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THE LAPLACIAN AND THE KOHN LAPLACIAN FOR THE SPHERE

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Introduction

In [2], Folland studied the Cauchy-Riemann ($\bar{\partial}_b$) complex on the sphere in $\mathbf{C}^{n+1}$, using the representation theory of the unitary group $U(n+1)$, hoping to use the analysis as an approximating model for general strongly pseudoconvex domains. The work was greatly complicated by the absence of an explicit formula for the associated Kohn Laplacian $\Box_b$, or its fundamental solution, and soon Folland embarked upon the notion of using the Heisenberg group $\mathbf{H}^n$ instead of the sphere as the model. This work was completed with Stein in [3].

It turns out that $\Box_b$ on $S_n = S^{2n+1}$ does indeed have a simple explicit form. We compute it directly from the definitions in this paper, obtaining the analogues of the Folland-Stein $\mathcal{L}_\alpha$ operators; and we show how it can be applied. Along the way, we shall meet the analogue of Lewy's unsolvable operator for S_1, and derive necessary and sufficient conditions for its local solvability, in analogy in [5]. We also prove local analytic hypoellipticity on (p, q)-forms when $0 < q < n$; this is now known on general strongly pseudoconvex manifolds ([7] and [8], independently), but the general proofs are very complex. At any rate, our main purpose is to show the reader a simple way of handling analysis on S_n in full analogy to that on $\mathbf{H}^n$, and we think our methods have much wider applicability.

The computation of $\Box_b$ is almost identical to that of the ordinary Laplacian on forms on the sphere $S^n \subset \mathbf{R}^{n+1}$ which does not seem to be in the literature. It is of lesser interest, being elliptic; but we think the comparison is instructive, so we include it. Although we do not refer to it, the reader should first read the first seven sections of [3] since $\mathbf{H}^n$ is easier.

Jiri Dadok and Reese Harvey [1] have, independently of us, computed a "fundamental solution" for $\Box_b$, without finding our formula for $\Box_b$. They used the work of Henkin and Skoda. We do not compute a fundamental

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solution, but instead show how to solve $\bar\partial_b^*$-closed forms explicitly, then solve the general case implicitly; this is good enough to deal with any application we know.

As is customary we restrict to $(0, q)$-forms on S_n, noting that all results go over to (p, q)-forms.

We announced this result in [4]. We would like to thank E. M. Stein and J. J. Kohn for very valuable discussions. Although we do not refer to it for proofs, this paper draws heavily on the work of Folland [2], both concretely and for inspiration.

1. The Computation

On $\mathbf{C}^{n+1}$ $(n > 0)$ we write $\partial_j = \partial/\partial z_j$, $\bar\partial_j = \partial/\partial \bar z_j (0 \leqslant j \leqslant n)$; on $\mathbf{R}^{n+1}$ we write $D_j = \partial/\partial x_j$ $(0 \leqslant j \leqslant n)$. If $J = (j_1, \cdots, j_q)$ where $0 \leqslant j_1, \cdots, j_q \leqslant n$, we write

$$d\bar z(J) = d\bar z_{j_1} \wedge \cdots \wedge d\bar z_{j_q}, \quad dx(J) = dx_{j_1} \wedge \cdots \wedge dx_{j_q}.$$

If $1 \leqslant k \leqslant n$, we also write

$$(k, J) = (k, j_1, \cdots, j_q), \quad d\bar z(k, J) = d\bar z((k, J)), \text{ etc.}$$

If B denotes the bundle of anti-holomorphic tangent vectors in $\mathbf{C}^{n+1}$, one defines $\bar\partial$: $C^\infty(\Lambda^q(B^*)) \to C^\infty(\Lambda^{q+1}(B^*))$ by $\bar\partial(f d\bar z(J)) = \Sigma_0^n \bar\partial_k f\, d\bar z(k, J)$ and extending linearly; similarly for forms defined only locally.

We place the metric on $\mathbf{C}^n$ with $\partial_1, \cdots, \partial_n, \bar\partial_1, \cdots, \bar\partial_n$ being orthonormal (volume element $2^{-(n+1)}$ times the usual one). On S_n we use the metric $\langle\, , \rangle$ which is the restriction of the metric on $\mathbf{C}^n$ (volume element $2^{-(n+1/2)}$ times the usual one). We put $\underset{\sim}{\partial} = (\partial_1, \cdots, \partial_n)$, $dz = (dz_1, \cdots, dz_n)$, $\cdot =$ dot product. Now if $X \in T\mathbf{C}^n$ at $z \in S_n$, $X \in TS_n$ iff $X(|z|^2) = 0$, iff $X \perp \frac{1}{2}(z \cdot \underset{\sim}{\partial} + \bar z \cdot \underset{\sim}{\bar\partial}) = \partial/\partial |z|^2$, iff $\langle d|z|^2, X\rangle = 0$ (where $d|z|^2 = z \cdot d\bar z + \bar z \cdot dz$). Put $\sigma = z \cdot d\bar z = \bar\partial |z|^2$. Let $T_{0,1} = \mathbf{C}T(S_n) \cap B$; one defines $\bar\partial_b$: $C^\infty(\Lambda^q(T^*_{0,1})) \to C^\infty(\Lambda^{q+1}(T^*_{0,1}))$ by letting the value of $\bar\partial_b \omega$ at any point z be the orthogonal projection of $\bar\partial \omega_1$ onto $\Lambda^{q+1}(T^*_{0,1})$ at z, where $\omega_1 \in C^\infty(\Lambda^q(B^*))$ in a neighborhood of S_n and $\omega_1 = \omega$ on S_n. Since two candidates for ω_1 differ by $(|z|^2 - 1)\varphi$, say, where φ is smooth, $\bar\partial_b$ is well-defined. Thus, if f is C^∞ in a neighborhood of S_n, then

$$\bar\partial_b f = \sum_0^n (\bar\partial_k f)\zeta_k, \tag{1.1}$$

where $\zeta_k = d\bar z_k - \bar z_k \sigma$. Similar considerations produce d: $C^\infty(\Lambda^q(T^* S_n)) \to C^\infty(\Lambda^{q+1}(T^* S_n))$ with

$$df = \sum (D_k f)\xi_k, \tag{1.1$'$}$$

where $\xi_k = dx_k - \frac{1}{2}x_k dR^2$, $\frac{1}{2}dR^2 = \Sigma\, x_j dx_j$. It is obvious that $\bar\partial_b$ (resp. d) is invariant under $U(n+1)$ (resp. $O(n+1)$).

It is already time for the key observation of this paper:

$$\bar\partial_b f = \sum_{0<j<k<n} \left(\overline{M}_{jk} f\right)\omega_{jk} \tag{1.2}$$

for $f \in C^\infty(S_n)$, where $\overline{M}_{jk} = z_j\bar\partial_k - z_k\bar\partial_j$, $\omega_{jk} = \bar z_j d\bar z_k - \bar z_k d\bar z_j$ $(0 < j, k < n)$. Similarly,

$$df = \sum_{j<k} (W_{jk} f) w_{jk}, \tag{1.2$'$}$$

$f \in C^\infty(S_n)$, where $W_{jk} = x_j D_k - x_k D_j$, $w_{jk} = x_j dx_k - x_k dx_j$. The proof is trivial from (1.1) and the advantage obvious–the $\overline{M}_{jk}$, ω_{jk} are tangential. Only the motivation must be supplied, and we turn to that now.

The M_{jk} $(0 \leqslant j, k \leqslant n)$ span $T_{1,0}$ at each $z \in S_n$, for $\Sigma_j\, z_j M_{jk} = \partial_k - \bar z_k(z \cdot \underset{\sim}{\partial})$, the projection of ∂_k onto $T_{1,0}(S_n)$ at z. We also put $L = (i/2)(z \cdot \underset{\sim}{\partial} - \bar z \cdot \underset{\sim}{\bar\partial})$, orthogonal to $T_{0,1} \oplus T_{1,0}$ at each point and equal to $J(\partial/\partial|z|^2)$, where J is the complex structure map: $J(\partial_j) = i\partial_j$, $J(\bar\partial_j) = -i\bar\partial_j$.

Now it is easy to see how the W_{jk} arise from $O(n+1)$; we have $W_{01}f(x) = d/d\theta\ f(x_0\cos\theta - x_1\sin\theta, x_0\sin\theta + x_1\cos\theta, x_2, \cdots, x_n)|_{\theta=0}$. To see the relation of the M_{jk} and L to $U(n+1)$ and $O(2n+2)$, we define a variety of maps. If $z \in \mathbf{C}^{n+1}$, put $z'' = (z_2, \cdots, z_n)$, $z' = (z_1, z'')$. Define $\rho(\theta)$, $\rho_{01}(\theta)$, h_0, $g_0 \in U(n+1)$ (for $\theta \in \mathbf{R}$) by: $\rho(\theta) = e^{i\theta}z$, $\rho_{01}(\theta)(z) = (z_0\cos\theta - z_1\sin\theta,\ z_0\sin\theta + z_1\cos\theta, z'')$, $h_0 z = (iz_0, z')$, $g_0 = h_1^{-1}$. Also define $j_0 \in O(2n+2)$ by $j_0(z) = (\bar z_0, z')$. If $T \in O(2n+2)$ and $f \in C^\infty(S_n)$, write $T^{\cdot}f = f \circ T$. Then we simply have $Lf = \frac{1}{2}(d/d\theta)\rho(\theta)^{\cdot}f|_{\theta=0}$. The M_{jk} arise in a more complex way: if $D_{01}f = (d/d\theta)\rho_{01}(\theta)^{\cdot}f|_{\theta=0}$ we have $D_{01} = N_{01} + \overline{N}_{01}$ where $N_{jk} = z_j\partial_k - \bar z_k\bar\partial_j \in T(S_n)$. Also $g_0^{\cdot}D_{01}h_0^{\cdot} = -i(N_{01} - \overline{N}_{01})$. Thus $N_{01} = \frac{1}{2}(D_{01} + ig_0^{\cdot}D_{01}h_0^{\cdot})$. Finally, $M_{01} = j_0^{\cdot}N_{01}j_0^{\cdot}$.

There are two significant consequences of this–first, $M_{jk}\Delta = \Delta M_{jk}$, etc., since elements of $O(2n+2)$ commute with Δ. Also, if $f \in C^\infty$, $\int_{S_n} M_{jk}f\, dS = 0$, etc., since dS is invariant under $O(2n+2)$ and hence, in particular, $\int \rho_{01}(\theta)^{\cdot}f$ is independent of θ.

We hope that this has helped to motivate (1.2). There is probably a better way to prove (1.2) than to compute from (1.1), but we do not see it.

At any rate, here is what we shall prove in this section. Put $\vartheta_b = \bar\partial_b^*$ (formal adjoint), $\square_b = \bar\partial_b\vartheta_b + \vartheta_b\bar\partial_b$. Then if $\eta = \Sigma_J\, f_J d\bar z(J) \in C^\infty(\Lambda^q)$ is tangential on S_n (summation over all increasing q-tuples J–that is, J with $J = (j_0, \cdots, j_{q-1})$, $0 \leqslant j_0 < \cdots < j_{q-1} \leqslant n$) we have

$$\square_b\eta = \sum_J (\mathfrak{L}_\alpha f_J) d\bar z(J) + \sigma \wedge \vartheta_b\eta, \tag{1.3}$$

where

$$\mathfrak{L}_\alpha = -\tfrac{1}{2}\sum_{j<k}\left(M_{jk}\overline{M}_{jk} + \overline{M}_{jk}M_{jk}\right) + i\alpha L + \tfrac{1}{4}(n-\alpha)(n+\alpha),$$

provided $\alpha = n - 2q$. We shall soon also see explicit formulas for $\vartheta_b\eta$, so (1.3) is completely explicit. Similarly, if $\delta = d^*$, $\Delta_s = d\delta + \delta d$, and $\eta = \sum f_J dx(J) \in C^\infty(\Lambda^q)$ is tangential on S^n, we have

$$\Delta_s\eta = \sum_J (\mathfrak{D}_\alpha f_J)dx(J) + dR^2 \wedge \delta\eta, \tag{1.3)$'$}$$

where $\mathfrak{D}_\alpha = -\Sigma_{j<k} W_{jk}^2 + (n-\alpha)(n+\alpha-2)/4$, $\alpha = n-2q$.

The similarity with $\mathbf{H}^n$ is striking, except for the special role played here by ϑ_b-closed forms. For the representation-theoretic reason for why these forms should be special, see Folland [2].

We have

$$\begin{aligned}\mathfrak{L}_\alpha = -\Delta + \sum_{j,k} z_j\bar{z}_k\partial_{j\bar{k}} + \tfrac{1}{2}(n+\alpha)\bar{z}\cdot\bar{\partial} + \tfrac{1}{2}(n-\alpha)z\cdot\underset{\sim}{\partial} \\ + \tfrac{1}{4}(n+\alpha)(n-\alpha),\end{aligned} \tag{1.4}$$

where $\partial_{j\bar{k}} = \partial^2/\partial z_j\partial\bar{z}_k$, $\Delta = \Sigma_j\,\partial_{j\bar{j}}$ (in our metric on $\mathbf{C}^{n+1}$). Each operator on the right side is invariant under $U(n+1)$, so $\mathfrak{L}_\alpha$ is also (as it must be if (1.3) is true). For example, to check $U^{\cdot}D = DU^{\cdot}$ where $D = \Sigma\, z_j\bar{z}_k\partial_{j\bar{k}}$, $U \in U(n+1)$, we need only check on second degree polynomials since both sides are second order differential operators. Put $V_1 = \mathrm{span}_{jk}\{z_j\bar{z}_k\}$, $V_2 = \mathrm{span}_{j,k}\{z_jz_k, z_j, \bar{z}_j, \bar{z}_j, 1\}$; then $D = \mathrm{id}$ on V_1, $D = 0$ on V_2, $U^{\cdot}$: $V_1 \to V_1$, $U^{\cdot}$: $V_2 \to V_2$, so the result follows. Similarly,

$$\mathfrak{D}_\alpha = -\Delta + \sum x_jx_kD_{jk} + nx\cdot D + \tfrac{1}{4}(n+\alpha-2)(n-\alpha) \tag{1.4)$'$}$$

is invariant under $O(n+1)$.

We begin the proof of (1.3). If $J = (j_1, \cdots, j_{q-1})$ is a q-tuple, call J *injective* if $j_k = j_l$ implies $k = l$. In this case, we write $(J; j_k) = (j_0, \cdots, j_{k-1}, j_{k+1}, \cdots, j_{q-1})$, $(J; j_k, j_l) = ((J; j_k); j_l)$ if $k \neq l$, $(J; m) = 0$ if $m \notin J$, $d\bar{z}(J; j_k) = d\bar{z}((J; j_k))$, etc. If J is a q-tuple, $q \geq 1$, we define the tangential $(q-1)$ form $\omega(J)$ by

$$d\bar{z}(J) = \zeta(J) + \sigma \wedge \omega(J). \tag{1.5}$$

If J is not injective, $\omega(J) = 0$. If J is injective, an expansion of $d\bar{z}(J)$ using $d\bar{z}_k = \zeta_k + \bar{z}_k\sigma$ shows

$$\omega(J) = \sum (-1)^k \bar{z}_{j_k}\zeta(J; j_k). \tag{1.6}$$

(We put $d\bar{z}(\phi) = \zeta(\phi) = \omega(\phi) = 0$.) Now

$$\omega(J) = \sum (-1)^k \bar{z}_{j_k}\, d\bar{z}(J; j_k). \tag{1.7}$$

To see this, suppose $J = (0, \cdots, q-1)$, $\omega'(J)$ is the right side of (1.7); then by (1.5) and (1.6), $\omega'(J) = \omega(J) + \sigma \wedge \psi$, where $\psi = \Sigma(-1)^k \bar{z}_k \omega(J; k)$; but

$$\psi = \sum_{j<k} (-1)^{j+k} \bar{z}_j \bar{z}_k \zeta(J; j, k) + \sum_{j>k} (-1)^{j+k-1} \bar{z}_j \bar{z}_k \zeta(J; k, j) = 0. \tag{1.8}$$

The $\omega(J)$ are due to Folland [2]. Note $\omega(j, k) = \omega_{jk}$. From (1.5), $J \to \omega(J)$ is alternating. We similarly have $dx(J) = \xi(J) + \frac{1}{2} dR^2 \wedge w(J)$ with similar formulas for the tangential form $w(J)$.

Some elementary relations:

$$\sum_0^n z_k \zeta(k, J) = 0, \tag{1.9}$$

$$\zeta(J) = \sum_0^n z_k \omega(k, J) \tag{1.10}$$

with obvious analogues for $\mathbf{R}^{n+1}$. For (1.9),

$$\sum z_k \zeta(k, J) = \left(\sum z_k \zeta_k\right) \wedge \zeta(J) = \left(\bar{\partial}_b |z|^2\right) \wedge \zeta(J) = 0.$$

For (1.10),

$$\sigma \wedge \sum_0^n z_k \omega(k, J) = \sum z_k \left[d\bar{z}(k, J) - \zeta(k, J) \right] = \sigma \wedge d\bar{z}(J) = \sigma \wedge \zeta(J)$$

as desired.

Put $\omega_0 = \omega(0, \cdots, n)$. Then $\sigma \wedge \omega_0 = d\bar{z}(0, \cdots, n)$, providing an orientation on $T^*_{0,1}$. In analogy to the Hodge operator, we define $*$: $\Lambda^q(T^*_{0,1}) \to \Lambda^{n-q}(T^*_{0,1})$ by $\psi \wedge *\phi = \langle \phi, \psi \rangle \omega_0$ ($\phi, \psi \in \Lambda^q$). Similarly, on S^n we let $*$ be the Hodge operator, $\psi \wedge *\phi = \langle \phi, \psi \rangle w(0, \cdots, n)$. If $e_0, \cdots, e_n$ is an orthonormal basis of T^*_{01} with $e_0 \wedge \cdots \wedge e_n = \omega_0$, we have

$$* \left(a e_0 \wedge \cdots \wedge e_{q-1}\right) = \bar{a} e_q \wedge \cdots \wedge e_n, \quad a \in \mathbf{C}.$$

This property defines $*$, so $** = (-1)^{q(n-q)}$. With this preparation, we can compute ϑ_b.

Lemma 1.1. (a) $\vartheta_b = -(-1)^{n(q+1)} * \bar{\partial}_b *$.

(b) (*Folland*) *If J is a j-tuple*, $\bar{\partial}_b \omega(J) = j\zeta(J)$, $\vartheta_b \zeta(J) = (n + 1 - j)\omega(J)$, *so* $\bar{\partial}_b \zeta(J) = \vartheta_b \omega(J) = 0$.

(c) *If J is a j-tuple, $f \in C^\infty(S_n)$, then*

$$\bar{\partial}_b(f\omega(J)) = \sum_{k,l} \bar{z}_l \overline{M_{kl}} f \omega(k, J) + jf\zeta(J).$$

If $J = (0, \cdots, j-1)$, then

$$\vartheta_b(f\omega(J)) = \sum_{i,m} (-1)^m z_i M_{im} f \omega(J; m).$$

(*Similarly, with d, δ, ξ, w, S^n, etc. in place of $\bar{\partial}_b$, ϑ_b, ζ, ω, S_n, etc.*.)

Proof. First we prove all the formulas for $\bar\partial_b$. By (1.1), $\zeta_k = \bar\partial_b \bar z_k$ so $\bar\partial_b \zeta(J) = 0$ for all J by the derivation formula. That $\bar\partial_b \omega(J) = j\zeta(J)$ follows from (1.6) and the derivation formula. Thus finally

$$\bar\partial_b(f\omega(J)) = S + jf\zeta(J), \quad S = \sum_{k<l} \overline{M_{kl}}f\omega_{kl} \wedge \omega(J).$$

Now

$$\begin{aligned}\sigma \wedge \omega_{kl} \wedge \omega(J) &= -\omega_{kl} \wedge (d\bar z(J) - \zeta(J)) \\ &= \bar z_l d\bar z(k, J) - \bar z_k d\bar z(l, J) - \bar z_l \zeta(k, J) + \bar z_k \zeta(l, J) \\ &\qquad\qquad \text{(expanding } \omega_{kl} \text{ by (1.6) then (1.7))} \\ &= \sigma \wedge \left[\bar z_l \omega(k, J) - \bar z_k \omega(l, J)\right].\end{aligned}$$

Thus

$$S = \sum_{k<l} \bar z_l \overline{M_{kl}} f\omega(k, J) - \sum_{k<l} \bar z_k \overline{M_{kl}} f\omega(l, J),$$

producing the formula of (c) when one interchanges k and l in the second sum.

For (a), put $\lambda = \sigma - \bar\sigma$, a global section of $(T_{0,1} \oplus T_{1,0})^{\perp}$. Then if $\gamma = \bar\omega_0 \wedge \lambda$, either $\gamma \wedge \omega_0$ or $-\gamma \wedge \omega_0$ is the volume form on S_n. Indeed, $d|z|^2 \wedge \lambda = \frac{1}{2}(\sigma + \bar\sigma) \wedge (\sigma - \bar\sigma) = \bar\sigma \wedge \sigma$; so

$$d|z|^2 \wedge \gamma \wedge \omega_0 = (-1)^n \bar\omega_0 \wedge \bar\sigma \wedge \sigma \wedge \omega_0 = dz(0, \cdots, n) \wedge d\bar z(0, \cdots, n),$$

as desired. It suffices then to show that $\int_{S_n} \gamma \wedge \bar\partial_b \eta = 0$ for any $\eta \in C^\infty(\Lambda^{n-1}(T^*_{0,1}))$. For we would then have that if $q \geq 1$, $\varphi \in C^\infty(\Lambda^q(T^*_{0,1}))$, and $\psi \in C^\infty(\Lambda^{q-1}(T^*_{0,1}))$, then

$$\begin{aligned}0 &= \int_{S_n} \gamma \wedge \bar\partial_b(\psi \wedge *\varphi) = \int_{S_n} \gamma \wedge \bar\partial_b \psi \wedge *\varphi + (-1)^{q-1} \int_{S_n} \gamma \wedge \psi \wedge \bar\partial_b *\varphi \\ &= \pm\left(\int \langle \varphi, \bar\partial_b \psi\rangle ds + (-1)^{n(q+1)} \int \langle *\bar\partial_b *\varphi, \psi\rangle ds\right),\end{aligned}$$

as desired. Now by (1.10) we may assume $\eta = f\omega(J)$ for some n-tuple J. We may also assume $J = (1, \cdots, n)$. Then

$$\bar\partial_b \eta = \sum \bar z_l \overline{M_{0l}} f\omega_0 + nf\zeta(J).$$

Since $\zeta(J) = z_0 \omega_0$ by (1.10), we have

$$\int \gamma \wedge \bar\partial_b \eta = \int\left(\sum_l \bar z_l \overline{M_{0l}} f + nz_0 f\right) dS = \sum_l \int \overline{M_{0l}}(\bar z_l f) dS = 0,$$

proving (a).

Problem. Prove an analogue of (a) on general compact *CR*-manifolds.

For (b), we must show that

$$(*) \qquad \vartheta_b \zeta(J) = (n + 1 - j)\omega(J).$$

$(*)$ is equivalent to

$(**)$ if $\alpha \in \operatorname{span}\{d\bar{z}^I | j\text{-tuples } I\}$ and $\alpha = \alpha_1 + \sigma \wedge \alpha_2, \alpha_1, \alpha_2$ tangential, then $\vartheta_b \alpha_1 = (n + 1 - j)\alpha_2$.

$(**)$ is unitarily invariant, so it suffices to prove $\vartheta_b \zeta(J) = (n + 1 - j)\omega(J)$ at the north pole $N = (1, 0, \cdots, 0)$.

First suppose $0 \notin J$. We may then assume $J = (1, \cdots, j)$, and must show $\vartheta_b \zeta(J) = 0$ at N. Let $\mathfrak{S} = \{f \in C^\infty(U)$ for some open $U \subset \mathbf{C}^{n+1}$ with $N \in U$ such that $f(N) = \partial_j f(N) = \bar{\partial}_j f(N) = 0$ for all $j\}$. An easy computation shows $\langle \zeta_k, \zeta_j \rangle = \delta_{jk} + a_{jk}, a_{jk} \in \mathfrak{S}, 1 \leqslant j, k \leqslant n$. By the Gram-Schmidt process, we can find an orthonormal basis $\{\beta_j | 1 \leqslant j \leqslant n\}$ of $T^*_{0,1}$ in a neighborhood of N in S_n with $\beta_j = (1 + b_j)(\zeta_j - \sum_{k=1}^{j-1} c_{jk}\zeta_k)$ with all $b_j, c_{jk} \in \mathfrak{S}$. Thus

$$*\zeta(J) = *(1 + e_J)\beta(J) = \pm(1 + \bar{e}_J)\beta(J^c) = \pm\zeta(J^c) + \sum_I g_I \zeta(I),$$

where $J^c = (j + 1, \cdots, n)$, Σ_I is sum over all $(n - j)$-tuples I and e_J, all $g_I \in \mathfrak{S}$. So $\vartheta_b \zeta(J) = 0$ at N by (1.1) and (a), if $0 \notin J$.

Now it follows from $\vartheta_b = \bar{\partial}_b^*$ and the derivation law that

$$(1.11) \qquad \vartheta_b(f\eta) = -\bar{\partial}_b \lrcorner \eta + f\vartheta_b \eta,$$

where $\varphi\lrcorner: \Lambda^q \to \Lambda^{q-1}$ is the adjoint of $\varphi \wedge : \Lambda^q \to \Lambda^{q+1}$ if φ is a 1-form. (If $e_0, \cdots, e_n$ is an orthonormal basis, J injective, $a \in \mathbf{C}$, then $ae_i \lrcorner e(J) = \pm \bar{a}e(J; i)$ for $i \in J$, 0 otherwise, the sign being chosen according to whether $e_i \wedge e(J; i) = \pm e(J)$.) So if $f \in C^\infty, J = (1, \cdots, j)$,

$$(1.12) \qquad \vartheta_b(f\zeta(J)) = -\sum (-1)^{k-1}\partial_k f\zeta(j; k) \quad \text{at } N.$$

Now suppose J is injective, $J = (0, K)$; we need to show

$$\vartheta_b(\zeta(J)) = (n + 1 - j)\zeta(K) \text{ at } N.$$

We may assume $K = (1, \cdots, j - 1)$. By (1.9), $\zeta(J) = -\bar{z}_0^{-1}\Sigma_{k=j}^n z_k \zeta(k, K)$; so the result and (b) follow from (1.12).

Finally, for (c) we have

$$\vartheta_b(f\omega(J)) = -\sum_{i<m} M_{im} f\omega_{im} \lrcorner \omega(J).$$

Now if φ is a tangential 1-form and $\varphi\lrcorner'$ is defined to be the adjoint of $\varphi \wedge$ acting on q-forms on $\mathbf{C}^n$, then it is obvious that $\varphi\lrcorner\psi = \varphi\lrcorner'\psi$ if ψ is tangential. Hence

$$\begin{aligned}\omega_{im} \lrcorner \omega(J) &= z_i d\bar{z}_m \lrcorner \omega(J) - z_m d\bar{z}_i \lrcorner \omega(J)\\ &= z_i(-1)^{m-1}\omega(J;\, m) - z_m(-1)^{i-1}\omega(J;\, i)\end{aligned}$$

as can be seen readily from (1.7), and we are done.

Proof of (1.3). It suffices to prove (1.3) at N. For (1.3) is equivalent to the following:

Suppose $V_1, \cdots, V_M \in \operatorname{span}\{d\bar{z}(I) | j\text{-tuples } I\}$ and $\eta = \Sigma_1^M f_k V_k$ is tangential. Then $\Box_b \eta = \Sigma(\mathcal{L}_\alpha f_k) V_k + \sigma \wedge \vartheta_b \eta$. This latter statement is clearly unitarily invariant.

Now $\{\omega(K) | 0 \in K, K \text{ a } (q+1)\text{-tuple}\}$ spans $\Lambda^q(T^*_{0,1})$ in a neighborhood of N. Indeed, by (1.10) it suffices to show that if I is a $(q+1)$-tuple, $0 \notin I$, then $\omega(I)$ is in the span near N. This follows from an application of (1.8) to $J = (0, I)$. Thus we may assume $\eta = f\omega(J)$, $J = (0, \cdots, q)$. By (1.10) it is clear we have only to use Lemma 1.1(c); we begin this chore now.

$$\begin{aligned}\vartheta_b \bar{\partial}_b(f\omega(J)) &= \vartheta_b\Big(\sum_{k,l} \bar{z}_l \overline{M}_{kl} f\omega(k, J) + (q+1) f \sum_k z_k \omega(k, J)\Big)\\ &= \sum{}'(-1)^{m-1} z_i M_{im}\big(\bar{z}_l \overline{M}_{kl} f\big)\omega((k, J);\, m)\\ &\quad + (q+1)\sum{}''(-1)^{m-1} z_i M_{im}(z_k f)\omega((k, J);\, m)\\ &\quad + \Big[\sum{}''' z_i M_{ik}\big(\bar{z}_l \overline{M}_{kl} f\big) + (q+1)\sum{}^{iv} z_i M_{ik}(z_k f)\Big]\omega(J),\end{aligned} \tag{1.13}$$

where Σ' denotes sum over $k > q$, $m \leqslant q$ and all i, l; Σ'' over $k > q$, $m \leqslant q$, all i; Σ''' over $k > q$, all i, l; Σ^{iv} over $k > q$, all i. Also

$$\begin{aligned}\bar{\partial}_b \vartheta_b(f\omega(J)) &= \bar{\partial}_b\Big(\sum_{i,m} (-1)^m z_i M_{im} f\omega(J;\, m)\Big)\\ &= \sum{}'(-1)^m \bar{z}_l \overline{M}_{kl}(z_i M_{im} f)\omega(f, (J;\, m))\\ &\quad + q\sum{}''(-1)^m z_k z_i M_{im} f\omega(k, (J;\, m))\\ &\quad + \Big[\sum{}^{v} \bar{z}_l \overline{M}_{kl}(z_i M_{ik} f) + q\sum{}^{vi} z_k z_i M_{ik} f\Big]\omega(J),\end{aligned} \tag{1.14}$$

where Σ^v denotes sum over $k \leqslant q$, all i, l; Σ^{vi} over $k \leqslant q$, all i. We might as well write $(k, J;\, m) = ((k, J);\, m) = (k, (J;\, m))$ in Σ' and Σ''.

Now one can unscramble all the commutators and arrive at (1.3), but it is far easier to recall that we are at N. At N, $\omega(k, J;\, m)$ $(k > q,\ m \leqslant q)$ is zero if

$m = 0$; if $m \neq 0$, $M_{im} = \delta_{i0}\partial_m$. Thus at N the coefficient of $\omega(k, J; m)$ $(k > j, 0 < m \leqslant q)$ in the sum of the right sides of (1.13) and (1.14) is

$$(-1)^{m-1}\Big[\sum_l \partial_m\big(\bar{z}_l\overline{M_{kl}}f\big) + \partial_m(z_k f) - \sum_i(-\bar{\partial}_k)(z_i M_{im} f) + 0\Big]$$
$$= (-1)^{m-1}\Big[-\sum_l \partial_m\big(|\bar{z}_l|^2\bar{\partial}_k f\big) + \sum_i \bar{\partial}_k\big(|z_i|^2\partial_m f\big)\Big].$$

We must have $i = l = 0$ for nonvanishing terms, whence the coefficient is zero.

The coefficient of $\omega(J)$ is

$$\sum_{\substack{l\\ k>q}} \partial_k\big(\bar{z}_l\overline{M_{kl}}f\big) + (q+1)\sum_{k>q}\partial_k(z_k f) - \sum_{\substack{i\\ 1<k<q}}\bar{\partial}_k(z_i M_{ik} f) = S_1 + S_2 + S_3.$$

Now $S_1 = \Sigma\, \partial_k(\bar{z}_l z_k \bar{\partial}_l f) - \Sigma\, \partial_k(|z_l|^2\bar{\partial}_k f)$. If $l = k$ in the first sum, the term vanishes; so $l \neq k$, $l = 0$ and

$$S_1 = \sum_{k>q}\partial_k\big(z_k\bar{\partial}_0 f\big) - \sum_{k>q}\partial_k\bar{\partial}_k f = (n-q)\bar{\partial}_0 f - \sum_{k>q}\partial_k\bar{\partial}_k f,$$
$$S_2 = (q+1)(n-q)f,$$
$$S_3 = -\sum \bar{\partial}_k\big(|z_i|^2\partial_k f\big) + \sum \bar{\partial}_k\big(z_i\bar{z}_k\partial_i f\big) = -\sum_{1<k<q}\bar{\partial}_k\partial_k f + q\partial_0 f.$$

Altogether at N

$$\Box_b\eta = \big[\big(-\Delta + \partial_{0\bar{0}} + (n-q)\bar{\partial}_0 + q\partial_0 + (q+1)(n-q)\big)f\big]\bar{z}_0 d\bar{z}(J;0)$$
$$(1.15)\qquad = \mathfrak{L}_\alpha(f\bar{z}_0)d\bar{z}(J;0),$$

since $(q+1)(n-q) - (n-q) = q(n-q)$. At N the right side of (1.3) is

$$\sum(-1)^k\mathfrak{L}_\alpha(f\bar{z}_k)d\bar{z}(J;k) + \sum_{m>0}(-1)^m\partial_m f(\sigma\wedge\omega(J;m)).$$

Now $\sigma\wedge\omega(J;m) = d\bar{z}(J;m) - \zeta(J;m) = d\bar{z}(J;m)$ at N if $m \neq 0$. We are left to check that $\mathfrak{L}_\alpha(f\bar{z}_k) + \partial_k f = 0$ at N if $k > 0$. But at N, $\mathfrak{L}_\alpha(f\bar{z}_k) = -\Delta(f\bar{z}_k)$ and we are done.

For (1.3)′ everything is the same until we reach (1.15) which is replaced by

$$\Delta_s\eta = \big[\big(-\Delta + D_{00} + nD_0 + (q+1)(n-q)\big)f\big]x_0 dx(J;0)$$
$$= \mathfrak{D}_\alpha(fx_0)dx(J;0),$$

since $(q+1)(n-q) - n = q(n-q-1)$. (1.3)′ then comes down to a check that $\mathfrak{D}_\alpha(fx_k) + 2D_k f = 0$ at N if $k = 0$, which is true.

We shall need to extend this result to $\mathcal{E}'_q$, the conjugate dual of $\mathfrak{B}^q = C^\infty(\Lambda^q(T_{0,1}(S_n)))$. Denote the sesquilinear pairing by $(\Omega|\varphi)(\Omega \in \mathcal{E}'_q, \varphi \in \mathfrak{B}^q$, linear in φ). Let $\mathcal{E}' = \mathcal{E}'_0$, and $\mathcal{C}^q = \{$all forms (not necessarily tangential)

$\varphi = \sum f_J d\bar{z}(J)$ with $f_J \in C^\infty(S_n)\}$. If Ω is a formal sum $\sum f_J d\bar{z}(J)$ (sum over all increasing q-tuples J, each $F_j \in \mathcal{E}_0'$), $\varphi \in \mathcal{C}^q$ as above, let us put $[\Omega|\varphi] = \sum(F_J|f_J)$. Then we can view $\Omega \in \mathcal{E}_q'$. Now $\mathcal{E}_q'$ is in bijective correspondence with the set of all such Ω satisfying $[\Omega|\sigma \wedge \varphi'] = 0$ for all $\varphi' \in \mathcal{C}^{q-1}$ (we set $(\Omega|\varphi) = [\Omega|\varphi]$ if $\varphi \in \mathfrak{B}^q$). We call such Ω *tangential*. For clearly a tangential Ω defines an element of $\mathcal{E}_q'$. If $\Omega = 0$ in $\mathcal{E}_q'$, it follows that $[\Omega|\varphi] = 0$ for all $\varphi \in \mathcal{C}^q$, whence all $F_J = 0$. Conversely, if $\Gamma \in \mathcal{E}_q'$ is supported in a small enough open set U that there exists a set of smooth q-forms $\{e(K)\}_K$ on U such that the forms are an orthogonal basis at each point of U, we surely have $\Gamma = \sum G_K e(K) = \sum F_I d\bar{z}(I)$ for certain G_K, $F_I \in \mathcal{E}'$ supported in U. Here $\sum G_K e(K)$ has the obvious meaning. This latter consideration shows that each $\Omega \in \mathcal{E}_q'$ can be written in the form $\sum_K H_K \zeta(K) = \sum_I P_I \omega(I)$ (certain H_K, $P_I \in \mathcal{E}'$).

Now $\bar{\partial}_b$, ϑ_b can be extended to maps from $\mathcal{E}_q'$ to $\mathcal{E}_{q+1}'$, $\mathcal{E}_{q-1}'$ respectively through duality (i.e., the relations $(\bar{\partial}_b\Omega|\varphi) = (\Omega|\vartheta_b\varphi)$, $(\vartheta_b\Omega|\varphi) = (\Omega|\bar{\partial}_b\varphi)$). If $\Omega = f\omega(J)$, we have Lemma 1.1(c) if f is smooth and hence if $f \in \mathcal{E}'$ since C^∞ is dense in $\mathcal{E}'$. Thus if $f_m \to f$ in $\mathcal{E}'$, all $f_m \in C^\infty$, $\Omega_m = f_m\omega(J)$, we have $\bar{\partial}_b\Omega_m \to \bar{\partial}_b\Omega$ in $\mathcal{E}_{q-1}'$ and, using (1.10), that $\vartheta_b\bar{\partial}_b\Omega_m \to \vartheta_b\bar{\partial}_b\Omega$; also $\vartheta_b\Omega_m \to \vartheta_b\Omega$ and $\Box_b\Omega_m \to \Box_b\Omega$. We must then have (1.3) for $\eta = \Omega$, whence for any $\eta \in \mathcal{E}_q'$ (η expressed in tangential form). Further, the right side of (1.3) is tangential.

We shall of course write $F(f) = \overline{(F|\bar{f})}$ if $F \in \mathcal{E}', f \in C^\infty$.

Remark. We indicate the interaction of $\mathcal{L}_\alpha$, $\mathfrak{D}_\alpha$ with spherical harmonics; we shall not be needing these facts later.

If P is a spherical harmonic in $\mathbf{R}^{n+1}$ of degree k (a harmonic polynomial, homogeneous of degree k),

$$\mathfrak{D}_\alpha P = \left(k + \frac{n-\alpha}{2}\right)\left(k + \frac{n+\alpha-2}{2}\right)P$$

as can be computed readily from (1.4)′, the case $\alpha = n$ being well known. If P is a bigraded spherical harmonic of type (p, q) on $\mathbf{C}^{n+1}$ (a harmonic polynomial which is a linear combination of terms of the form $z^\rho\bar{z}^\gamma$, ρ, γ multi-indices with $|\rho| = p$, $|\gamma| = q$), then

$$\mathcal{L}_\alpha P = \left(p + \frac{n+\alpha}{2}\right)\left(q + \frac{n-\alpha}{2}\right)P,$$

which agrees with formulas in [2]. Since the restrictions of bigraded spherical harmonics to S_n span a dense subspace of $L^2(S_n)$, one conjectures that $\mathcal{L}_\alpha$ is invertible if and only if $\pm\alpha \neq n, n+2, \cdots$ as on $\mathbf{H}^n$; we shall see this next.

Since $M_{jk}\Delta = \Delta M_{jk}$, M_{jk} maps (p, q)-harmonics to $(p-1, q+1)$-harmonics; this observation was what led the author to (1.2).

2. Solving $\mathfrak{L}_\alpha$, $\mathfrak{D}_\alpha$

In this section we determine fundamental solutions for $\mathfrak{L}_\alpha$, $\mathfrak{D}_\alpha$; it is not immediately apparent how this helps to solve $\Box_b$, Δ_s, but we show how it does in the next section. We have no more use for the forms $\zeta(J)$, $\xi(J)$ and free the letters ζ, ξ for other purposes.

It will help to remember Gauss's hypergeometric equation:

$$s(1-s)f'' + [c-(a+b+1)s]f' - abf = 0, \tag{2.1}$$

where $s \in \mathbf{C}, f = f(s)$, $a, b, c \in \mathbf{C}$. Put $(a)_0 = 1$, $(a)_m = a(a+1)\cdots(a+m-1)$ for $m \in \mathbf{N}$, then $F(a, b, c; s) = \sum_{m=0}^{\infty} p_m s^m$ is a solution of (2.1) in $\{|s| < 1\}$ if $p_m = (a)_m(b)_m/[(c)_m m!]$ and $c \notin \mathbf{Z}^- = \{0, -1, -2, \cdots\}$. If also $a, b \notin \mathbf{Z}^-$, we have, for example $(a)_m = \Gamma(a+m)/\Gamma(a)$. From this,

$$p_m = \Gamma(c)\Gamma(a)^{-1}\Gamma(b)^{-1}m^{d-1}(1 + O(m^{-1})), \tag{2.2}$$

where $d = a + b - c$, since $p_m = p_m\Gamma(m)^2/\Gamma(m)^2$ and, e.g., $\Gamma(a+m)/\Gamma(m) = m^a(1 + O(m^{-1}))$ (from the Stirling series for $\log \Gamma$). (For *real a* this is easy: from the convexity of $\log \Gamma$, $(m + a - 1)^a < \Gamma(a+m)/\Gamma(m) < m^a$ if $0 < a < 1$, $m \in \mathbf{N}$. $\log \Gamma$ is convex because $(\log \Gamma)''(x) = \sum_{n=0}^{\infty}(x+n)^{-2}$.) Now if $l \in \mathbf{R}$, $l \notin \mathbf{Z}^-$, put $q_{ml} = (l)_m/m!$, then $\sum q_{ml}s^m = (1-s)^{-l}$. If $l = 0$, put $q_{ml} = 1/m$, then $\sum q_{ml}s^m = \log(1-s)$. Suppose $d > 0$ and put $q_m = q_{m,d}$. Then $q_m = \Gamma(d)^{-1}m^{d-1}(1 + O(m^{-1}))$. Put $B = \Gamma(c)\Gamma(d)\Gamma(a)^{-1}\Gamma(b)^{-1}$; then $p_m = Bq_m(1 + O(m^{-1}))$. If $d \neq 1$, $q_{m,d-1} = q_m(d-1)/(d+m-1)$; if $d = 1$, $q_{m,0} = q_m/m$. Thus $O(q_m m^{-1}) = O(q_{m,d-1})$. Finally,

$$\lim_{s\to 1}(1-s)^d F(a, b, c; s) = B. \tag{2.3}$$

This will be useful. If $d = 0$, we have directly from (2.2) that $F(a, b, c; s) = C\log(1-s) + g(s)$, where g is continuous in $\{|s| \leqslant 1\}$; while if $d < 0$, $|F(a, b, c; s)|$ is continuous in $|s| \leqslant 1$. Put $G(a, b, c; s) = F(a, b, c; (1+s)/2)$, $\beta = (n+\alpha)/2$, $\gamma = (n-\alpha)/2$, and $j_{\alpha,\xi}(x) = G(\beta - 1, \gamma; n/2; x \cdot \xi)$, $\varphi_{\alpha,\zeta}(z) = (1 - z\cdot\bar{\zeta})^{-\beta}(1 - \bar{z}\cdot\zeta)^{-\gamma}$ for $x, \xi \in S^n$, $z, \zeta \in S_n$. Write $\psi_\alpha = \psi_{\alpha,N}$, $\varphi_\alpha = \varphi_{\alpha,N}$. (Here $1 - z\cdot\bar{\zeta}$ always lies in the right half plane, and we use the principal branch of the power functions.)

Theorem 2.1. $\mathfrak{D}_\alpha\psi_{\alpha,\xi} = b_\alpha\delta_\xi$, $\mathfrak{L}_\alpha\varphi_{\alpha,\zeta} = c_\alpha\delta_\zeta$ *where*
$b_\alpha = 2^n\pi^{n/2}\Gamma(n/2)\Gamma(\beta-1)^{-1}\Gamma(\gamma)^{-1}$, $c_\alpha = 2^{-n+1/2}\pi^{n+1}\Gamma(\beta)^{-1}\Gamma(\gamma)^{-1}$.

Proof. We may assume ξ or $\zeta = N$ and just write ψ_α, φ_α. Now if $g = g(x_0)$, then $-\mathfrak{D}_\alpha g = (1 - x_0^2)g'' - nx_0g' - (\beta - 1)\gamma g$. (By the way, this equation is closely related to the Legendre equation.) If $f(u) = g(2u-1)$, then $-\mathfrak{D}_\alpha g = u(1-u)f'' + [n/2 - nu]f' - (\beta-1)\gamma f$ (differentiation with respect to u). Thus $\mathfrak{D}_\alpha\psi_\alpha = 0$ away from N. For $-1 < s < 1$ we apply Green's theorem in the region $\Omega = \{x|x_0 \leqslant s\}$, put $\Gamma_\delta = \{x|x_0 = s\}$, and recall $\Delta_s = \mathcal{D}_n$ on 0-forms. Thus if $f \in C^\infty(S_n)$, then

$$\int_{\Gamma_s} (f\partial\psi_\alpha/\partial n - \psi_\alpha \partial f/\partial n)\,dm = \int_{\Omega_s} (f\mathcal{D}_n\psi_\alpha - \psi_\alpha\mathcal{D}_n f)\,dS$$

$$= \int_{\Omega_s} (f\mathcal{D}_\alpha\psi_\alpha - \psi_\alpha\mathcal{D}_\alpha f)\,dS$$

$$= -\int_{\Omega_s} \psi_\alpha \mathcal{D}_\alpha f\, dS,$$

since $\mathcal{D}_\alpha = \mathcal{D}_n + (\beta - 1)\gamma$. ($m$ = measure on Γ_s). Now

$$\int_{\Gamma_s} dm = \frac{2(1 - s^2)^{(n-1)/2}\pi^{n/2}}{\Gamma(n/2)},$$

since the radius of Γ_s is $(1 - s^2)^{1/2}$. Thus $\int_{\Gamma_s} \psi_\alpha(\partial f/\partial n)dm \to 0$ as $s \to 1$ by (2.3) (in this case $d = (n-2)/2$; note $1 - s^2 = (1+s)(1-s)$). By orthogonal invariance $\partial\psi_\alpha/\partial n$ is constant along Γ_s. At $(x_0, x_1, 0, \cdots, 0) \in \Gamma_s$, $\partial/\partial n = x_0 D_1 - x_1 D_0 = W_{01}$ so here

$$\frac{\partial\psi_\alpha}{\partial n} = -x_1 D_0 \psi_\alpha = -(1 - s^2)^{1/2}\left[\frac{(\beta - 1)\gamma}{n}\right] G\left(\beta, \gamma + 1; \frac{n}{2} + 1; x_0\right).$$

Letting $s \to 1$ we find

$$\int_{S^n} \psi_\alpha \mathcal{D}_\alpha f\, dS$$

$$= f(N)2^{n+1}\frac{(\beta - 1)\gamma}{n}\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})}\lim_{s\to 1}\left(\frac{1 - s^2}{4}\right)^{n/2} G\left(\beta, \gamma + 1; \frac{n}{2} + 1; s\right)$$

$$= b_\alpha f(N)$$

by (2.3). Since $\mathcal{D}_\alpha$ is obviously self-adjoint, we then have only to show that ψ_α is integrable on S^n. By (2.3) it suffices to show $(1 - x_0)^{-d}$ or equivalently $h(x_0) = (1 - x_0^2)^{-d}$ is integrable near N where $d = n/2 - 1$ (at least if $n \geqslant 3$). But if $x_0 = \cos\theta$, $E = S^n \cap \{x_0 \geqslant 0\}$, then $\int_E h(x_0)dS = c\int_0^{\pi/2}(\sin\theta)^{-n+2}(\sin\theta)^{n-1}\,d\theta$. $n = 1$ or 2 is similarly dismissed.

For $\mathcal{L}_\alpha$, we do not know if there is an effective analogue of Green's theorem; so in analogy to [3] we begin by defining, for $A > 1$, $\Phi_A = (A - z_0)^{-\beta}(A - \bar{z}_0)^{-\gamma}$. If $f = f(z_0)$ we have

$$-\mathcal{L}_\alpha f = (1 - |z_0|^2)\partial_{0\bar{0}} f - \beta\bar{z}_0\bar{\partial}_0 f - \gamma z_0 \partial_0 f - \beta\gamma f,$$

so that $\mathcal{L}_\alpha\Phi_A = \psi_A = \beta\gamma(A^2 - 1)(A - z_0)^{-(\beta+1)}(A - \bar{z}_0)^{-(\gamma+1)}$. We must prove the following two results: (a) $|1 - z_0|^{-n}$ is integrable on S_n, (b) $\int \psi_A\, dS \to c_\alpha$ as $A \to 1$. For we shall know from (a) that $\Phi_A \to \varphi$ as distributions by dominated convergence, and from (b) that $\mathcal{L}_\alpha\varphi = c_\alpha\delta$ since the mass of Ψ_A

concentrates at N. Both facts are consequences of the following: if $a, b \in \mathbf{C}$, $A > 1, f(A, a, b; z_0) = (A - z_0)^{-a}(A - \bar{z}_0)^{-b}$, then

$$\int_{S_n} f(A, a, b; z_0)dS = 2^{-n+1/2}\pi^{n+1}n!A^{-(a+b)}F(a, b; n + 1; A^{-2}). \tag{2.4}$$

Indeed, if (2.4) is known, then for (a), if $a = b = n/2$ we should have $d = -1$ so that $\lim_{A \searrow 1} \int_{S_n} f(A, a, b; z_0)\, dS$ exists. But, if $z_0 = x_0 + iy_0$, then

$$f(A, a, b; z_0) = |A - z_0|^{-n} = \left[(A - x_0)^2 + y_0^2\right]^{-n/2}.$$

Thus $f(A, a, b; z_0) \nearrow |1 - z_0|^{-n}$ which is hence integrable by the monotone convergence theorem. Also (b) follows from (2.4) by (2.3).

To prove (2.4) we recall a well-known device: suppose s, t are $(n + 1)$-tuples of nonnegative integers, and put $p = \frac{1}{2}(|s| + |t|)$, where $|s| = \Sigma\, s_j$. Then

$$\begin{aligned} I = \int_{S_n} \zeta^s\bar{\zeta}^t dS(\zeta) &= (p + n)!^{-1}\int_0^\infty e^{-|z|^2}I(|z|^2)^{n+p}d|z|^2 \\ &= 2^{3/2}(p + n)!^{-1}\int_{\mathbf{C}^n} z^s\bar{z}^t e^{-|z|^2}dV \\ &= 2^{-n+1/2}\pi^{n+1}\delta_{st}s!\,(|s| + n)!^{-1}. \end{aligned}$$

(Write the last integral as a product. Here $\delta_{st} = \Pi\, \delta_{s_jt_j}$, $s! = \Pi s_j!$ and the twos are caused by our choice of volume element.) In particular,

$$\int_{S_n} z_0^q\bar{z}_0^r dS = 2^{-n+1/2}\pi^n\delta_{qr}q!\,(q + n)!^{-1}.$$

If we expand $(1 - z_0A^{-1})^{-a}$ and $(1 - \bar{z}_0A^{-1})^{-b}$ in power series and integrate termwise, we then find (2.4), as desired.

It is perhaps curious that (2.3) plays such a crucial role in two such different ways. However, one should be aware that hypergeometric functions come up in many different contexts in analysis on the sphere.

If we are on S_n, $\pm\alpha \neq n, n + 2, \cdots$ (i.e., when $c_\alpha \neq 0$), we say α is admissible and put $\Phi_\alpha = c_\alpha^{-1}\varphi_\alpha$. If we are on S^n, $\pm\alpha \neq n, n + 2, \cdots$ and $\alpha \neq -n + 2$, we say α is admissible and put $\Psi_\alpha = b_\alpha^{-1}\psi_\alpha$.

Now there is no natural notion of convolution of general functions on S_n, but if $f \in L^\infty(S_n)$, $\Phi \in L^1(S_n)$, and $\Phi = \Phi(z_0)$ we can define $f * \Phi \in L^\infty(S_n)$ by

$$f * \Phi(z) = \int_{S_n} f(\zeta)\Phi(z \cdot \bar{\zeta})dS(\zeta).$$

If $U \in U(n + 1)$, one trivially computes $U\cdot(f * \Phi) = Uf * \Phi$ (recall $U\cdot g = g \circ U$). From this–and the way that the N_{jk}, L arise from $U(n + 1)$ (see the discussion after (1.2))–one finds readily that if f is smooth, $N_{jk}(f * \Phi)(z)$

exists for all z and equals $(N_{jk}f) * \Phi(z)$; similarly for $\bar{N}_{jk}$ and L. Thus $f * \Phi$ is smooth if f is, since the N_{jk}, $\bar{N}_{jk}$, L span $T(S_n)$ at each point. (Indeed, $\sum \bar{z}_j N_{jk} + 2i\bar{z}_k L = \partial_k - \bar{z}_k(z \cdot \partial)$ so that $T_{1,0}$ is in the span, etc.) Further, since $N_{jk}\bar{N}_{jk} + \bar{N}_{jk}N_{jk} = M_{jk}\bar{M}_{jk} + \bar{M}_{jk}M_{jk}$ as a computation shows, one has $\mathcal{L}_\alpha(f * \Phi) = \mathcal{L}_\alpha f * \Phi$.

If h is a function on S_n, put $\tilde{h}(z) = h(\bar{z})$. It is obvious that if $f, g \in L^\infty$, and Φ as above, we have

$$\int (g * \Phi)(z)f(z)dV(z) = \int g(\zeta)(f * \tilde{\Phi})(\zeta)dV(\zeta).$$

Thus, if $g \in \mathcal{E}'(S_n)$ we can define $g * \Phi \in \mathcal{E}'$ by $g * \Phi(f) = g(f * \tilde{\Phi})$ ($f \in C^\infty$). If Φ is smooth on S_n, one easily finds that $g * \Phi$ is smooth and $g * \Phi(z) = g(\Phi_z)$ where $\Phi_z(\zeta) = \Phi(z \cdot \bar{\zeta})$. If, instead, g is zero on some open set U and Φ is smooth (respectively, real analytic) away from N, it is easy to see that $g * \Phi$ is smooth (resp. real analytic) on U.

We define K_α: $\mathcal{E}' \to \mathcal{E}'$ by $K_\alpha(g) = g * \Phi_\alpha$ if α is admissible. Then we have

Corollary 2.2. (a) $\mathcal{L}_\alpha K_\alpha = K_\alpha \mathcal{L}_\alpha = I$. (b) $\mathcal{L}_\alpha$ *is locally solvable, injective on* $\mathcal{E}'$ *and hypoelliptic*. (c) $\mathcal{L}_\alpha$ *is* (*local*) *analytic hypoelliptic* (*i.e., if* $f \in \mathcal{E}'$, $\mathcal{L}_\alpha f = u$ *is real analytic on* U, *then* f *is real analytic on* U).

Proof. For (a), the duals on $\mathcal{E}'$ of $\mathcal{L}_\alpha$, K_α: $C^\infty \to C^\infty$ are $\mathcal{L}_{-\alpha}$, $K_{-\alpha}$; so we have only to show the identity on C^∞. But if $f \in C^\infty$, then $\mathcal{L}_\alpha K_\alpha f = \mathcal{L}_\alpha(f * \Phi_\alpha) = \mathcal{L}_\alpha f * \Phi_\alpha = K_\alpha \mathcal{L}_\alpha f$, while $K_\alpha \mathcal{L}_\alpha f(z) = \Phi_{\alpha,z}(\mathcal{L}_\alpha f)$ (see notation above) $= (\mathcal{L}_{-\alpha}\Phi_{\alpha,z})(f) = c_\alpha^{-1}(\mathcal{L}_{-\alpha}\varphi_{-\alpha,z})(f) = \delta_z(f) = f(z)$. This gives (b) in a simple standard way–see [3] for the same proofs on $\mathbf{H}^n$. (c) is also standard but we include a proof. $\mathcal{L}_\alpha$ has many noncharacteristic surfaces at each point–e.g., by (1.4) it is easy to see that $\{\operatorname{Re} z_1 = 0\}$ is noncharacteristic at N. Thus by Cauchy-Kowalewski, if $p \in U$, we can find g real analytic in a neighborhood of p with $\mathcal{L}_\alpha g = u$ there. With ρ a suitable cutoff function, $\mathcal{L}_\alpha(f - \rho g) = 0$ near p, whence $f = \rho g + K_\alpha \mathcal{L}_\alpha(f - \rho g)$ is real analytic at p.

Everything since Theorem 2.1 has an obvious analogue for $\mathcal{D}_\alpha$, α admissible.

We shall see in the next section how to derive analogous results for Δ_s, $\Box_b$ on q-forms when $\alpha = n - 2q$ is admissible. As for nonadmissible values, there is no point in worrying about n-forms; since $\Box_b * = (-1)^n * \Box_b$ by Lemma 1.1(a), questions about n-forms are at once reduced to questions about 0-forms. So we need only concern ourselves with $\mathcal{L}_n$ and $\mathcal{D}_n$ since $\mathcal{D}_{-(n-2)} = \mathcal{D}_n$.

In anticipation of an analogy with [5], we begin by setting, for $0 < r < 1$, $S_r(z) = C_n(1 - rz_0)^{-(n+1)}$, $C_n = 2^{n+1/2}n!(2\pi^{n+1})^{-1}$. If $Q_r(z) = C_n'(1 - rz_0)^{-n}$, $C_n' = -2C_n/n$, one has $S_r = (iL - n/2)Q_r$. Now $Q_r \to C_n'\varphi_n$ in $\mathcal{E}'$ as $r \to 1$;

thus S_r has a limit in $\mathcal{E}'$, say S_1, and

(2.5) $$S_1 = C_n'(iL - n/2)\varphi_n.$$

It is then apparent that $f * S_r$ has a limit in C^∞ if $f \in C^\infty$, and–by duality–in $\mathcal{E}'$ if $f \in \mathcal{E}'$; this limit one calls $f * S_1 = C_b f$ (C_b = Cauchy-Szego map). We define Cf on $B^{n+1} = \{|z| < 1\}$ by putting $Cf(z) = f * S_{|z|}(z/|z|)$. Then Cf is holomorphic on B^{n+1}. If $f = 0$ near $P \in S_n$, it is easy to see that Cf has a holomorphic extension past P.

It is known that $f \in L^2 \Rightarrow C_b f \in L^2$; we shall not need this.

If $z \in S_n$, let $\theta(z) = \log[(1 - z_0)/(1 - \bar{z}_0)]$, $z_0 \neq 1$. We let $\Phi_n(z) = 2^{n-1/2}\Gamma(n)\pi^{-n-1}(1 - z_0)^{-n}\theta(z)$, $K_n g = g * \Phi_n (g \in \mathcal{E}')$.

Theorem 2.3. (a) $\mathcal{L}_n K_n = K_n \mathcal{L}_n = I - C_b$.

(b) *Suppose* $f \in \mathcal{E}'$. *Then* $\bar\partial_b(C_b f) = 0$; $f = C_b f$, *iff* $\bar\partial_b f = 0$; $C_b^2 = C_b$; $(f|g) = 0$ *for all* $g \in C^\infty \cap \ker \bar\partial_b$, *iff* $C_b f = 0$, *iff* $f \in \vartheta_b \mathcal{E}_1'$.

(c) *If* $P \in S_n$, $f \in \mathcal{E}'$, *then* (i) *there exists* $u \in \mathcal{E}'$ *with* $\Box_b u = f$ *near* P *iff* (ii) $C_b f$ *is real analytic near* P, *iff* (iii) Cf *has a holomorphic extension past* P, *iff* (iv) *there exist* $v_{jk} \in \mathcal{E}'$, $(0 \leq j < k \leq n)$, *with* $f = \sum M_{jk} v_{jk}$ *near* P. (*When* $n = 1$, *this reduces to* $f = M_{01} v_{01}$ *near* P; M_{01} *is the analogue of Lewy's unsolvable operator on* S_1.)

(d) *If* $f \in \mathcal{E}'$ *is real analytic near* P, *then* Cf *has a holomorphic extension past* P.

(e) *Suppose* $f \in \vartheta_b \mathcal{E}_1'$. *Then there is a unique* $u \in \vartheta_b \mathcal{E}_1'$ *such that* $\Box_b u = f$; *further* u *is smooth or real analytic on any open set where* f *is. In particular,* $C^\infty = V_1 \oplus V_2$ (*orthogonal direct sum*) *where*

$$V_1 = \ker \bar\partial_b|_{C^\infty} = \ker \Box_b|_{C^\infty} = C_b(C^\infty),$$
$$V_2 = \vartheta_b \mathcal{B}^1 = \Box_b C^\infty = (I - C_b)C^\infty.$$

Proof. For (a) one has only to show $\mathcal{L}_n \Phi_n = \delta - S$, for then one can argue as Corollary 2.2(a). To prove this, one differentiates $\mathcal{L}_\alpha \varphi_\alpha = c_\alpha \delta$ with respect to α at $\alpha = n$. That is, one divides this equation by $\alpha - n$ and lets (real) $\alpha \to n$, using (2.5) and the dominated convergence theorem with the estimate $|(e^{i\theta\alpha} - 1)/\alpha| = 2|\sin(\theta\alpha/2)/\alpha| \leq \theta$. For (b), we have $\bar\partial_b(f * S_r) = 0$ for $r < 1$, so $\bar\partial_b C_b f = 0$. Thus $f = C_b f \Rightarrow \bar\partial_b f = 0$, while if $\bar\partial_b f = 0$, $0 = \vartheta_b \bar\partial_b f = (I - C_b)f$. Since $\bar\partial_b(C_b f) = 0$, $C_b f = C_b^2 f$. If $(f|g) = 0$ for all $g \in C^\infty \cap \ker \bar\partial_b$, $(f|C_b G) = 0$ for all $G \in C^\infty$, so $(C_b f|G) = 0$ for all G, and therefore $C_b f = 0$. If $C_b f = 0$, then $f = (I - C_b)f = \vartheta_b \bar\partial_b f \in \vartheta_b \mathcal{E}_1'$. If $f \in \vartheta_b \mathcal{E}_1'$, clearly $(f|g) = 0$ for all $g \in C^\infty \cap \operatorname{Ker} \bar\partial_b$. For (c), (iii) $\Rightarrow$ (ii) is trivial, (ii) $\Rightarrow$ (i) follows from (a) and Cauchy-Kowalewski as in Theorem 3 of [5], (i) $\Rightarrow$ (iv) follows from the relation $\mathcal{L}_n = \sum_{j<k} M_{jk} \overline{M_{jk}}$ which in turn follows from $[M_{jk}, \overline{M_{jk}}] = -\bar{z}_j \bar\partial_j - \bar{z}_k \bar\partial_k + z_j \partial_j + z_k \partial_k, j \neq k$, and (iv) $\Rightarrow$ (iii) by Theorem 2

of [5] with $A_b = M_{jk}$, one having only to note that (iii) is a local condition on f and that $CM_{jk} = 0$ since $M_{jk}\Phi_z = 0$ if $\Phi_z(\zeta) = S_r(z \cdot \bar{\zeta})$. (d) follows from (i) $\Rightarrow$ (iii) and Cauchy-Kowalewski. For (e), if $u_0 = K_n f$, then $\Box_b u_0 = f$ by (b) and (a); thus $\Box_b u = f$ if $u = (I - C_b)u_0$ since $\bar{\partial}_b C_b = 0$, and $u \in \vartheta_b \mathcal{E}_1'$. If $\Box_b v = 0$, and $v \in \vartheta_b \mathcal{E}_1'$, then $v = (I - C_b)v = K_n \Box_b v = 0$. If f is smooth near P, so is u_0. Then $u_0 = u_1 + u_2$ where u_1 is smooth and $u_2 = 0$ near P. Thus $(I - C_b)u_1$ is smooth near P and $(I - C_b)u_2$ is real analytic near P, and hence u is smooth near P. If f is real analytic near P, we can, by Cauchy-Kowalewski, find $g \in \mathcal{E}'$ real analytic near P with $\mathcal{L}_n g = f$ near P. Then $\mathcal{L}_n(u - g) = 0$ near P, and $u = (I - C_b)u = K_n \mathcal{L}_n(u - g) + (I - C_b)g$ which is real analytic near P by (d), completing the proof.

For $\mathfrak{D}_n$ one has a similar result. We put, if $n > 1$,

$$\Psi_n(x) = B_n \sum_{m=1} \frac{(n-1)_m}{(n/2)_m m}\left(\frac{1 + x_0}{2}\right)^m,$$

where $B_n = \Gamma(n - 1)2^{-n}\pi^{-n/2}\Gamma(n/2)^{-1}$, while if $n = 1$,

$$\Psi_1(x) = B_1 \sum_{m=1}^{\infty} \frac{(m-1)!}{(1/2)_m m}\left(\frac{1 + x_0}{2}\right)^m,$$

where $B_1 = (2\pi)^{-1}$. Let $Q(x) = nB_n$. It is easy to see by induction on n that nB_n equals the reciprocal of the area of S^n, i.e., $nB_n = \Gamma((n + 1)/2)/2\pi^{(n+1)/2}$, so that if we define $H: \mathcal{E}' \to \mathcal{E}'$ by $Hf = f * Q$, H simply projects L^2 onto the constant functions. Thus we have

Theorem 2.4. (a) $\Psi_n \in L^1(S^n)$. *Define* $J_n: \mathcal{E}' \to \mathcal{E}'$ *by* $J_n f = f * \Psi_n$. *Then* $\mathfrak{D}_n J_n = J_n \mathfrak{D}_n = I - H$.

(b) *Suppose* $f \in \mathcal{E}'$. *Then* $(f|Q) = 0$ *iff* $Hf = 0$, *iff* $g \in \delta \mathcal{E}_1'$.

(c) $\mathfrak{D}_n$ *is locally solvable, hypoelliptic and analytic hypoelliptic*, $C^\infty = V_1 \oplus V_2$ *where* $V_1 = \ker d|_{C^\infty} = \ker \Delta_s|_{C^\infty} = HC^\infty$, $V_2 = \delta(C^\infty(\Lambda^1)) = \Delta_s C^\infty = (I - H)C^\infty$.

Proof. For (a), again one divides $\mathfrak{D}_\alpha \psi_\alpha = b_\alpha \delta$ by $\alpha - n$, and lets (real) $\alpha \to n$, except when $n = 1$ when one divides by $(1 - \alpha)^2$ and lets (real) $\alpha \to 1$. When $n > 1$ one has only to check the validity of applying the dominated convergence theorem to assert that

$$\sum_{m=1}^{\infty} s_{m,\alpha}[(1 + x_0)/2]^m \to \sum_{m=1}^{\infty} s_m[(1 + x_0)/2]^m$$

in L^1 if $s_{m,\alpha} = (\beta - 1)_m(\gamma + 1)_m/[(n/2)_m m!]$, ($\beta, \gamma$ as usual), and $s_m = (n - 1)_m/[(n/2)_m m]$. Invoking the estimate

$$(a)_m = [\Gamma(m)/\Gamma(a)][\Gamma(a + m)/\Gamma(m)] < m^a\Gamma(m)/\Gamma(a),$$

if a is real, one finds $|s_{m,\alpha}| < m^{n/2-2}\Gamma(\beta - 1)^{-1}\Gamma(\gamma + 1)^{-1} < C(d)_m/m!$ with $d = n/2 - 1$ (some $C > 0$) if $n > 2$, so that the convergence is dominated by $C'(1 - x_0)^{-d}$ as desired. There are obvious modifications for $n = 1$ or 2. (b) and (c) follow from (a) by methods we have already seen.

3. Solving $\square_b$, Δ_s

The cases $q = 0$ and $q = n$ are explained by Theorems 2.3 and 2.4. For $0 < q < n$ we let $\mathfrak{T}'_q = \{f \in \mathcal{E}'_q | \vartheta_b f = 0\}$. Clearly, $\square_b$: $\mathfrak{T}'_q \to \mathfrak{T}'_q$; and if $\eta = \sum f_J d\bar{z}(J) \in \mathfrak{T}'_q$ is tangential, we have $\square_b\eta = \sum(\mathcal{L}_\alpha f_J)d\bar{z}(J)$, $\alpha = n - 2q$. We would be very disappointed if $\eta' = \sum(K_\alpha f_J)d\bar{z}(J)$ were not tangential and in $\mathfrak{T}'_q$ so that $\square_b\eta' = \eta$. We are forced to conjecture and prove the following lemma.

Lemma 3.1. *Suppose $\eta = \sum f_J d\bar{z}(J)$ is tangential and in $\mathfrak{T}'_q$, $\Phi = \Phi(z_0) \in L^1(S_n)$.*

(a) *Then $\eta' = \sum(f_J * \Phi)d\bar{z}(J)$ is tangential and in $\mathfrak{T}'_q$.*

(b) *If $\Phi \equiv 1$, then $\eta' = 0$.*

Proof. (a) Φ may be approximated in L^1 by a sequence of C^∞ functions h_m; setting

$$\Phi_m(z) = \int_{U(n)} h_m(z_0, Uz')d\mu(U),$$

where $z = (z_0, z')$, and μ is Haar measure on $U(n)$, we find $\Phi_m \to \Phi$ in L^1 also. Thus we may assume Φ is smooth on S_n.

First, we show that η' is tangential. For more clarity, we think of Φ as a function on S_n and define φ: $\bar{D} \to \mathbf{C}$ where $D = \{z_0 | |z_0| < 1\}$ by $\varphi(z_0) = \Phi(z_0, z')$ if $(z_0, z') \in S_n$. Then φ is smooth on D and continuous on $\bar{D}$; thus, by use of cutoff functions, we may assume φ is smooth on $\mathbf{C}$ and has compact support within D.

Now since η' has smooth coefficients, we have only to show that $\langle\eta', \sigma \wedge \beta\rangle = 0$ for each $z \in S_n$ and each $\beta \in \Lambda^{q-1}(T^*_{0,1}(\mathbf{C}^n))$ where $\langle\ \rangle$ represents the inner product on $\Lambda^q(T^*_{0,1})$ at z. We may assume $\beta = d\bar{z}(J)$, $J = (0, \cdots, q-2)$; $\beta = 1$ if $q = 1$. Then we must show that

$$\sum_{k=q-1}^{n} z_k \overline{f_{(k,J)}(\Phi_z)} = 0$$

for all z, or that for each z, $[\eta|\tau_z] = 0$ where

$$\tau_z(\zeta) = \sum_{k=q-1}^{n} z_k\bar{\varphi}(z \cdot \bar{\zeta})d\bar{\zeta}(k, J),$$

writing $d\bar{\zeta}$ in place of the usual $d\bar{z}$. Now select $\psi \in C^\infty(\mathbf{C})$ (coordinate w) with $\partial\psi/\partial\bar{w} = \bar{\varphi}$, e.g.,

$$\psi(w) = (2\pi i)^{-1} \int_{\mathbf{C}^n} \bar{\varphi}(v) \frac{dv \wedge d\bar{v}}{v - w};$$

see [6, p. 25]. Fix $z \in S_n$, $\zeta \in \mathbf{C}^n$, and put $\mu(\zeta) = \psi(z \cdot \zeta) d\bar{\zeta}(J)$. Then $\bar{\partial}\mu = \tau_z$. Since $\mu = \mu_1 + \sigma \wedge \mu_2$, where μ_1 restricted to S_n is tangential, we have $\bar{\partial}\mu = \bar{\partial}\mu_1 + \bar{\partial}|z|^2 \wedge \bar{\partial}\mu_2 = \bar{\partial}_b\mu_1 + \sigma \wedge \mu_3$, say; so $[\eta|\tau_z] = 0$ as desired.

We next prove $\vartheta_b\eta' = 0$. Put $\Phi^\zeta(z) = \Phi(z \cdot \bar{\zeta})$, and $\alpha = n - 2q$. We claim $\mathfrak{L}_\alpha(\Phi^\zeta)(z)$ actually depends only on $z \cdot \bar{\zeta}$, equalling $\Psi(z \cdot \bar{\zeta})$, say. Indeed, if $\zeta_0 \in S_n$ satisfies $z \cdot \bar{\zeta} = N \cdot \bar{\zeta}_0$, we can find $U \in U(n+1)$ with $UN = z$, $U\zeta_0 = \zeta$. Since $\mathfrak{L}_\alpha$ is invariant under $U(n+1)$, we have

$$\mathfrak{L}_\alpha(\Phi^\zeta)(z) = \mathfrak{L}_\alpha(U^*\Phi^\zeta)(N) = \mathfrak{L}_\alpha(\Phi^{U^*\zeta})(N) = \mathfrak{L}_\alpha(\Phi^{\zeta_0})(N)$$

as desired. Ψ is of course smooth in z if $\zeta = N$. Now by (1.3), $\Box_b\eta' = \Sigma(f_J' * \Psi)d\bar{z}(J) + \sigma \wedge \vartheta_b\eta'$. The first term on the right side is tangential by the first part of the proof; so $\sigma \wedge \vartheta_b\eta'$ is tangential, and hence $\vartheta_b\eta' = 0$.

For (b), if $\Phi = 1$, $f_J * \Phi = \overline{(\eta'|\gamma)}$ where $\gamma = d\bar{z}(J) = \pm\bar{\partial}(\bar{z}_k d\bar{z}(J; k))$ if $k \in J$, so $\eta' = 0$.

There is of course an $\mathbf{R}^{n+1}$ analogue of this lemma, which is easier since one only has to solve $d\psi/dx = \bar{\varphi}$ on $\mathbf{R}$.

With the lemma proved, we are completely in the clear. Let $\mathfrak{T}_0' = \vartheta_b\mathfrak{S}_1'$.

Lemma 3.2. *$\Box_b|_{\mathfrak{T}_q'}$ is an isomorphism for $0 \leqslant q < n$, hypoelliptic and analytic hypoelliptic, with an analogous situation for Δ_s.*

Proof. This follows at once from Corollary 2.2, Theorem 2.3(e) and Lemma 3.1(a). The only small worry is the case $q = n - 1$ of the $\mathbf{R}^{n+1}$ analogue; but if $\eta = \Sigma f_J dx(J)$ is tangential and $\delta\eta = 0$, then $\Sigma Hf_J dx(J) = 0$ by Lemma 3.1(b), so this case is also covered by Theorem 2.4.

Theorem 3.3. *$\Box_b$ on $\mathfrak{S}_q'$ is an isomorphism for $0 < q < n$, hypoelliptic and analytic hypoelliptic, with an analogous situation for Δ_s.*

Proof. We prove $\Box_b$ is injective. Suppose $u \in \mathfrak{S}_q'$, $\Box_b u = 0$. Then $\vartheta_b\Box_b u = \Box_b\vartheta_b u = 0$ and $\vartheta_b u \in \mathfrak{T}_{q-1}'$; so $\vartheta_b u = 0$; so $u \in \mathfrak{T}_q'$ and $\Box_b u = 0$; so $u = 0$.

We prove $\Box_b$ is onto. Suppose $f \in \mathfrak{S}_q'$. Then $\vartheta_b f \in \mathfrak{T}_{q-1}'$ and therefore there is a unique $v_1 \in \mathfrak{T}_{q-1}'$ with $\Box_b v_1 = \vartheta_b f$. Also there is a unique $v_2 \in \mathfrak{T}_{q-1}'$ with $\Box_b v_2 = v_1$. Thus $v_1 = \vartheta_b v_3$ wtih $v_3 = \bar{\partial}_b v_2$. Now $\vartheta_b(f - \Box_b v_3) = 0$. So $f - \Box_b v_3 \in \mathfrak{T}_q'$, and there is a unique $v_4 \in \mathfrak{T}_q'$ with $f - \Box_b v_3 = \Box_b v_4$. Hence $u = v_3 + v_4$ is the unique solution of $\Box_b u = f$.

Since v_1, v_2, v_3, and v_4 are all smooth or real analytic on any open set where f is, the remaining statements follow.

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