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BLASCHKE'S THEOREM FOR CONVEX HYPERSURFACES

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1. Introduction

The aim of this paper is to prove some generalizations of the following theorem.

Theorem A [6]. *Let M and $\tilde{M}$ be compact connected oriented hypersurfaces in R^n with positive curvatures. Assume that the second fundamental form of $\tilde{M}$ at $\tilde{m}$ is great than or equal to the second fundamental form of M at m whenever the Gauss' map of $\tilde{M}$ at $\tilde{m}$ is equal the Gauss' map of M at m. Then, up to a translation, $\tilde{M}$ is included in the convex region bounded by M.*

We will prove a similar theorem when M and $\tilde{M}$ are complete rather than compact. Actually, with additional hypothesis on the curvatures of M and $\tilde{M}$ our proof holds for hypersurfaces of a Hilbert space.

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2. Notation and main results

Except when explicitly stated, M and $\tilde{M}$ will denote complete connected oriented hypersurfaces in R^n with positive curvatures (that is, at each two-dimensional subspace σ of T_mM, the sectional curvature $K(\sigma)$ is strictly positive). Such manifolds are convex by Sacksteder's theorem [7]. We will denote the Gauss' normal map of M by N, its inverse (when it exists) by n, and the second fundamental form of M by $\widetilde{II}$, and we will consider M and $\tilde{M}$ oriented by outward normals. We will also use $\tilde{N}$, $\tilde{n}$ and $\widetilde{II}$ to denote the corresponding objects in $\tilde{M}$.

Definition. We say that two hypersurfaces M and $\tilde{M}$ are *internally tangent* at a point $m \in M \cap \tilde{M}$ if $N(m) = \tilde{N}(\tilde{m})$.

Now we can state the main result of our work.

Theorem 1. *Let M and $\tilde{M}$ be two complete hypersurfaces in R^n such that if $N(m) = \tilde{N}(\tilde{m})$ then*

$$II_m(v) \leqslant \widetilde{II}_{\tilde{m}}(v)$$

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for all $v \in T_m M \cong T_{\tilde{m}} \tilde{M}$. Assume that M and $\tilde{M}$ are internally tangent at a point $m_0 \in M \cap \tilde{M}$. Then $\tilde{M}$ is included in the convex body of M.

Definition. We say that a convex hypersurface $\tilde{M}$ *rolls freely inside* the convex hypersurface M if whenever $\tilde{M}$ is internally tangent to M at a point, then $\tilde{M}$ lies in the convex region bounded by M.

Corollary. *Let M be a hypersurface in R^n such that its principal curvatures are bounded above. Then the sphere with radius equal to the inverse of the supremum of the principal curvatures of M rolls freely inside M. Moreover M is tangent to the sphere at a point or along a geodesic arc.*

Remark 1. Assume that $N(M) \cap \tilde{N}(\tilde{M}) \neq \varnothing$. It is easy to see that M and $\tilde{M}$ are internally tangent up to a translation. Therefore, if $N(M) \cap \tilde{N}(\tilde{M}) \neq \varnothing$ in Theorem 1, we can drop out the hypothesis "internally tangent" and replace its conclusion by "up to a translation $\tilde{M}$ is included in the convex body of M".

Historical comments. W. Blaschke [1, pp. 114–117] proved Theorem 1 for closed curves in R^2. H. Karcher [5] formulated and proved, for closed curves in the sphere, a proposition analogous to the corollary. D. Koutroufiotis [4] proved Theorem 1 for complete curves in R^2 and complete hypersurfaces in R^3 (but his proof is different from the one presented here). Finally J. Rauch [6], by using Blaschke's techniques, proved Theorem 1 for compact hypersurfaces in R^n. Our proof is inspired in [6] that in its turn was inspired in [1].

3. Proof of the main result

First a sketch of the proof. We know from H. Wu [8] that N is a diffeomorphism from M onto its image and that $N(M)$ is an open convex set in the unit sphere in R^n. On the other hand, we will prove (Lemma 2) that $\tilde{M}$ is included in the convex body of M if and only if $h(x) = \langle n(x) - \tilde{n}(x), x\rangle$, $x \in N(M)$, does not change sign. By restricting h to an arc of great circle C, $C \subseteq N(M)$, we obtain a second order differential equation for h the solution of which has a constant sign on C. The result follows by convexity of $N(M)$.

We now start the proof. By defining $II_m(v) = \langle dN_m \cdot v, v\rangle$, $m \in M$, $v \in T_m M$, and by noting the previous orientation convention we have that the principal curvatures are positive.

Lemma 1. *Let M and $\tilde{M}$ satisfy the hypothesis of Theorem* 1. *Then $N(M) \subseteq \tilde{N}(\tilde{M})$.*

Proof. Assume that $\tilde{M}$ is bounded. Then $\tilde{N}(\tilde{M}) = S^{n-1}$, and there is nothing to prove. Suppose that $\tilde{M}$ is unbounded and that $N(M) \not\subseteq \tilde{N}(\tilde{M})$. Let $y \in N(M) - \tilde{N}(\tilde{M})$. It follows from the convexity of $N(M)$ and the fact that M and $\tilde{M}$ are internally tangent at the point m_0, that there exists a minimal geodesic $\gamma\colon [0, l] \to S^{n-1}$ with $\gamma([0, l]) \subseteq N(M)$, $\gamma(l) = y$ and $\gamma(0) = m_0$. Since $\tilde{N}(\tilde{M})$ is connected, there exists a point $t_0 \in (0, l)$ such that $(\gamma[0, t_0)) \subseteq \tilde{N}(\tilde{M})$ and $\gamma(t_0)$ belongs to the boundary of $\tilde{N}(\tilde{M})$. Therefore we can define a curve $e\colon [0, t_0) \to \tilde{M}$ by setting $\tilde{e}(t) = \tilde{n}(\gamma(t))$. Since $\tilde{M}$ is unbounded

$$\lim_{t \to t_0} d(\tilde{e}(t), \tilde{e}(0)) = \infty, \tag{1}$$

where d denotes the distance function in $\tilde{M}$. We claim that there exists a sequence of points $(t_n) \subset (0, l)$ with $\lim_n t_n = t_0$, such that if $v_n = \tilde{e}'(t_n)/|\tilde{e}'(t_n)|$ then

$$\lim_n \langle d\tilde{N}_{\tilde{e}(t_n)} \cdot v_n, v_n \rangle = 0. \tag{2}$$

In fact, since $\tilde{N}(\tilde{e})$ is a geodesic in S^{n-1} we have

$$\int_0^{t_0} |d\tilde{N}_{\tilde{e}(t)} \cdot \tilde{e}(t)|\, dt \leqslant 2\pi.$$

By using in succession the equality

$$\int_0^{t_0} |d\tilde{N}_{\tilde{e}(t)} \cdot \tilde{e}'(t)|\, dt = \int_0^{t_0} |d\tilde{N}_{\tilde{e}(t)} \cdot v|\; |\tilde{e}'(t)|\, dt,$$

where $v = \tilde{e}'(t)/|\tilde{e}'(t)|$, and using Schwarz's inequality in the integrand, we obtain

$$\int_0^{t_0} |\tilde{e}'(t)| \langle d\tilde{N}_{\tilde{e}(t)} \cdot v, v \rangle\, dt \leqslant 2\pi. \tag{3}$$

Our claims follows from (1) and (3). Now let $e\colon [0, l) \to M$ be defined by $e(t) = n(\gamma(t))$. Thus from the fact that $e(t_0)$ is in M and that the principal curvatures of M are positive, it follows that

$$\lim_n \langle dN_{e(t_n)} \cdot v_n, v_n \rangle > 0, \tag{4}$$

where $v_n = e'(t_n)/|e'(t_n)|$, and t_n is given by (2). Since by hypothesis $II_{e(t_n)}(v_n) \leqslant II_{\tilde{e}(t_n)}(v_n)$ we have

$$\lim_n \langle dN_{e(t_n)} \cdot v_n, v_n \rangle \leqslant \lim_n \langle d\tilde{N}_{\tilde{e}(t_n)} \cdot v_n, v_n \rangle. \tag{5}$$

But from (2) the second member of (5) is zero, which contradicts the inequality (4). Therefore $N(M) \subseteq \tilde{N}(\tilde{M})$, and this completes the proof of Lemma 1.

Remark 2. Let M and $\tilde{M}$ be two hypersurfaces such that $N(m) = \tilde{N}(\tilde{m})$ implies that $II_m(v) = \widetilde{II}_{\tilde{m}}(v)$ for all $v \in T_m M \cong T_{\tilde{m}} M$. Then the above proof shows that up a rigid motion, $N(M) = \tilde{N}(\tilde{M})$.

Remark 3. It is interesting to notice the following fact which is contained in the above proof: let $e\colon (a, b) \to M$ (M complete hypersurface in R^n, noncompact and not necessarily convex) be a differentiable curve of infinite length, such that $N \circ e\colon (a, b) \to S^{n-1}$ has finite length. Then either

$$\lim_{t\to b}\Bigl(\inf_{\substack{s>t\\ 1\leqslant i\leqslant n-1}} k_i(e(s))\Bigr) \leqslant 0,$$

or

$$\lim_{t\to a}\Bigl(\inf_{\substack{s<t\\ 1\leqslant i\leqslant n-1}} k_i(e(s))\Bigr) \leqslant 0,$$

where $k_i(e(s))$ denotes the ith principal curvature at the point $e(s)$.

Since the Gauss' normal map is a diffeomorphism and $N(M)$ is included in $\tilde N(\tilde M)$, we can define on $N(M)$ a map $h\colon N(M) \to R$ by setting $h(y) = \langle n(y) - \tilde n(y), y\rangle$, $y \in N(M)$. The lemma below characterizes through the function h the fact that $\tilde M$ is included in the convex body of M.

Lemma 2. *Let M and $\bar M$ satisfy the hypothesis of Theorem* 1. *Then h does not change sign if and only if $\tilde M$ is included in the convex body of M.*

Proof. Suppose that $\tilde M$ is included in the convex body of M. The vector radius $\tilde m m$, for all $\tilde m \in M$ and all $m \in M$, points to the positive half-space determined by $T_m M$ and $N(m)$. Thus, if $y = N(m)$, we have $h(y) \geqslant 0$.

Conversely, suppose that $h \geqslant 0$, and denote by K the convex body of M. If $\tilde M - K \neq \varnothing$, we will show that there exists $y \in N(M)$ such that $h(y) < 0$, which contradicts the hypothesis and concludes the proof. In fact, let $m_1 \in M$ be a minimum for the function $\beta(m) = |m - \tilde m|$ for a fixed $\tilde m \in M - K$. Thus $\tilde m - m_1$ is parallel to a certain $y = N(m_1)$. We set $\tilde m - m_1 = \lambda y$ where $|y| = 1$, and observe that $\lambda > 0$. Since $\tilde M$ is convex, we have

$$0 \leqslant \langle \tilde n(y) - \tilde m, y\rangle.$$

Hence

$$0 \leqslant \langle \tilde n(y) - m_1, y\rangle + \langle m_1 - \tilde m, y\rangle = -h(y) - \lambda,$$

implying $h(y) < 0$.

The proof of the lemma below can be found in [6].

Lemma 3. *Let T and S be two positive* (*invertible*) *operators in a hilbert space H. If $T \geqslant S$, then $T^{-1} \leqslant S^{-1}$.*

Proof of Theorem 1. Let m_0 be the point where M and $\tilde M$ are internally tangent. Set $y_0 = N(m_0) = \tilde N(m_0)$. Parametrize a great circle C passing through y_0 by $\alpha(s)$, where s is the arc length in such a way that $\alpha(0) = y_0$. Set $h(s) = h(\alpha(s))$. We will show that

$$h'' + h = u,$$

where $u \geqslant 0$ and

$$h'(0) = h(0) = 0. \tag{7}$$

First we claim that the *support function* p restricted to C,

$$p(s) = \langle n(\alpha(s)), \alpha(s)\rangle,$$

satisfies the equation

$$p'' + p = \langle \alpha', n'\rangle.$$

In fact

$$p' = \langle n', \alpha\rangle + \langle n, \alpha'\rangle = \langle n, \alpha'\rangle,$$

because $n' = dn \cdot \alpha'$ is orthogonal to α for all s. Since α is a parametrization by arc length we obtain, by derivation of the last equation,

$$p'' + p = \langle \alpha', n'\rangle,$$

which was our claim. If we restrict $\tilde{p}$ to α, we obtain similarly

$$\tilde{p}'' + \tilde{p} = \langle \alpha', \tilde{n}'\rangle.$$

It follows that

$$h'' + h = \langle \alpha', n' - \tilde{n}'\rangle.$$

By using Lemma 3, the fact that $n' = dn \cdot \alpha'$, $\tilde{n}' = d\tilde{n} \cdot \alpha'$, and the hypothesis of the second fundamental forms we obtain

$$h'' + h = u,$$

where u is a nonnegative function in s. Moreover from the fact that M and $\tilde{M}$ are internally tangent at the point $m_0 = n(y_0) = \tilde{n}(y_0)$ we have that the last equation satisfies the initial conditions

$$h(0) = h'(0) = 0.$$

This proves (6) and (7). It is easy to see, by derivation, that

$$h(s) = \int_0^s u(t)\,\mathrm{sen}(s - t)\,dt$$

is the solution to (6) which satisfies (7). We notice that if $-\pi \leqslant s \leqslant \pi$, then $h(s) \geqslant 0$. Since $N(M)$ is included in a hemisphere [8] and s is the arc length of a geodesic in S^{n-1}, we obtain that h is nonnegative on α. But $N(M)$ is convex [8]. Therefore h is nonnegative on $N(M)$. By Lemma 2 this concludes the proof of Theorem 1.

Proof of corollary. The first part is an immediate consequence of Theorem 1. In fact, in this case M is complete and $\tilde{M}$ is a sphere of radius $1/a$, where a is an upper bound for the principal curvatures, in particular the supremum. The second claim follows immediately from the following facts:

(i) If $h(s_0) = 0$ for any s_0 then $h = 0$ in $[0, s_0]$.

(ii) If two hypersurfaces M and $\tilde{M}$ in R^n are tangent along a curve C, then the geodesic curvature of C is the same whether with respect to M or $\tilde{M}$. This concludes the proof of corollary.

At this point it is interesting to remark that the function $h(s) = \int_0^s u(t) \operatorname{sen}(s - t)\, dt$ allows us to conclude that if the second fundamental forms are equal for all points then the hypersurfaces M and $\tilde{M}$ coincide (see Remark 2).

4. Generalizations

A careful observation shows that the proof of Theorem 1 still holds for hypersurfaces in a Hilbert space once the following two facts are true: (i) the Gauss' normal map N is a diffeomorphism onto its image, (ii) $N(M)$ is convex.

By using a result of R. L. de Andrade [2] we can make sure of the two facts mentioned above if we assume that the hypersurfaces have sectional curvatures bounded away from zero (that is, for each point $m \in M$ there exists $\delta(m) > 0$ such that $K(\sigma) \geqslant \delta(m)$ for all two-dimensional subspace $\sigma \subseteq T_m M$ where $K(\sigma)$ is the sectional curvature of the σ-plane). Therefore we can obtain

Theorem 1′. *Let M and $\tilde{M}$ be connected, convex, complete, oriented hypersurfaces in a Hilbert space H with sectional curvatures bounded away from zero and such that if $N(M) = \tilde{N}(\tilde{M})$ then*

$$\langle dN_m \cdot v, v \rangle \leqslant \langle d\tilde{N}_{\tilde{m}} \cdot v, v \rangle,$$

for all $v \in T_m M \cong T_{\tilde{m}} \tilde{M}$. Assume that M and $\tilde{M}$ are internally tangent at a point $m_0 \in M \cap \tilde{M}$. Then $\tilde{M}$ is included in the convex body of M.

The following corollaries are proved in a way similar to the corollary of Theorem 1.

Corollary 1. *Let M and $\tilde{M}$ be as in the Theorem* 1′, *and assume that M is bounded. Then $\tilde{M}$ is bounded.*

Corollary 2. *Let M be a connected, convex, complete, oriented hypersurface in a Hilbert space with sectional curvatures bounded away from zero and such that*

$$a = \operatorname{Sup}\{\langle dN_m \cdot v, v \rangle, v \in T_m M, |v| = 1, m \in M\}$$

is finite. Then the sphere with radius $1/a$ rolls freely inside M, and is tangent to M at a point or along a geodesic arc.

Corollary 3. *Let M be a connected, convex, complete, oriented hypersurface in a Hilbert space H with sectional curvatures bounded away from zero and such that*

$$a = \operatorname{Inf}\{\langle dN_m \cdot v, v \rangle, m \in M, v \in T_m M, |v| = 1\}$$

is not zero. Then the hypersurface M rolls freely inside the sphere of radius $1/a$. Moreover, M is bounded with diameter smaller than $2\pi/a$, and is tangent to the sphere at a point or along a geodesic arc.

A crucial point in the proof of Theorem 1 is the fact that the hypersurfaces have positive curvatures. A natural question is whether the same theorem would be true when we consider nonnegative rather than positive curvatures.

If two curves in R^2 have subsets with zero curvatures, the theorem is not true as we see in the followng example.

Example. We consider two curves in R^2 by given

$$\gamma_1(t) = (pt, t^4) t \in R \text{ and } p > 1,$$

$$\gamma_2(s) = \begin{cases} (s, (s-1)^4), & s \in R, \quad s \geqslant 1, \\ (s, 0), & s \in R, \quad |s| \leqslant 1, \\ (s, (s+1)^4), & s \in R, \quad s \leqslant -1. \end{cases}$$

If N_1 (resp. N_2) is the unit outward normal on γ_1 (resp. γ_2), we have

$$N_1(t) = \frac{1}{(p^2 + 16t^6)^{1/2}}(4t^3, -p),$$

$$N_2(s) = \begin{cases} (0, -1), & \text{if } |s| \leqslant 1, \\ \dfrac{1}{(1 + 16(s-1)^6)^{1/2}}(4(s-1)^3, -1), & \text{if } s \geqslant 1, \\ \dfrac{1}{(1 + 16(s+1)^6)^{1/2}}(4(s+1)^3, -1), & \text{if } s \leqslant -1. \end{cases}$$

Therefore $N_1(t) = N_2(s)$ if and only if $s \geqslant 1$ and $t = \sqrt[3]{p}\,(s-1)$ or $s \leqslant -1$ and $t = \sqrt[3]{p}\,(s+1)$. Since

$$k(\gamma_1(t)) = \frac{12t^2}{(p^2 + 16t^6)^{1/2}},$$

and

$$k(\gamma_2(s)) = \begin{cases} 0, & \text{if } |s| \leqslant 1, \\ \dfrac{12(s-1)^2}{(1 + 16(s-1)^6)^{1/2}}, & \text{if } s \geqslant 1, \\ \dfrac{12(s+1)^2}{(1 + 16(s+1)^6)^{1/2}}, & \text{if } s \leqslant -1, \end{cases}$$

we have that

$$k(\gamma_1(t)) \geqslant k(\gamma_2(s)),$$

whenever $N_1(t) = N_2(s)$. But the points (pt, t^4) with $t > 1/(p-1)$ are not in the convex region bounded by $\gamma_2(R)$. This example shows that the Koutroufiotis' result [4], for curves in R^2, is the best possible.

It would be interesting to prove or to find a counterexample for the following statement.

Let M and $\tilde{M}$ be connected, complete, oriented, convex hypersurfaces in R^n such that if $N(m) = \tilde{N}(\tilde{m})$ then

$$\widetilde{II}_m(v) \leqslant II_{\tilde{m}}(v)$$

for all $v \in T_m M \approx T_{\tilde{m}}\tilde{M}$. Assume that M has nonnegative curvatures, $\tilde{M}$ has positive curvatures, and M and $\tilde{M}$ are internally tangent at a point $m_0 \in M \cap \tilde{M}$. Then $\tilde{M}$ is included in the convex body of M.

We refer to [4] for a proof of this fact when M and $\tilde{M}$ are curves in R^2.

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