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AN ORTHOGONAL TRANSFORMATION GROUP OF $(8k-1)$-SPHERE

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0. Introduction

We give an example of an orthogonal transformation group of $(8k-1)$-sphere with codimension-two principal orbits and an action possessing just two isolated singular orbits (cf. [1, p. 214], [4]). This example shows that a theorem of Hsiang and Lawson [2, Theorem 6] should be properly modified. So we give a modified theorem in a correct form. Finally, we give another example due to T. Asoh, which shows that another theorem of Hsiang and Lawson [2, Theorem 5] should be properly modified.

1. An example

Let ν_m, ν_n be the standard representations of $\mathbf{Sp}(m)$ and $\mathbf{Sp}(n)$ on $\mathbf{H}^m$ and $\mathbf{H}^n$ respectively, where $\mathbf{H}^m$, $\mathbf{H}^n$ are the right quaternionic vector spaces. Let $(\mathbf{H}^n)^*$ denote the dual vector space of $\mathbf{H}^n$, which is a left quaternionic vector space. It is well known that $\mathbf{H}^m \otimes_{\mathbf{H}} (\mathbf{H}^n)^*$ is a real $4mn$-dimensional vector space, and $\nu_m \otimes_{\mathbf{H}} \nu_n^*$ is a real representation of $\mathbf{Sp}(m) \times \mathbf{Sp}(n)$ on $\mathbf{R}^{4mn} = \mathbf{H}^m \otimes_{\mathbf{H}} (\mathbf{H}^n)^*$. This representation can be regarded as follows.

Let $M(m, n; \mathbf{H})$ denote the set of all $m \times n$ quaternionic matrices. For an $m \times n$ quaternionic matrix X, let X^* denote the transpose of the conjugate of X. Then

$$\mathbf{Sp}(m) = \{A \in M(m, m; \mathbf{H}) \colon A^*A = I \text{ the unit matrix}\},$$

the representation space $\mathbf{H}^m \otimes_{\mathbf{H}} (\mathbf{H}^n)^*$, is identified with $M(m, n; \mathbf{H})$, and the representation $\psi = \nu_m \otimes_{\mathbf{H}} \nu_n^*$ can be expressed by

$$\psi((A, B)) \cdot X = AXB^*; A \in \mathbf{Sp}(m), B \in \mathbf{Sp}(n), X \in M(m, n; \mathbf{H}).$$

Put

$$\langle X, Y\rangle = \operatorname{trace} X^* Y, \quad \operatorname{Re}\langle X, Y\rangle = \text{real part of } \langle X, Y\rangle$$

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for $X, Y \in M(m, n; \mathbf{H})$. $\mathrm{Re}\langle X, Y\rangle$ is an $\mathbf{Sp}(m) \times \mathbf{Sp}(n)$-invariant inner product of the real vector space $M(m, n; \mathbf{H})$. For an $m \times n$ quaternionic matrix X, let $\operatorname{rank} X$ be the maximum number of linearly independent column vectors of X as the right quaternionic vectors.

Example. We shall consider a real $8k$-dimensional representation $\psi_k = \nu_k \otimes_{\mathbf{H}} (\nu_2^* | \mathbf{Sp}(1) \times \mathbf{Sp}(1))$ of $\mathbf{Sp}(k) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1)$ on $M(k, 2; \mathbf{H})$. Suppose $k \geqslant 2$ in the following. For a $k \times 2$ quaternionic matrix X, let X_1, X_2 denote the first and the second column vector of X respectively. Then the representation ψ_k can be expressed by

$$\psi_k((A, q_1, q_2)) \cdot (X_1, X_2) = (AX_1\bar{q}_1, AX_2\bar{q}_2)$$

for $A \in \mathbf{Sp}(k)$, $q_i \in \mathbf{Sp}(1)$, $X = (X_1, X_2) \in M(k, 2; \mathbf{H})$. Straightforward computations show the following:

(i) Suppose that $\operatorname{rank} X = 2$ and $\langle X_1, X_2\rangle \neq 0$ for $X = (X_1, X_2)$. Then the isotropy group at X is conjugate to

$$\left\{\left[\left[\begin{array}{cc|c} q & 0 & 0 \\ 0 & q & \\ \hline \multicolumn{2}{c|}{0} & * \end{array}\right], q, q\right] : q \in \mathbf{Sp}(1)\right\},$$

and the orbit through X is $(8k - 3)$-dimensional, which is diffeomorphic to $\mathbf{Sp}(k)/\mathbf{Sp}(k-2) \times S^3$.

(ii) Suppose that $\operatorname{rank} X = 2$ and $\langle X_1, X_2\rangle = 0$ for $X = (X_1, X_2)$. Then the isotropy group at X is conjugate to

$$\left\{\left[\left[\begin{array}{cc|c} q_1 & 0 & 0 \\ 0 & q_2 & \\ \hline \multicolumn{2}{c|}{0} & * \end{array}\right], q_1, q_2\right] : q_i \in \mathbf{Sp}(1)\right\},$$

and the orbit through X is $(8k - 6)$-dimensional, which is diffeomorphic to $\mathbf{Sp}(k)/\mathbf{Sp}(k-2)$.

(iii) Suppose that $\operatorname{rank} X = 1$ and $\langle X_1, X_2\rangle \neq 0$ for $X = (X_1, X_2)$. Then the isotropy group at X is conjugate to

$$\left\{\left(\begin{pmatrix} q & 0 \\ 0 & * \end{pmatrix}, q, q\right) : q \in \mathbf{Sp}(1)\right\},$$

and the orbit through X is $(4k + 2)$-dimensional, which is diffeomorphic to $S^{4k-1} \times S^3$.

(iv) Suppose that $\operatorname{rank} X = 1$ and $\langle X_1, X_2\rangle = 0$ for $X = (X_1, X_2)$. Then the isotropy group at X is conjugate to

$$\left\{\left(\begin{pmatrix} q_1 & 0 \\ 0 & * \end{pmatrix}, q_1, q_2\right) : q_i \in \mathbf{Sp}(1)\right\} \text{ for } X_1 \neq 0$$

or

$$\left\{\left(\begin{pmatrix} q_2 & 0 \\ 0 & * \end{pmatrix}, q_1, q_2\right) : q_i \in \mathbf{Sp}(1)\right\} \text{ for } X_2 \neq 0,$$

and the orbit through X is a $(4k-1)$-sphere.

Remark. (a) The representation ψ_k induces an $\mathbf{Sp}(k) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1)$ action on a sphere S^{8k-1}. The principal orbits of this action are of codimension two, and this action possesses just two isolated singular orbits which are diffeomorphic to a $(4k-1)$-sphere. (b) The representation ψ_k is an example of a reducible compact linear group of cohomogeneity 3 (in the sense of Hsiang and Lawson [2]). This example shows that a theorem of Hsiang and Lawson [2, Theorem 6] should be properly modified.

2. Linear groups of cohomogeneity 3

The theorem of Hsiang and Lawson [2, Theorem 6] can be modified as follows.

Theorem. *Let (G, ψ) be a reducible maximal compact connected linear group of cohomogeneity* 3. *Then it is one of the following*:

(i) $\psi = \psi' + \theta^1$, *(G, ψ') is a compact linear group of cohomogeneity* 2 (*cf.* [2, *Theorem* 5]) *and θ^1 is a* 1-*dimensional trivial representation.*

(ii) $G = \mathbf{SO}(k) \times G'$, $\psi = \rho_k + \psi'$ *for* $k \geqslant 2$, *and (G', ψ') is a compact linear group of cohomogeneity* 2.

(iii) $G = \mathbf{SO}(k)$ *and* $\psi = 2\rho_k$ *for* $k \geqslant 3$.

(iv) $G = \mathbf{Sp}(k) \times \mathbf{Sp}(1) \times \mathbf{Sp}(1)$ *and* $\psi = \nu_k \otimes_{\mathbf{H}} (\nu_2^* | \mathbf{Sp}(1) \times \mathbf{Sp}(1))$ *for* $k \geqslant 2$.

(v) $G = \mathbf{SU}(k) \times \mathbf{U}(1) \times \mathbf{U}(1)$ *and* $\psi = [\mu_k \otimes_{\mathbf{C}} (\mu_2^* | \mathbf{U}(1) \times \mathbf{U}(1))]_{\mathbf{R}}$ *for* $k \geqslant 2$.

(vi) $G = \mathbf{Spin}(9)$ *and* $\psi = \Delta_9 + \rho_9$.

(vii) $G = \mathbf{Sp}(2) \times \mathbf{Sp}(1)$, $\psi = (\nu_2 \otimes_{\mathbf{H}} \nu_1^*) + \pi$, *and* $\pi: \mathbf{Sp}(2) \to \mathbf{SO}(5)$ *is a surjection.*

(viii) $G = \mathbf{U}(2)$, $\psi = [\mu_2]_{\mathbf{R}} + \pi'$, *and* $\pi': \mathbf{U}(2) \to \mathbf{SO}(3)$ *is a surjection.*

(ix) *G is a circle group acting on* $\mathbf{R}^4$.

Proof. We first discount the special cases (i), (ix). Since (G, ψ) is reducible, we have $\psi = \psi_1 + \psi_2$. Put

$$n_i = \deg \psi_i, \quad G_i' = \psi_i(G),$$
$$G_1 = (\ker \psi_2)^0, \quad G_2 = (\ker \psi_1)^0,$$

where K^0 denotes the identity component of K. Then there is a closed connected normal subgroup H of G such that

$$G = (G_1 \times G_2) \circ H \text{ (essential direct product).}$$

Thus H is locally isomorphic to $\psi_i(H)$, and $G_i' = G_i \circ \psi_i(H)$. Consider the G-orbit of $u = (x_1, x_2) \in \mathbf{R}^{n_1} \oplus \mathbf{R}^{n_2}$. We can assume

$$\dim G(u) = n_1 + n_2 - 3.$$

Since

$$G(u) \subset G_1'(x_1) \times G_2'(x_2) \subset S^{n_1-1} \times S^{n_2-1},$$

we can assume that

$$\dim G_1'(x_1) = n_1 - 1, \quad \dim G_2'(x_2) = n_2 - 1 \text{ or } n_2 - 2.$$

(a) Suppose $\dim G_2'(x_2) = n_2 - 2$. Then we have

$$G(u) = G_1'(x_1) \times G_2'(x_2),$$

and hence $G = G_1' \times G_2'$ and $G_1' = \mathbf{SO}(n_1)$ by the maximality of G. This is the case (ii).

(b) Suppose $\dim G_i'(x_i) = n_i - 1$ for $i = 1, 2$. First we shall show $G_i(x_i) \neq S^{n_i-1}$ for $i = 1, 2$. There is a differentiable fibration

$$G_{x_2}/G_u \to G/G_u \to G/G_{x_2}$$

where

$$G/G_{x_2} = G_2'(x_2) = S^{n_2-1},$$

$$G_{x_2}/G_u = G_{x_2}(u) = G_{x_2}(x_1) \times \{x_2\}, \text{ and } G_1 \subset G_{x_2}.$$

If $G_1(x_1) = S^{n_1-1}$, then $\dim G(u) = n_1 + n_2 - 2$ which is a contradiction. Therefore G_i is non-transitive on S^{n_i-1}, but G_i' acts transitively on S^{n_i-1} for $i = 1, 2$. Hence we have from a theorem of Montgomery and Samelson [3, Theorem I′] that

$$H(x_i) = S^{n_i-1} \text{ for } i = 1, 2.$$

The H-action on S^{n_i-1} is almost effective. It follows from the classification of compact linear groups of cohomogeneity one (i.e., the transitive actions on spheres) that the only possible cases of (H, ψ_i) are as follows:

$$(\mathbf{SO}(k), \rho_k), \quad (\mathbf{SU}(k), [\mu_k]_{\mathbf{R}}), \quad (\mathbf{U}(k), [\mu_k]_{\mathbf{R}}), \quad (\mathbf{Sp}(k), [\nu_k]_{\mathbf{R}}),$$

$$(\mathbf{Sp}(k) \times \mathbf{Sp}(1), \nu_k \otimes_{\mathbf{H}} \nu_1^*), \quad (\mathbf{Spin}(7), \Delta_7), \quad (\mathbf{Spin}(9), \Delta_9),$$

$$(\mathbf{G}_2, \omega); \ \deg \Delta_7 = 8, \ \deg \Delta_9 = 16 \text{ and } \deg \omega = 7.$$

Suppose $H \neq G$ (i.e., $G_1 \neq 1$ or $G_2 \neq 1$). Since G_i' acts effectively on S^{n_i-1}, we have that

$$H = \mathbf{SU}(k) \text{ or } \mathbf{Sp}(k).$$

This is the case (iv) or (v) if $(H, \psi_1) = (H, \psi_2)$, and the case (vii) or (viii) if $(H, \psi_1) \neq (H, \psi_2)$. Suppose $H = G$ (i.e., $G_1 = 1$ and $G_2 = 1$),. Then, by the maximality of G, this is the case (iii) if $(H, \psi_1) = (H, \psi_2)$, and the case (vi) if $(H, \psi_1) \neq (H, \psi_2)$.

Remark. The cases (iv), (v), (vi), (vii), (viii) are missing in the theorem of Hsiang and Lawson [2, Theorem 6]. The case (viii) has been explained in a book of Bredon [1, p. 213], the cases (vi), (vii) have been treated by Uchida and Watabe [4].

3. Concluding remark

Here we give another example due to T. Asoh. This example shows that another theorem of Hsiang and Lawson [2, Theorem 5] should be properly modified.

There is a homomorphism σ: $\mathbf{SU}(2) \to \mathbf{Sp}(2)$ defined by

$$\sigma\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} = \begin{bmatrix} a^3 + jb^3 & -\sqrt{3}\,(a^2\bar{b} - j\bar{a}b^2) \\ \sqrt{3}\,(a^2 b - jab^2) & a^2\bar{a} - 2ab\bar{b} + jb^2\bar{b} - 2ja\bar{a}b \end{bmatrix},$$

where j is a quaternion such that $j^2 = -1$ and $aj = j\bar{a}$ for each complex number a. In fact, the complexification (forgetting the quaternionic structure) of σ is equivalent to the third symmetric product $S^3(\mu_2)$ of the standard representation μ_2 of $\mathbf{SU}(2)$ on $\mathbf{C}^2$.

We shall consider a real 8-dimensional representation $\psi = (\nu_2|\sigma\mathbf{SU}(2)) \otimes_{\mathbf{H}} \nu_1^*$ of $G = \sigma(\mathbf{SU}(2)) \times \mathbf{Sp}(1)$ on $M(2, 1; \mathbf{H})$. Let G_t denote the isotropy group at $\binom{t}{j}$ for each real number t. Put

$$x = \sigma\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \times (-i) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \times (-i),$$

$$y = \sigma\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \times (j) = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix} \times (j).$$

Let H be a subgroup of G generated by x, y. Then H is a finite group of order 8. It is seen that

$$G_t = H \quad \text{for } 0 < t < 1/\sqrt{3}\,.$$

Moreover, G_0 and $G_{1/\sqrt{3}}$ are 1-dimensional subgroups of G.

Remark. This example is missing in the theorem of Hsiang and Lawson [2, Theorem 5 (ii)].

References

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