mathscr{R}$. A major portion of this section (§§6.2–6.4) is devoted to finding properties of these subsets. For example, we show that $\delta_\infty\Lambda(\mathscr{R})$ is locally a line, $\Lambda_1(\mathscr{R}) \cup \tilde{\Sigma}^o$ is a projective surface, and $\Lambda(\mathscr{R}) \cap \tilde{\Sigma}$ is a closed subset of $\tilde{\Sigma}$. One of the principal results of this section is the proper concavity of $\Lambda(\mathscr{R})$ (see §3.1), from which we obtain a simplistic crescent.

6.1. Let us start with the easier case that there is a great-disk crescent in $\check{\Sigma}$. Call it $\mathscr{R}$. The crescent $\mathscr{R}$, itself, is properly concave. The two concavity conditions (a) and (b$'$) are obviously true. Condition (c) follows from §2.4. Clearly, $\tilde{\Sigma}$ may or may not be a subset of $\mathscr{R}$. Suppose that $\tilde{\Sigma} \not\subset \mathscr{R}$. Then $\mathrm{bd}\mathscr{R} \cap \tilde{\Sigma} \neq \emptyset$. By the concavity lemma, $\mathscr{R}$ is a simplistic crescent. Suppose that $\tilde{\Sigma} \subset \mathscr{R}$. If $\delta\tilde{\Sigma} = \emptyset$, then $\delta\mathscr{R} = \tilde{\Sigma}_\infty$, a contradiction to the definition of crescents. Hence, $\delta\tilde{\Sigma} \neq \emptyset$. Since a boundary component of $\tilde{\Sigma}$ is a subset of $\delta\mathscr{R}$, the crescent $\mathscr{R}$ is simplistic.

6.2. Now, we consider the case where there is no great-disk crescent. Two crescents in $\check{\Sigma}$ are said to be *simply equivalent* if they overlap. This relation generates an equivalence relation for the collection of crescents in $\tilde{\Sigma}$. Given a crescents $\mathscr{R}$, let us define the following nonempty subsets (see Figure 7):

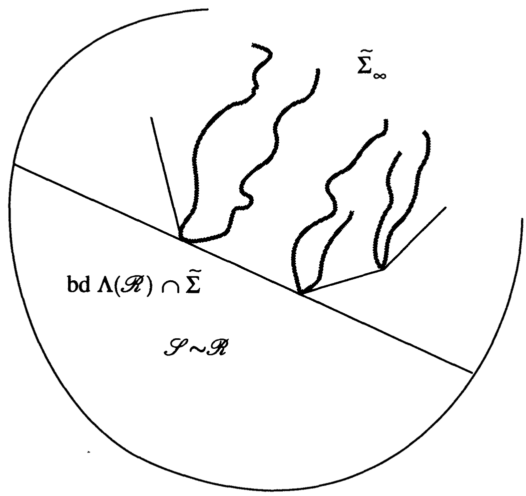

FIGURE 7. A PICTURE OF $\Lambda(\mathscr{R})$.

$$\Lambda(\mathscr{R}) = \bigcup_{\mathscr{S} \sim \mathscr{R}} \mathscr{S}, \quad \delta_\infty \Lambda(\mathscr{R}) = \bigcup_{\mathscr{S} \sim \mathscr{R}} \alpha_{\mathscr{S}},$$
$$\Lambda_1(\mathscr{R}) = \bigcup_{\mathscr{S} \sim \mathscr{R}} (\mathscr{S} - \nu_{\mathscr{S}}).$$

For example, in $\check{\Sigma}_i$, for each $i = 2, 3, 4, 5$ (see §1.2), $\Lambda(\mathscr{R})$, given a crescent $\mathscr{R}$, is identical with $\check{\Sigma}_i$. In $\check{\Sigma}_4$, we have for a crescent $\mathscr{R}$

$$\Lambda_1(\mathscr{R}) = B_2^o \cup B_3^o \cup \overline{-e_3 e_2 - e_2}^o, \qquad \delta_\infty \Lambda(\mathscr{R}) = \overline{-e_3 e_2 - e_2}^o$$

In $\check{\Sigma}_5$, we have for a crescent $\mathscr{R}$

$$\Lambda_1(\mathscr{R}) = \check{\Sigma}_5 - \{[0, 0]\}, \qquad \delta_\infty \Lambda(\mathscr{R}) = \{[\pi/2, \theta] | \theta \in \mathbf{R}\}.$$

The subsets $\Lambda(\mathscr{R})$, $\delta_\infty \Lambda(\mathscr{R})$, and $\Lambda_1(\mathscr{R})$ for a crescent $\mathscr{R}$ in $\check{\Sigma}$ have many properties. By definition, $\Lambda(\mathscr{R})$ and $\Lambda_1(\mathscr{R})$ are path connected. Since $\alpha_{\mathscr{S}} \subset \tilde{\Sigma}_\infty$ for every crescent $\mathscr{S}$, it follows that $\delta_\infty \Lambda(\mathscr{R}) \subset \tilde{\Sigma}_\infty$. By Theorem 2.6, $\delta_\infty \Lambda(\mathscr{R})$ is locally a line; hence, a unique great circle $\mathbf{S}^1$ includes $\mathbf{dev}(\delta_\infty \Lambda(\mathscr{R}))$. By the same theorem, there exists a component $A_{\mathscr{R}}$ of $\mathbf{S}^2 - \mathbf{S}^1$ such that

$$\mathbf{dev}(\Lambda(\mathscr{R})) \subset \mathrm{Cl}(A_{\mathscr{R}}) \quad \text{and} \quad \mathbf{dev}(\Lambda(\mathscr{R}) - \mathrm{Cl}(\delta_\infty \Lambda(\mathscr{R}))) \subset A_{\mathscr{R}}$$

hold. For every deck transformation ϑ preserving $\Lambda(\mathscr{R})$, the projective automorphism $h(\vartheta)$ preserves $A_{\mathscr{R}}$, where $h: \pi_1(\Sigma) \to \mathrm{Aut}(\mathbf{S}^2)$ is the holonomy homomorphism.

For a crescent $\mathscr{S}$ equivalent to $\mathscr{R}$, the subset $\mathscr{S} - \nu_{\mathscr{S}}$ is open and is a disk with boundary $\alpha_{\mathscr{S}}$ included in $\delta_\infty \Lambda(\mathscr{R})$. Thus, the subset $\Lambda_1(\mathscr{R})$ is a connected differentiable surface with boundary $\delta_\infty \Lambda(\mathscr{R})$. Since $\mathscr{S} - \nu_{\mathscr{S}}$ has a projective structure compatible with the projective structure of $\tilde{\Sigma}$, also so the surface $\Lambda_1(\mathscr{R})$ has. Consequently, $\Lambda_1(\mathscr{R}) \cup \tilde{\Sigma}^o$ is a projective surface with boundary $\delta_\infty \Lambda(\mathscr{R})$ and with the projective structure extending the projective structure on $\tilde{\Sigma}^o$.

Next, let us suppose that $x \in \mathrm{bd}\, \Lambda(\mathscr{R}) \cap \tilde{\Sigma}$. A sequence $\{x_i\}$ of points of $\Lambda(\mathscr{R})$ converges to x. A lune crescent $\mathscr{R}_i$ equivalent to $\mathscr{R}$ contains x_i for each i. Now, let us apply the results of the appendix as in §5 to obtain a crescent containing x and equivalent to $\mathscr{R}$. Hence, $x \in \Lambda(\mathscr{R})$. Therefore, $\mathrm{bd}\, \Lambda(\mathscr{R}) \cap \tilde{\Sigma} \subset \Lambda(\mathscr{R})$, and $\Lambda(\mathscr{R}) \cap \tilde{\Sigma}$ is a closed subset of $\tilde{\Sigma}$.

Furthermore, $\mathrm{bd}\, \Lambda(\mathscr{R}) \cap \tilde{\Sigma} \subset \tilde{\Sigma}^o$. Suppose not. Let $x \in \mathrm{bd}\, \Lambda(\mathscr{R}) \cap \delta\tilde{\Sigma}$. Then $x \in \mathscr{S}$ for a crescent $\mathscr{S}$ equivalent to $\mathscr{R}$. Since the component of $\delta\tilde{\Sigma}$ containing x is a subset of $\nu_{\mathscr{S}}$ by Lemma 3.4, $x \in \mathrm{int}\, \mathscr{S}$. Thus, $x \in \mathrm{int}\, \Lambda(\mathscr{R})$. This is a contradiction.

Next, $\Lambda(\mathscr{R})$ satisfies the *maximum property*, i.e., if given a segment $\alpha \subset \tilde{\Sigma}$, and $\delta\alpha$ is a subset of $\Lambda(\mathscr{R})$, then $\alpha \subset \Lambda(\mathscr{R})$. This is not proved in this paper although this is a straightforward consequence of the properties of lines (§4.8) and the fact that $\Lambda_1(\mathscr{R}) \cup \tilde{\Sigma}^o$ is a projective surface with geodesic boundary $\delta_\infty \Lambda(\mathscr{R})$.

Perhaps the most important property of $\Lambda(\mathscr{R})$ is its proper concavity. The first concavity condition (a) follows from Lemma 6.3. Next, let α be an arbitrary component of $\mathrm{bd}\, \Lambda(\mathscr{R}) \cap \tilde{\Sigma}$. Choose an arbitrary point on α. Then it belongs to a crescent $\mathscr{S}$ where $\mathscr{S} \sim \mathscr{R}$. It is straightforward to show that $\alpha \subset \nu_{\mathscr{S}} \cap \tilde{\Sigma}$, so that the second concavity condition (b$'$) is satisfied. Suppose that $\mathrm{int}(\Lambda(\mathscr{R}) \cap \tilde{\Sigma}) \cap \vartheta(\Lambda(\mathscr{R})) \neq \varnothing$ holds for a deck transformation ϑ. Since by Lemma 6.4(2), $\Lambda(\mathscr{R}) \cap \Lambda_1(\vartheta(\mathscr{R})) \neq \varnothing$, by the same lemma $\Lambda(\mathscr{R}) = \Lambda(\vartheta(\mathscr{R})) = \vartheta(\Lambda(\mathscr{R}))$. Hence we have proved the last concavity condition (c).

6.3. The following lemma is a consequence of the dipping intersection property and the maximum property. Let $x \in \mathrm{bd}\, \Lambda(\mathscr{R}) \cap \tilde{\Sigma}$. Since $x \in \tilde{\Sigma}^o$, we let $B(x)$ to be a tiny disk of x included in $\tilde{\Sigma}^o$.

Lemma. $\mathrm{bd}\, \Lambda(\mathscr{R}) \cap B(x)^o$ *is a connected open line passing through* x *with endpoints in* $\delta B(x)$.

Proof. Let $O = B(x)^o - \Lambda(\mathscr{R})$. Since $B(x)^o \not\subset \Lambda(\mathscr{R})$, we have $O \neq \varnothing$. Suppose that $\mathscr{S} \cap B(x)^o \neq \varnothing$ for a crescent $\mathscr{S}$ equivalent to $\mathscr{R}$.

Since $B(x) \not\subset \mathscr{S}$, it follows that $B(x)$ is dipped into $(\mathscr{S}, \nu_{\mathscr{S}})$ nicely. This shows that O and, hence, $\mathrm{Cl}(O)$ are nonempty convex subsets. Since $B(x)^o$ is convex, the maximum property of $\Lambda(\mathscr{R})$ yields the convexity of $\Lambda(\mathscr{R}) \cap B(x)^o$. Since $B(x)^o$ is open,

$$\mathrm{bd}\,\Lambda(\mathscr{R}) \cap B(x)^o = (\Lambda(\mathscr{R}) \cap B(x)^o) \cap (\mathrm{Cl}(O) \cap B(x)^o).$$

Due to the convexity of $\Lambda(\mathscr{R}) \cap B(x)^o$ and $\mathrm{Cl}(O) \cap B(x)^o$, $\mathrm{bd}\,\Lambda(\mathscr{R}) \cap B(x)^o$ is a separating convex line with two endpoints in $\delta B(x)$.

6.4. Lemma. *Let ϑ be a deck transformation.*

(1) $\vartheta(\Lambda(\mathscr{R})) = \Lambda(\vartheta(\mathscr{R}))$, $\vartheta(\Lambda_1(\mathscr{R})) = \Lambda_1(\vartheta(\mathscr{R}))$, *and* $\vartheta(\delta_\infty\Lambda(\mathscr{R})) = \delta_\infty\Lambda(\vartheta(\mathscr{R}))$.

(2) *If* $\mathrm{int}(\Lambda(\mathscr{R}) \cap \tilde{\Sigma}) \cap \vartheta(\Lambda(\mathscr{R})) \neq \varnothing$, *then* $\Lambda(\mathscr{R}) \cap \Lambda_1(\vartheta(\mathscr{R})) \neq \varnothing$.

(3) *If* $\Lambda(\mathscr{R}) \cap \Lambda_1(\vartheta(\mathscr{R})) \neq \varnothing$, *then* $\mathscr{R} \sim \vartheta(\mathscr{R})$.

Proof. (1) This is implied by the fact that $\mathscr{S} \sim \mathscr{R}$ if and only if $\vartheta(\mathscr{S}) \sim \vartheta(\mathscr{R})$ given a crescent $\mathscr{S}$.

(2) Let x be a point belonging to

$$\mathrm{int}(\Lambda(\mathscr{R}) \cap \tilde{\Sigma}) \cap \vartheta(\Lambda(\mathscr{R})) = \mathrm{int}(\Lambda(\mathscr{R}) \cap \tilde{\Sigma}) \cap \Lambda(\vartheta(\mathscr{R})).$$

The open subset $\mathrm{int}(\Lambda(\mathscr{R}) \cap \tilde{\Sigma})$ includes a tiny disk $B(x)$ of x, and a crescent $\mathscr{S}$ equivalent to $\vartheta(\mathscr{R})$ contains x. Recall (§2.4) that $B(x)$ is either a subset of $\mathscr{S}$ or dipped into $(\mathscr{S}, \nu_{\mathscr{S}})$ nicely. Since $B(x) \cap \mathscr{S}^o \neq \varnothing$ in each of the cases, we have

$$\Lambda(\mathscr{R}) \cap \mathscr{S}^o \neq \varnothing \quad \text{and} \quad \Lambda(\mathscr{R}) \cap \Lambda_1(\vartheta(\mathscr{R})) \neq \varnothing.$$

(3) Since $\Lambda(\mathscr{R}) \cap \Lambda_1(\vartheta(\mathscr{R})) \neq \varnothing$, there are two crescents $\mathscr{S}$ and $\mathscr{T}$ such that

$$\mathscr{S} \sim \mathscr{R}, \quad \mathscr{T} \sim \vartheta(\mathscr{R}), \quad \text{and} \quad \mathscr{S} \cap (\mathscr{T} - \nu_{\mathscr{T}}) \neq \varnothing.$$

Since $\mathscr{S}^o$ is dense in $\mathscr{S}$, the openness of $\mathscr{T} - \nu_{\mathscr{T}}$ implies that $\mathscr{S}^o \cap (\mathscr{T} - \nu_{\mathscr{T}}) \neq \varnothing$. Hence, $\mathscr{S} \sim \mathscr{T}$ and $\mathscr{R} \sim \vartheta(\mathscr{R})$.

6.5. We now complete the second step (II). A simplistic crescent can be obtained from $\Lambda(\mathscr{R})$ in a very similar manner to §6.1 unless we have $\tilde{\Sigma} \subset \Lambda(\mathscr{R})$ and $\delta\tilde{\Sigma} = \varnothing$. We show that this cannot happen. Recall what $A_{\mathscr{R}}$ denotes from §6.2. Since $\tilde{\Sigma}$ is invariant under deck transformations, we have

$$\vartheta(\Lambda(\mathscr{R})) = \Lambda(\mathscr{R}) \quad \text{and} \quad h(\vartheta)(A_{\mathscr{R}}) = A_{\mathscr{R}}$$

for every deck transformation ϑ on $\tilde{\Sigma}$ and the holonomy homomorphism h of Σ. The component $A_{\mathscr{R}}$, an open great disk, has a natural affine structure (see §1.3), and $h(\vartheta)$ restricts to an affine automorphism on $A_{\mathscr{R}}$

with respect to the affine structure for every deck transformation ϑ. Thus Σ has an affine structure. The corollary to Theorem 1 [19] implies $\chi(\Sigma) = 0$ which is a contradiction.

7. π-Annulus

Finally, we obtain a π-annulus with a projective map to Σ from the previously obtained simplistic crescent using a single technique whether the crescent is a great disk or a lune. This completes the third step (III) and, hence, the proof of the main theorem.

7.1. Let us go over some properties of simplistic crescents. Let $\mathscr{R}$ be a simplistic crescent. Choose a simplistic line τ of $\mathscr{R}$. Let ϑ be the deck transformation corresponding to τ. The relation

$$\tau = \vartheta(\tau) \subset \vartheta(\nu_{\mathscr{R}}) = \nu_{\vartheta(\mathscr{R})}$$

and the orientability of Σ imply that $\mathscr{R}^o \cap \vartheta(\mathscr{R}) \neq \varnothing$. By Theorem 2.6, we have

$$\mathscr{R} = \vartheta(\mathscr{R}), \quad \nu_{\mathscr{R}} = \vartheta(\nu_{\mathscr{R}}), \quad \text{and} \quad \alpha_{\mathscr{R}} = \vartheta(\alpha_{\mathscr{R}}).$$

Since $\mathscr{R}$ is tame, there is the inverse map of $\mathbf{dev}|\mathscr{R}$ denoted by $(\mathbf{dev}|\mathscr{R})^{-1}\colon \mathbf{dev}(\mathscr{R}) \to \check{\Sigma}$. Thus

$$h(\vartheta)|\mathbf{dev}(\mathscr{R}) = \mathbf{dev} \circ \vartheta \circ (\mathbf{dev}|\mathscr{R})^{-1}.$$

Since $\langle\vartheta\rangle$ acts on $\mathscr{R}$ and properly discontinuously and freely on $\mathscr{R} \cap \tilde{\Sigma}$, the subgroup $\langle h(\vartheta)\rangle$ acts on $\mathbf{dev}(\mathscr{R})$ and properly discontinuously and freely on $\mathbf{dev}(\mathscr{R} \cap \tilde{\Sigma})$.

Finally, at least one endpoint of each simplistic line of $\mathscr{R}$ is an endpoint of $\nu_{\mathscr{R}}$. This can be seen from the following discussion: Suppose that $\check{\Sigma}$ includes a convex segment α invariant under a deck transformation φ. Then φ has at most three fixed points on α unless the function $\varphi'\colon \alpha \to \alpha$ obtained from φ by restricting the domain and the range is the identity map.

7.2. The process of obtaining a π-annulus starts now. Let τ be a simplistic line of a simplistic crescent $\mathscr{R}$; let ϑ be the deck transformation corresponding to τ. Let $\tau' = \nu^o_{\mathscr{R}} - \mathrm{Cl}(\tau)$. It follows that τ' is a connected open line ending at an endpoint of $\nu_{\mathscr{R}}$ or is the empty set since an endpoint of τ is an endpoint of $\nu_{\mathscr{R}}$. There are the following possibilities:

(i) $\tau' \subset \tilde{\Sigma}_\infty$.
(ii) $\tau' \not\subset \tilde{\Sigma}_\infty$, and $\tau' \cap \tilde{\Sigma}_\infty = \varnothing$.
(iii) $\tau' \not\subset \tilde{\Sigma}_\infty$, and $\tau' \cap \tilde{\Sigma}_\infty \neq \varnothing$.

We deduce by elimination that only (ii) is possible and that (ii) implies the existence of a π-annulus.

(i) Since a component $\mathscr{R}^o$ of $\widetilde{\Sigma} - \tau$ is convex, the following lemma contradicts the nonconvexity of Σ.

Lemma. *Let M be a compact projective surface with negative Euler characteristic, and let α be a simplistic line in the universal cover $\widetilde{M}$ such that a component M_1 of $\widetilde{M} - \alpha$ is a convex. Then M is convex.*

Proof. Let d' be an original metric of $\widetilde{M}$ (see §1.11). Since $\chi(M) < 0$, for every positive real number R and a point x of $\widetilde{M}$, there is a deck transformation φ that maps x to $\varphi(x)$ in M_1 such that $d'(\varphi(x), \alpha) > R$. (To see this, use hyperbolic geometry.)

Let x and y be two arbitrary points belonging to $\widetilde{M}$; then a deck transformation φ maps x into M_1 such that $d'(\varphi(x), \alpha) > 2d'(x, y)$. Thus, $\varphi(y)$ belongs to M_1. Since M_1 is convex, M_1 includes a convex segment $\overline{\varphi(x)\varphi(y)}$ connecting $\varphi(x)$ and $\varphi(y)$; thus, the segment $\varphi^{-1}\overline{(\varphi(x)\varphi(y))}$ connects x and y. Hence, $\widetilde{M}$ is convex. (Compare this argument to one in Chapter 2 of Benzécri [1].)

(ii) Since Σ is orientable, $h(\vartheta)$ is orientation preserving. Let

$$\Omega = \mathbf{dev}(\mathscr{R}^o) \cup \mathbf{dev}(\tau) \cup \mathbf{dev}(\tau').$$

Then $\langle h(\vartheta)\rangle$ acts properly discontinuously and freely on Ω. By definition, the quotient projective surface Λ of Ω under the action of $\langle h(\varphi)\rangle$ is a π-annulus. The projective map $\widetilde{\Phi}\colon \Omega \to \Sigma$ defined by $\widetilde{\Phi} = \mathbf{p} \circ (\mathbf{dev}|\mathscr{R})^{-1}|\Omega$ induces a projective map $\Phi\colon \Lambda \to \Sigma$ here $\mathbf{p}\colon \widetilde{\Sigma} \to \Sigma$ is the universal covering map.

(iii) There is a component α of $\nu^o_{\mathscr{R}} \cap \widetilde{\Sigma}$ with both endpoints in $\nu^o_{\mathscr{R}}$. By Lemma 7.3, similarly to 6.1, the component α is simplistic. This contradicts a result of §7.1 since α has both endpoints in $\nu^o_{\mathscr{R}}$.

7.3. Suppose that we are in case (iii). Let p and q be two endpoints of $\nu_{\mathscr{R}}$ so that p is an endpoint of τ as well. Since $\langle\vartheta\rangle$ acts freely on τ, $\langle\vartheta\rangle$ acts on τ'. Since $\tau' \cap \widetilde{\Sigma}_\infty \neq \varnothing$, any neighborhood of q intersects $\tau' \cap \widetilde{\Sigma}_\infty$; thus, q is a boundary point of $\tau' \cap \widetilde{\Sigma}_\infty$.

Lemma. *$\mathscr{R}$ is concave.*

Proof. Clearly, $\mathscr{R}$ satisfies the first two concavity conditions (a) and (b). The third condition (c) follows from the statement that if $\mathscr{R}$ and $\varphi(\mathscr{R})$ overlap for a deck transformation φ, then $\mathscr{R} = \varphi(\mathscr{R})$. Recall (§2.4) that this statement is true if $\mathscr{R}$ is a great disk. Hence, we need to prove this statement when $\mathscr{R}$ is a lune.

So, suppose that $\mathscr{R}$ is a lune. Suppose, to the contrary, that there is a deck transformation φ such that $\mathscr{R}^o \cap \varphi(\mathscr{R}) \neq \emptyset$ and $\mathscr{R} \neq \varphi(\mathscr{R})$. Condition (iii) and a result of §2.4 show that $\mathscr{R} \not\subset \varphi(\mathscr{R})$ and $\varphi(\mathscr{R}) \not\subset \mathscr{R}$. Thus, $\mathscr{R}$ and $\varphi(\mathscr{R})$ intersect transversally by Theorem 2.6. Let x be the transversal intersection point of $\nu_{\mathscr{R}}$ and $\varphi(\nu_{\mathscr{R}})$. Either p or q is a point of $\alpha_{\varphi(\mathscr{R})}$; either $\varphi(p)$ or $\varphi(q)$ is a point of $\alpha_{\mathscr{R}}$. Suppose that $q \in \alpha_{\varphi(\mathscr{R})}$. Then $\nu_{\mathscr{R}}$ includes a relative segment neighborhood α of q that is the closure of a component of $\nu_{\mathscr{R}} - \{x\}$. Here, $\alpha^o \subset \varphi(\mathscr{R}^o)$. Since q is a boundary point of $\tilde{\Sigma}_\infty \cap \nu^o_{\mathscr{R}}$, it follows that $\alpha^o \cap \tilde{\Sigma}_\infty \neq \emptyset$; this contradicts $\varphi(\mathscr{R}^o) \subset \tilde{\Sigma}$. Therefore, $p \in \alpha_{\varphi(\mathscr{R})}$ and, similarly, $\varphi(p) \in \alpha_{\mathscr{R}}$.

Both $\mathbf{dev}(\mathscr{R})$ and $\mathbf{dev}(\varphi(\mathscr{R}))$ are lunes such that a common great circle $\mathbf{S}^1$ includes $\mathbf{dev}(\alpha_{\mathscr{R}})$ and $\mathbf{dev}(\alpha_{\varphi(\mathscr{R})})$, and these lunes are included in a great disk bounded by $\mathbf{S}^1$. However, since $p \in \alpha_{\varphi(\mathscr{R})}$ and $\varphi(p) \in \alpha_{\mathscr{R}}$, the projective automorphism $h(\varphi)$ is orientation reversing; this is a contradiction.

Appendix: Sequences of convex disks

First, we discuss for $\mathbf{S}^2$ the convergence of a sequence of convex disks, the limit of a convergent sequence, and the boundary and the interior of the limit. Next, we consider sequences of convex disks in a projective completion. Subsequences may not converge since the projective completion is not compact in general. However, when a sequence has a common convex open disk in every sequence element, a subsequence will have a "limit." So, a certain criterion assuring the existence of a common convex open disk is presented first. Finally, certain sequences are shown to "converge."

1. We need to use the following fact from spherical geometry: Given a great circle $\mathbf{S}^1$ in $\mathbf{S}^2$ and a point x belonging to $\mathbf{S}^2 - \mathbf{S}^1$, if a line from x to $\mathbf{S}^1$ is perpendicular to $\mathbf{S}^1$, and its $\mathbf{d}$-length is less than or equal to $\pi/2$, then its $\mathbf{d}$-length equals $\mathbf{d}(x, \mathbf{S}^1)$. (Recall that $\mathbf{d}$ is the standard metric of the standard sphere $\mathbf{S}^2$.) Another fact which we use is as follows: Suppose that $x \in B$ for a convex disk B. Then $\mathbf{d}(x, \delta B) \leq \pi/2$. We have $\mathbf{d}(x, \delta B) = \pi/2$ if and only if B is the great disk of which x is the center.

Let $\mathbf{d}^H$ be the Hausdorff distance of compact subsets of $\mathbf{S}^2$. Given any two compact disks in $\mathbf{S}^2$ arbitrarily close to each other, there may be a point which is "deep" in the interior of one of the disk but is not in the other. The following lemma shows that this cannot happen for two convex disks. More precisely, given two convex disks A and B in $\mathbf{S}^2$ if

$\mathbf{d}^H(A, B) < \varepsilon$ for a positive constant ε, then $A^o - B^o$ and $B^o - A^o$ lie inside 2ε-neighborhoods of δA and δB respectively.

Lemma. *Let A and B be two convex disks in $\mathbf{S}^2$. Suppose that $A^o - B^o$ contains a point x such that $\mathbf{d}(x, \delta A) > \varepsilon$ for a positive constant ε. Then $\mathbf{d}^H(A, B) > \varepsilon/2$.*

Proof. Since $x \notin B^o$, a great disk H contains x and satisfies $B \cap H^o = \emptyset$ (see §1.3). The proof is reduced to the claim that $A \cap H$ contains a point y such that $\mathbf{d}(y, \mathbf{S}^2 - H^o) > \varepsilon/2$. Let α be the diameter of H passing through x. Let $\beta = \alpha \cap A$. The subset β is a connected segment in the convex disk $A \cap H$, and the endpoints of β are contained in $\delta(A \cap H)$. Since $\beta \ni x$, and at least one of the endpoints of β belongs to δA, it follows that $\mathbf{d}$-length$(\beta) > \varepsilon$. Thus $\beta \subset \alpha$ and $\varepsilon < \pi$ imply that the segment β includes a point y such that $\varepsilon/2 < \mathbf{d}(y, \delta\alpha) \leq \pi/2$. Since α is perpendicular to δH, $\mathbf{d}(y, \mathbf{S}^2 - H^o) > \varepsilon/2$.

2. A sequence of compact convex subsets of $\mathbf{S}^2$ always has a convergent subsequence $\{D_i\}$ with respect to $\mathbf{d}^H$, the limit is always a compact convex subset, and the dimension of the limit is less than or equal to $\liminf_{i\to\infty} \dim(D_i)$. We deduce from this that the limit of a convergent sequence of convex segments is either a convex segment or a set of consisting of a point; however, the limit cannot be a great circle. We may also deduce that the limit of a convergent sequence of convex disks is a great disk, a lune, a simply convex disk, a convex segment, or a set consisting of a point; the limit cannot be $\mathbf{S}^2$ itself or a great circle.

Now, interesting questions arise: what are the properties of the boundary and the interior of the limit? Suppose that a sequence of convex disks $\{D_i\}$ converges to a convex disk D. Then the following statements concerning the boundary and the interior hold: δD is the limit of $\{\delta D_i\}$, and $\bigcup_{i=1}^{\infty} D_i^o \supset D^o$.

The boundary statement is an easy consequence of the following statement: given two convex disks A and B, if $\mathbf{d}^H(A, B) \leq \varepsilon$ for a positive real number ε, then $\mathbf{d}^H(\delta A, \delta B) \leq 2\varepsilon$. This statement is proved. Suppose that $\mathbf{d}^H(\delta A, \delta H) > 2\varepsilon$. Then either δA contains a point x such that $\mathbf{d}(x, \delta B) > 2\varepsilon$ or δB contains a point y such that $\mathbf{d}(\delta A, y) > 2\varepsilon$. Let us assume the former without loss of generality. If $x \notin B$, then $\mathbf{d}(x, B) > 2\varepsilon$ and, hence, $\mathbf{d}^H(A, B) > 2\varepsilon$. If $x \in B$, then $x \in B^o$ and, by Lemma 1, $\mathbf{d}^H(A, B) > \varepsilon$. These are contradictions. Thus, the statement is proved.

The proof of the interior statement is as follows: Let $x \in D^o$. Then $\mathbf{d}(x, \delta D) > \varepsilon$ for a positive constant ε. Let N be a positive integer such that $\mathbf{d}^H(D, D_i) < \varepsilon/2$ whenever $i > N$. By Lemma 1, $x \in D_i^o$ whenever $i > N$. Thus $\bigcup_{i=1}^{\infty} D_i^o \supset D^o$.

3. As in §1 of the main text, let M be a projective surface, which may or may not be compact, and let $\widetilde{M}$ be its universal cover. Let **dev** be the developing map; let $\check{M}$ be the projective completion. Let μ and $\mathbf{d}$ be the spherical Riemannian and the spherical metrics associated with **dev** respectively. We study sequences of convex disks in $\check{M}$.

Lemma. *Let $\{D_i\}$ be a sequence of convex disks in $\check{M}$. Let $x \in \widetilde{M}$, and let $B(x)$ be a tiny disk of x. Suppose that the following properties hold:*

(1) *There is a segment ν_i in δD_i for each i.*

(2) $B(x) \cap (\delta D_i - \nu_i^o) = \varnothing$ *for each i.*

(3) *A sequence $\{x_i\}$ converges to x where $x_i \in \nu_i$ for each i.*

(4) *The sequence $\{\mathbf{n}_i\}$ converges where $\mathbf{n}_i$ is the outer-normal vector to ν_i at x_i with respect to μ for each i.*

Then there exist a positive integer N and a convex open disk $\mathscr{P}$ in $B(x)$ such that $\mathscr{P} \subset D_i$ whenever $i > N$.

Proof. We may suppose without loss of generality that $B(x) \not\subset D_i$ and that $x_i \in \operatorname{int} B(x)$ for each i. This means that $B(x)$ is dipped into (D_i, ν_i) nicely for each i by Corollary 1.9.

Now, we show that $D_i \cap B(x)$ includes a common convex $\mathbf{d}$-ball of a fixed radius for sufficiently large i. Let c be a number satisfying $\mathbf{d}(x, \operatorname{bd} B(x)) > c$, $0 < c < \pi/2$. There exists a positive integer N_1 such that $\mathbf{d}(x, x_i) < c/4$ whenever $i > N_1$. Thus, $\mathbf{d}(x_i, \operatorname{bd} B(x)) > c/2$ whenever $i > N_1$. For each i, the convex disk $D_i \cap B(x)$ includes a maximal segment κ_i that is inward normal to $\nu_i \cap B(x)$ at x_i with respect to μ. Let $i > N_1$. Since κ_i ends at a point of $\operatorname{bd} B(x)$, it follows that $\mathbf{d}$-length$(\kappa_i) > c/2$. Let y_i be the point of κ_i such that $\mathbf{d}(x_i, y_i) = c/4$. Then y_i satisfies the following properties:

$$\mathbf{d}(y_i, \nu_i \cap B(x)) = c/4 \quad (\text{by } \S 1),$$

and

$$\mathbf{d}(y_i, \operatorname{bd} B(x)) \geq \mathbf{d}(x_i, \operatorname{bd} B(x)) - \mathbf{d}(x_i, y_i) > c/4.$$

Since

$$\delta(D_i \cap B(x)) \subset (\nu_i \cap B(x)) \cup \operatorname{bd} B(x),$$

it follows that

$$\mathbf{d}(y_i, \delta(D_i \cap B(x))) \geq \min\{\mathbf{d}(y_i, \nu_i \cap B(x)), \mathbf{d}(y_i, \operatorname{bd} B(x))\} > c/8.$$

Thus, the $\mathbf{d}$-ball $B(y_i, c/8)$ with radius $c/8$ around y_i satisfies

$$B(y_i, c/8) \subset (D_i \cap B(x))^o = D_i^o \cap B(x)^o.$$

We conclude that D_i^o includes the convex open disk $B(y_i, c/8)^o$ whenever $i > N_1$.

Since $\{\mathbf{n}_i\}$ converges to an element of $UT_x(\widetilde{M})$, the sequence $\{y_i\}$ converges to a point y of $B(x)$. Let N be a positive integer greater than N_1 such that

$$\mathbf{d}(y, y_i) < c/16 \text{ whenever } i > N.$$

Then $B(y_i, c/8) \supset B(y, c/16)$ whenever $i > N$. Let $\mathscr{P}$ be $B(y, c/16)^o$, which is a convex open disk. This completes the proof.

4. We say that a compact subset D_∞ of $\mathbf{S}^2$ is the *resulting set* of a sequence $\{D_i\}$ of compact subsets of $\check{M}$ if $\{\mathbf{dev}(D_i)\}$ converges to D_∞. Let $\{D_i\}$ and $\{B_i\}$ be sequences of convex disks in $\check{M}$ with resulting sets D_∞ and B_∞ respectively; let $\{K_i\}$ be a sequence of compact subsets in $\check{M}$ with the resulting set K_∞. We say that $\{D_i\}$ *subjugates* $\{K_i\}$ if $D_i \supset K_i$ for each i and that $\{B_i\}$ *dominates* $\{D_i\}$ if $B_i^o \cap D_i \neq \varnothing$ for each i and $B_\infty \supset D_\infty$. Moreover, $\{K_i\}$ is *ideal* if there is a positive integer N for each compact subset F of $\widetilde{M}$ such that $F \cap K_i = \varnothing$ whenever $i > N$.

Convergence Theorem. *Suppose that $\{D_i\}$ is a sequence with a common convex open disk $\mathscr{P}$, that $\{D_i\}$ subjugates $\{K_i\}$ and that $\{B_i\}$ dominates $\{D_i\}$. Then $\check{M}$ includes two convex disks D^u and B^u and a compact subset K^u with the following properties*:

(1) *$D^u \supset \mathscr{P}$, and $\mathbf{dev}(D^u) = D_\infty$.*

(2) *$B^u \supset D^u$, and $\mathbf{dev}(B^u) = B_\infty$.*

(3) *$D^u \supset K^u$, and $\mathbf{dev}(K^u) = K_\infty$.*

(4) *If $\{K_i\}$ is ideal, then $K^u \subset \widetilde{M}_\infty$.*

Proof. (1) Since $\mathbf{dev}(D_i) \supset \mathbf{dev}(\mathscr{P})$ for each i, it follows that $D_\infty \supset \mathbf{dev}(\mathscr{P})$. Hence, D_∞ is a convex disk. By §2, $\bigcup_{i=1}^\infty \mathbf{dev}(D_i)^o \supset D_\infty^o$. By Lemma 1.6, $\mathbf{dev}|\bigcup_{i=1}^\infty D_i^o$ is an imbedding onto $\bigcup_{i=1}^\infty \mathbf{dev}(D_i)^o$. Thus, $\bigcup_{i=1}^\infty D_i^o$ includes a convex disk D^* such that $\mathbf{dev}|D^*$ is an imbedding onto D_∞^o. If we let $D^u = \mathrm{Cl}(D^*)$, then (1) follows.

(2) Clearly, $B_\infty \supset \mathbf{dev}(\mathscr{P})$. Let $\mathscr{P}'$ be a compact convex disk in $\mathscr{P}^o$; let $\mathscr{P}'' = (\mathscr{P}')^o$. Then, for each point x of $\mathbf{dev}(\mathscr{P}'')$, $\mathbf{d}(x, \delta B_\infty) > \varepsilon$ for a small positive constant ε. By Lemma 1, there is a positive integer N such that $\mathbf{dev}(B_i) \supset \mathbf{dev}(\mathscr{P}'')$ whenever $i > N$. Since $\mathbf{dev}|B_i \cup D_i$ is an imbedding onto $\mathbf{dev}(B_i) \cup \mathbf{dev}(D_i)$ and $D_i \supset \mathscr{P}$, it follows that $B_i \supset \mathscr{P}''$ whenever $i > N$. (2) follows from (1).

(3) Since $K_i \subset D_i$ for each i, $K_\infty \subset D_\infty$. Let

$$K^u = (\mathbf{dev}|D^u)^{-1}(K_\infty).$$

Then (3) follows.

(4) We show that $K^u \subset \widetilde{M}_\infty$. To the contrary, suppose that $x \in K^u \cap \widetilde{M}$. Suppose further that $x \in \widetilde{M}^o$; thus, there is a tiny disk $B(x)$ satisfying $x \in B(x)^o$ and such that $\mathbf{dev}(B(x))^o \cup D_\infty^o$ is star-shaped from a point y of $\mathbf{dev}(\mathscr{P})$. (A *star-shaped* subset of $\mathbf{S}^2$ from a point is a subset such that each of its elements can be connected by a simply convex segment in it from the point.) Since $B(x)^o \cap D^* \neq \varnothing$, Lemma 1.6 shows that $\mathbf{dev}|B(x)^o \cup D^*$ is an imbedding onto $\mathbf{dev}(B(x))^o \cup D_\infty^o$. For each i, the following statements hold:

(i) Both D_i^o and $B(x)^o \cup D^*$ include $\mathscr{P}$.

(ii) Both $\mathbf{dev}(D_i)^o$ and $\mathbf{dev}(B(x))^o \cup D_\infty^o$ are star-shaped from the point y of $\mathbf{dev}(\mathscr{P})$.

By Lemma 1.6 $\mathbf{dev}|D_i^o \cup B(x)^o \cup D^*$ is an imbedding onto $\mathbf{dev}(D_i)^o \cup \mathbf{dev}(B(x))^o \cup D_\infty^o$ for each i.

Since $\{\mathbf{dev}(K_i)\}$ converges to K_∞, and $\mathbf{dev}(B(x))^o$ is an open neighborhood of $\mathbf{dev}(x)$, there is a positive integer N such that

$$\mathbf{dev}(K_i) \cap \mathbf{dev}(B(x))^o \neq \varnothing \quad \text{whenever } i > N.$$

Let i be an integer greater than N. The open disk $B(x)^o$ includes a nonempty subset δ_i defined by

$$\delta_i = (\mathbf{dev}|B(x)^o)^{-1}(\mathbf{dev}(K_i) \cap \mathbf{dev}(B(x))^o).$$

We claim that $\delta_i \subset K_i$. Let p be a point belonging to δ_i. There is a sequence $\{q_j\}$ of points of $\mathbf{dev}(D_i)^o$ converging to $\mathbf{dev}(p)$, and a positive integer N_1 such that $q_j \in \mathbf{dev}(B(x))^o$ whenever $j > N_1$. Let $p_j = (\mathbf{dev}|D_i^o)^{-1}(q_j)$ for each j. The final statement of the above paragraph shows that $p_j \in B(x)^o$ whenever $j > N_1$. Since $\mathbf{dev}|D_i$ is an isometry, the Cauchy sequence $\{p_i\}$ converges to a point p^u belonging to $B(x) \cap D_i$, where $\mathbf{dev}(p^u) = \mathbf{dev}(p)$. By the injectivity of $\mathbf{dev}|B(x)$, $p^u = p$. Since $\delta_i \subset D_i$, $\delta_i \subset K_i$ by the injectivity of $\mathbf{dev}|D_i$. However, since by definition $\delta_i \subset B(x)$ whenever $i > N$, this contradicts the ideal property of $\{K_i\}$. Thus, $x \notin \widetilde{M}^o$.

Finally, suppose that $x \in \delta\widetilde{M}$. Let us extend $\widetilde{M}$ by attaching a small open disk in $\mathbf{S}^2$ around x by a projective map. We take out few points to make it into a manifold. The resulting projective surface still has a convex boundary. Now the previous argument applies and yields a contradiction again.

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