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A COMPLEX ANALOGUE OF HARTMAN-NIRENBERG CYLINDER THEOREM

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1. Introduction

Hartman and Nirenberg [3] proved, in 1959,

Theorem (*Hartman-Nirenberg*). *Let M^n be a connected complete Riemannian hypersurface in an $(n+1)$-dimensional Euclidean space R^{n+1}. If the rank of the Gauss map is ≤ 1 everywhere, then M^n is cylindrical.*

This theorem is the first global determination of a flat hypersurface M in Euclidean space. Indeed the condition about the rank of Gauss map is equivalent to the flatness of M. Classically, we had only the local classification of flat surfaces. In this paper, we shall show the complex version of the above theorem.

Let M^n be a complex n-dimensional complete connected Kählerian hypersurface isometrically and holomorphically immersed by f into an $(n+1)$-complex space C^{n+1}.

Let $\tilde{\phi}: M^n \to CP^n$ be a mapping from M^n to the complex projective n-space CP^n which assigns to a point x in M^n the normal plane of $f(M^n)$ at $f(x)$ in C^{n+1}, which we can identify with a point in CP^n by the parallel displacement in C^{n+1}. We call this mapping *the Gauss map* for the complex hypersurface M^n in C^{n+1}.

Let ξ be any unit normal vector field around x, and denote by A the tensor field of type (1,1) given by

$$\tilde{\nabla}_X \xi = -A_\xi X + \hat{\nabla}_X \xi ,$$

where $\tilde{\nabla}$ is the canonical connection of C^{n+1} and $\hat{\nabla}$ is the normal connection induced by $\tilde{\nabla}$. Then we have:

(1.1) $\tilde{\phi}_*(X) = 0$ if and only if $AX = 0$, where $\tilde{\phi}_*$ is the Jacobian of $\tilde{\phi}$;

(1.2) the rank of $\tilde{\phi}_*$ is equal to that of A;

(1.3) the Gauss map $\tilde{\phi}$ is anti-holomorphic.

For the proof of (1.1), (1.2) and (1.3), see K. Nomizu and B. Smyth [5]. Now our main theorem is stated as follows.

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Theorem. *Let M^n be an n-dimensional Kählerian hypersurface of C^{n+1} immersed into C^{n+1} holomorphically and isometrically. Then the following conditions are equivalent*:

(1.4) *The rank of $\tilde{\phi}_*$ is ≤ 2 everywhere, where $\tilde{\phi}_*$ is the Jacobian of $\tilde{\phi}$;*

(1.5) *$\tilde{\phi}$ maps M^n into some complex projective line, say CP^1, in CP^n;*

(1.6) *The manifold M^n is cylindrical, i.e., there exist an $(n-1)$-dimensional Kählerian manifold M_1^{n-1} and a Kählerian curve M_2^1 such that there exists a holomorphic isometry $g: M_1^{n-1} \times M_2^1 \to M^n$ whose composition with f, i.e., $f \circ g$, restricted to $M_1^{n-1} \times \{y\}$ in $M_1^{n-1} \times M_2^1$, for each y, i.e., $f \circ g | M^{n-1} \times \{y\}$, maps $M_1^{n-1} \times \{y\}$ holomorphically and isometrically onto an $(n-1)$-dimensional complex plane which is parallel to each other in C^{n+1}, and $f \circ g$ restricted to $\{x\} \times M_2^1$ in $M_1^{n-1} \times M_2^1$, i.e., $f \circ g | \{x\} \times M_2^1$, maps $\{x\} \times M_2^1$ into a 2-dimensional complex subspace of C^{n+1} which is perpendicular to $f \circ g(M^{n-1} \times \{y\})$ at $f \circ g(x, y))$ for each $x \in M_1^{n-1}$ and $y \in M_2^1$.*

This theorem is the answer to the problem proposed in [5].

2. Preparations

Let $\alpha(X, Y)$, for X and $Y \in TM$, be the second fundamental form of an isometric immersion $f: M^n \to M^N(c)$, where M^n is a Riemannian manifold, TM is the tangent space of M^n, and $M^N(c)$ is the space form of constant curvature c. For any x in M, the subspace $RN(x)$ of the tangent space $TX(x)$ of M^n at x defined by $RN(x) = \{X \in TM(x) : \alpha(X, Y) = 0$, for all $Y\}$ is called *the relative nullity space* of f at x, and the dimension $\nu(x)$ of $RN(x)$ is called *the relative nullity* of f at x. Following Chern and Kuiper [2], we also call $\nu = \min \nu(x)$ for $x \in M$ the index of relative nullity of f. It is well known that the subset G of M defined by $G = \{x \in M^n : \nu(x) = \nu\}$ is open, and we can define on G *the relative nullity distribution* which assigns to x in G the relative nullity space $RN(x)$. It is also well known that the distribution is differentiable, involutive and totally geodesic; for more details of this see [1] in which we have shown that the maximal integral submanifolds of the distribution, i.e., the leaves, are complete if M^n is complete. It was also shown that if M^n is a complete Kählerian manifold of complex dimension n, and $M^N(c)$ is the complex space form of holomorphic sectional curvature c and of complex dimension N, then the leaves are totally geodesic Kählerian submanifolds in both M^n and M^N.

In particular, in our case here, each leaf is a complex $(n-1)$-dimensional plane, since $G = \{x \in M^n$: the rank of $\tilde{\phi}_*(x) = 2\}$ by (1.1), (1.2). Now we shall introduce the notion of conullity operater which was defined by Rosenthal [6].

Let x be a point in a leaf in G. For any η_x in the relative nullity space $RN(x)$ at x define a linear operator, say $\bar{A}_{\eta_x}$ of the orthogonal complement $RN(x)^\perp$ of $RN(x)$ in TM_x, by

$$\bar{A}_{\eta_x} X = P_n^{\perp}(\nabla_X \eta)_x \,, \tag{2.1}$$

where ∇ is the connection in M^n, η is an extension of η_x in a neighborhood of x, and $P_x^{\perp}$ is the projection of TM_x onto $RN(x)^{\perp}$. The following Propositions 2.1 and 2.2 are due to Rosenthal [7].

Proposition 2.1. *$\bar{A}_{\eta_x}$ depends only on the vector η_x and not on the extensions.*

Proof. Let g be a C^{∞} function on M^n, and η an extension of η_x on a neighborhood of x. It suffices to show that $\bar{A}_{(g\eta)_x} X = g(x) \cdot \bar{A}_{\eta_x} X$. By the definition of the operator, for X in $RN(x)^{\perp}$, $\bar{A}_{(g\eta)_x} X = P_x^{\perp}(\nabla_X g\eta)_x = P_x^{\perp}(Xg \cdot \eta + g\nabla_X \eta)_x = g(x) P_x^{-}(\nabla_X \eta)_x = g(x) \bar{A}_{\eta_x} X$.

Proposition 2.2. *Let α be the second fundamental form of M^n in C^{n+1}. Then $\alpha(X, \bar{A}_{\eta_x} Y) = \alpha(Y, \bar{A}_{\eta_x} X)$ for any X, Y in $RN(x)^{\perp}$.*

Proof. For X and Y in $RN(x)^{\perp}$ and η in $RN(x)$, we have $\tilde{R}(X, Y)\eta = R(X, Y)\eta + \alpha(X, \nabla_Y \eta) - \alpha(Y, \nabla_X \eta)$, where R and $\tilde{R}$ are the curvature tensor fields of M^n and C^{n+1}, respectively. Since $\tilde{R} = 0$, $\alpha(X, \nabla_Y \eta) = \alpha(Y, \nabla_X \eta)$ holds. From this last equality and the definition of $\bar{A}$, we obtain the equality in Proposition 2.2.

Proposition 2.3. *For any x in G and any η_x in $RN(x)$, $\bar{A}_{\eta_x}$ is a complex linear function of $RN(x)^{\perp}$.*

Proof. This proposition is slightly more general than the one in [7]. Let J be the complex structure of M^n. Then as is seen in [1, Proposition 2.3.1], $RN(x)$ and $RN(x)^{\perp}$ are invariant subspaces of J. First of all, we have $\alpha(X, \bar{A}_{\eta_x} JY) = \alpha(JY, \bar{A}_{\eta_x} X) = J\alpha(Y, \bar{A}_{\eta_x} X)$ by Proposition 2.2. and the fact that M^n is a Kählerian submanifold. On the other hand, $\alpha(X, J\bar{A}_{\eta_x} Y) = J\alpha(X, \bar{A}_{\eta_x} Y)$. So we have

$$\alpha(X, \bar{A}_{\eta_x} JY) - \alpha(X, J\bar{A}_{\eta_x} Y) = \alpha(X, (\bar{A}_{\eta_x} J - J\bar{A}_{\eta_x})Y) = 0 \,.$$

Suppose that $\bar{A}_{\eta_x} J - J\bar{A}_{\eta_x} \neq 0$. Then there exists Y' in $RN(x)^{\perp}$ such that $(\bar{A}_{\eta_x} J - J\bar{A}_{\eta_x})Y' \neq 0$. However, $(\bar{A}_{\eta_x} J - J\bar{A}_{\eta_x})Y'$ is in $RN(x)^{\perp}$, so there must exist X' in $RN(x)^{\perp}$ such that $\alpha((A_{\eta_x} J - JA_{\eta_x})Y', X') \neq 0$. This is a contradiction. Hence $\bar{A}$ and J commute. q.e.d.

Notice that by Proposition 2.3 we have the following expression of $\bar{A}_{\eta_x}$ with respect to a unitary frame:

$$\bar{A}_{\eta_x} = \begin{bmatrix} \alpha(x) & -\beta(x) \\ \beta(x) & \alpha(x) \end{bmatrix} . \tag{2.2}$$

3. Lemmas

Lemma 3.1. *Under the assumptions in § 1, $\bar{A}_{\eta_x} = 0$ for all x in G and all η_x in $RN(x)$.*

Proof. We claim that all eigenvalues of $\bar{A}_{\eta_x}$ are zero. As is mentioned in

§ 1, we have shown that the leaves of the relative nullity distribution are complete; see [1, Theorem 1.8.1]. Therefore the argument in the proof of [6, Theorem 3.1] is applicable to our Lemma 3.1, and consequently the real eigenvalues of $\bar{A}_{\eta_x}$ are zero. In order to show that the complex eigenvalues, if any, are also zero, let $a + bi$ be a complex eigenvalue. Then there exists a vector X in $RN(x)^{\perp}$ such that $\bar{A}_{\eta_x}X = aX + bJX$. Consider a new vector ξ_x in $RN(x)$ given by $\xi_x = a\eta_x - bJ\eta_x$, so that $\bar{A}_{\xi_x}X = (a^2 + b^2)X$. Again by the same argument as mentioned above, $a^2 + b^2 = 0$, i.e., $a = b = 0$, so that $a + bi = 0$. Now by (2.2), $\alpha \pm \beta i$ are the only possible eigenvalues of $\bar{A}_{\eta_x}$. Thus $\alpha \pm \beta i = 0$ implies that $\alpha = \beta = 0$, i.e., $\bar{A}_{\eta_x} = 0$.

Lemma 3.2. *The distributions RN and $RN^{\perp}$ (the distribution defined by the orthogonal complement of RN) are parallel. In particular, $RN^{\perp}$ is involutive.*

Proof. Let η and ζ be in RN, and X and Y in $RN^{\perp}$. Since RN is totally geodesic, $g(\tilde{\nabla}_{\zeta}\eta, X) = 0$. Also $g(\tilde{\nabla}_Y\eta, X) = g(\bar{A}_{\eta}(Y), X) = 0$ by Lemma 3.1. Therefore RN is parallel, and so is $RN^{\perp}$ automatically.

Lemma 3.3. *Let M^n be given as in the introduction of this paper. Then the set $G = \{x \in M^n; \nu(x) = 2(n-1)\} = \{x \in M^n;$ the rank of the Gauss map $= 2\}$ is open and dense in M^n.*

Proof. By upper semi-continuity of ν, G is open. Suppose that $M - G$ contains an interior point, say x. Then we have a minimal geodesic $\gamma(t)$ in M which joins x to a point y in G, i.e., $\gamma(0) = y$ and $\gamma(t_0) = x$.

Let e_1 and Je_1 at y be such that $g(R(e_1, Je_1)Je_1, e_1) \neq 0$. By the parallel displacement along γ, we have real analytic vector fields $e_1(t)$ and $Je_1(t)$ along $\gamma(t)$.

Define a function K: $[0, t_0] \to R$ by

$$K(t) = g(A(t)(e_1(t)), e_1(t))^2 + g(JA(t)(e_1(t)), e_1(t))^2 , \tag{3.1}$$

where A is a $(1, 1)$-tensor field defined by $\tilde{\nabla}_X\xi = -AX + \hat{\nabla}_X\xi$. Then clearly K is a real analytic function and $-K(t)$ is the holomorphic sectional curvature of the plane spanned by e_1 and Je_1 at t, $0 \leq t \leq t_0$.

Since y is in G, $K(0) \neq 0$. Therefore on $[0, t_0]$, K is not identically zero so that it must have at most finite zeros. This contradicts the assumption that x is an interior point of $M - G$.

Lemma 3.4. *Let x be any point in $M - G$. Then there exists an ε-ball around x such that in the ball any geodesic starting at x is either entirely in $M - G$ or intersects with $M - G$ at finitely many points.*

Proof. Let $e_1, \cdots, e_{2n}$ be a unitary frame at x such that $e_{i+n} = Je_i$, $1 \leq i \leq n$. Then for small $\varepsilon > 0$, we can take an ε-ball where we can define a real analytic frame by the parallel displacement of $e_1, \cdots, e_{2n}$ along each geodesic starting at x. For convenience, we will denote the frame field on the ball by the same letters.

Let γ be any geodesic segment such that $\gamma(0) \doteq x$ and the whole segment is in the ball. If γ is not in $M - G$ entirely, then there is a point y on γ such that $y = \gamma(t_0)$ is in G. Thus we can find a pair of vectors, say e_i and Je_i, among $e_1, \cdots, e_{2n}$ such that the function $K(t)$ in (3.1) with e_1 replaced by e_i is non-zero at y. Since $K(t)$ is a real analytic function, there exist at most finitely many zeros on $\gamma(t)$.

Lemma 3.5. *Let x be any point in M. Then there exists a complex $(n - 1)$-dimensional plane, say $p(f(x))$, in C^{n+1} such that:*

(3.2) *$p(f(x))$ is tangent to $f(M)$, i.e., there exists a complex $(n - 1)$-dimensional plane $p(x)$ in TM_x such that $f_*(p(x)) = p(f(x))$,*

(3.3) *$p(f(x))$ is parallel to a fixed complex $(n - 1)$-dimensional plane, say C^{n-1}, in C^{n+1} for all x in M.*

Proof. To define the above fixed plane C^{n-1} in C^{n+1}, take a fixed point x_0 in G, and consider the image of the leaf passing through x_0 by f, which is a complex $(n - 1)$- plane in C^{n+1} by [1, Theorem 2.3.1]. So we may define C^{n-1} to be the $(n - 1)$-dimensional plane passing through the origin of C^{n+1} and parallel to the image plane of the leaf containing x_0.

If x is in G, then define $p(f(x))$ to be the image plane of the leaf passing through x. Since f is an isometric immersion, (3.2) is satisfied.

If x is in $M - G$, by Lemmas 3.3 and 3.4 we can find a connected component of G, say G', such that there exists a geodesic segment of $\gamma'(t)$, $0 \leq t \leq \varepsilon'$, which, except $\gamma'(0) = x$, belongs to G'. By Lemma 3.2 the image planes $p(f(\bar{x}))$ where $\bar{x}$ is in G' are parallel in C^{n+1}. Thus define $p(f(x))$ to be the point set limit of the planes $p(f(\gamma'(t)))$, $0 \leq t \leq \varepsilon'$, as t approaches 0. Notice here that such a limit plane as above is also parallel to the planes in $f(G')$.

Next we show that the definition of $p(f(x))$, $x \in M - G$, does not depend on the choice of the connected component of G. Let G'' be another connected component of G such that there exists a geodesic segment $\gamma''(t)$, $0 \leq t \leq \varepsilon''$, starting at x and belonging to G'' except at $\gamma''(0) = x$. Let p'' be the plane defined as $p(f(x))$ by G''. Note that these planes are tangent to M in the sense of (3.2), because M is complete and f is an isometric immersion.

Let $(x^1, \cdots, x^{2n})$ be a local coordinate system around x in M, and for convenience let $f(x)$ be the origin of C^{n+1}. Then we can regard $f_*(TM_x)$ as a $2n$-dimensional subspace passing through the origin in $R^{2n+2} = C^{n+1}$.

Let $e_1, \cdots, e_{2n}, e_{2n+1}, e_{2n+2}$ be a basis of R^{2n+2} such that $e_1 = f_*(\partial/\partial x^1), \cdots,$ $e_{2n} = f_*(\partial/\partial x^{2n})$, and e_{2n+1} and e_{2n+2} are orthogonal to $f_*(T_xM)$. Define $\tilde{p}\colon R^{2n+2} \to f_*(TM_x)$ to be the natural projection. Then the Jacobian of $\tilde{p} \circ f\colon M \to TM_x$ at x is nothing but the identity matrix with respect to the basis introduced above. Thus $\tilde{p} \circ f$ is a diffeomorphism on a small neighborhood U of x where f is an isometry. Therefore $\tilde{p}$ is a diffeomorphism on $f(U)$.

Since the projection $\tilde{p}$ preserves parallelism for affine subspaces in R^{2n+2}, $\tilde{p}(p(f(x)))$ is parallel to $\tilde{p}(p(f(\gamma'(t))))$ in $f^*(TM_x)$ for all $\gamma'(t)$, $0 \leq t \leq \varepsilon'$, in U as $2(n - 1)$-dimensional subspaces. Note that $\tilde{p}$ is a local diffeomorphism on $f(U)$, and $p(f(x)) \subset f(M)$, $p(f(\gamma'(t))) \subset f(M)$, so the $\tilde{p}$-images of these planes have the same dimension as that of $p(f(x))$ and $p(f(\gamma'(t)))$ for $\gamma'(t)$ in U.

Suppose $p'' \neq p(f(x))$ as complex $(n-1)$-dimensional subspace of $f_*(TM_x)$. This assumption makes sense because p'' and $p(f(x))$ are actually in $f_*(TM_x)$. Then we have a complex line H in p'' such that $H \cap f(p(x)) = \{0\}$, and H and $f(p(x))$ span $f_*(TM_x)$. Under the above condition, we know that any complex $(n-1)$-dimensional affine subspace of $f_*(T_*M_x)$ which is parallel to $p(f(x))$ must intersect H. So for sufficiently small $t_0 > 0$, $\tilde{p}(p(f(\gamma'(t_0))))$ must intersect H in $\tilde{p} \circ f(U)$. Therefore $H \cap p(f(\gamma'(t_0))) \neq \emptyset$ in $f(U)$. Since this is impossible, we have shown $p'' = p(f(x))$.

To show each $p(f(x))$ is parallel to C^{n-1}, take a minimal geodesic segment $\gamma(t)$ between x and x_0 such that $\gamma(0) = x_0$ and $\gamma(t_*) = x$. Then by the same argument as in the proofs of Lemmas 3.3 and 3.4, we find finitely many points, say $0 < t_1 <, \cdots, < t_k \leq t_*$, which are in $M - G$. By the above argument, we know that $p(f(\gamma(0))) = p(f(x_0))$, $p(f(\gamma(t_1))), \cdots, p(f(\gamma(t_k)))$ and $p(f(\gamma(t_*)))$ are parallel to each other, hence $p(f(x))$ is parallel to C^{n-1} in C^{n+1}.

4. Proof of the theorem

Proposition 4.1. (1.4) *in the theorem implies* (1.5).

Proof. For convenience, let $(Z^0, \cdots, Z^n)$ be the natural coordinate system in C^{n+1} such that C^{n-1} in Lemma 3.5 is given as the set $\{(Z^0, \cdots, Z^{n-2}, 0, 0) \in C^{n+1}\}$.

By the definition of Gauss map and by Lemma 3.5 the point $\tilde{\phi}(x)$ for $x \in M$ must be identified with a complex line given by the parallel transformation in C^{n+1} from a complex line orthogonal to $p(f(x))$. Thus $\phi(M)$ must be in the complex projective line CP^1 in CP^n which corresponds to the linear subspace $\{(0, \cdots, 0, Z^{n-1}, Z^n)\}$ in C^{n+1}. q.e.d.

It is almost obvious that (1.6) implies (1.4). To show that (1.5) implies (1.6), we start with the following lemma.

Lemma 4.1. *Under the conditions* (1.5) *in the theorem, we can find* C^∞-*distributions on* M, *say* D *and* $D^\perp$, *whose complex dimensions are* $n-1$ *and* 1, *respectively, and which satisfy the following*:

(4.1) D *and* $D^\perp$ *are of* C^∞ *and invariant by* J,

(4.2) D *and* $D^\perp$ *are parallel*,

(4.3) $D^\perp(x)$ *is orthogonal to* $D(x)$ *at each* x *in* M.

Proof. For convenience, let $CP^1 \subset CP^n$ be given as in the above Proposition 4.1, and let $\tilde{\phi}(M) \subset CP^1 \subset CP^n$.

Since f is an isometric immersion, for any x in M, TM_x contains a complex $(n-1)$-dimensional plane whose image by f_* is parallel to C^{n-1} in C^{n+1}.

Define $D(x)$ to be the $(n-1)$-plane in TM_x, and $D^\perp(x)$ to be the plane in TM_x orthogonal to $D(x)$ at each x. Then (4.1) and (4.3) are clear by the definition of D and $D^\perp$. To show (4.2) for any $X \in TM$ and Y in D, we have

$$\tilde{\nabla}_{f_*(X)} f_*(Y) = f_*(\nabla_X Y) + \alpha(X, Y) .$$

By the definition of D, we have $\tilde{\nabla}_{f_*(X)} f_*(Y) \subset f_*(D)$. Thus

$$\nabla_X Y = f_*^{-1}(f_*(\nabla_X Y)) = f_*^{-1}(\tilde{\nabla}_{f_*(X)} f_*(Y)) \subset D .$$

Hence D is parallel.

Since the parallel transformation preserves the Riemannian metric, $D^\perp$ is parallel. q.e.d.

Now applying the de Rham decomposition theorem [4], we have the local product structure. To extend it globally, let $(\tilde{M}, P)$ be the universal covering manifold of M with the projection P. Then we can put the canonical Kählerian structure induced from that of M in $\tilde{M}$.

Define distributions $\tilde{D}$ and $\tilde{D}^\perp$ on $\tilde{M}$ as follows: $\tilde{D}(\tilde{x})$ is the subspace of $T\tilde{M}_{\tilde{x}}$ which is mapped isometrically to $D(P(\tilde{x}))$ in $TM_{P(\tilde{x})}$ by P_*, and $\tilde{D}^\perp(\tilde{x})$ is the orthogonal complement of $\tilde{D}(\tilde{x})$ in $T\tilde{M}_{\tilde{x}}$.

Since P is an isometric immersion, $P_*(\tilde{D}^\perp(\tilde{x})) = D^\perp(x)$, and therefore

(4.4) $\tilde{D}$ and $\tilde{D}^\perp$ are of C^∞ and invariant by the complex structure in $\tilde{M}$,

(4.5) $\tilde{D}$ and $\tilde{D}^\perp$ are parallel,

(4.6) $\tilde{D}^\perp(\tilde{x})$ is orthogonal to $\tilde{D}(\tilde{x})$ at $\tilde{x}$.

Hence by the de Rham decomposition theorem for Kählerian manifolds [4, Vol. II], we have an $(n-1)$-dimensional Kählerian manifold $\tilde{M}_1^{n-1}$ and a 1-dimensional Kählerian manifold $\tilde{M}_2^1$ such that there exists a holomorphic isometry $\tilde{q}\colon \tilde{M}_1^{n-1} \times \tilde{M}_2^1 \to M$ mapping each $(\tilde{M}_1^{n-1}, \tilde{x}_2)$, for $\tilde{x}_2 \in \tilde{M}_2^1$, to the leaf of $\tilde{D}$ passing through $(\tilde{x}_1, \tilde{x}_2)$, $\tilde{x}_1 \in \tilde{M}_1^{n-1}$, holomorphically and isometrically.

When we consider $\tilde{M}$ as a submanifold of C^{n+1} immersed by $f \circ p$, we will easily see that each leaf of the foliation by $\tilde{D}$ is totally geodesic in C^{n+1} as well as in M and $\tilde{M}$. Completeness of the leaves is also obtained from completeness of $\tilde{M}$ by the same argument as in [4], once we know that the leaves are totally geodesic. Thus $f \circ p \circ \tilde{q} \,|\, (\tilde{M}_1^{n-1}, \tilde{x})$, i.e., the restriction of $f \circ p \circ \tilde{q}$ to $(\tilde{M}_1^{n-1}, \tilde{x}_2)$, maps $(\tilde{M}_1^{n-1}, \tilde{x}_2)$ holomorphically and isometrically onto a complex $(n-1)$-dimensional plane which is parallel to C^{n-1} in C^{n+1}.

Since $(\tilde{x}_1, \tilde{M}_2^1)$, for $\tilde{x}_1$ in $\tilde{M}_1^{n-1}$, is orthogonal to $(\tilde{M}_1^{n-1}, \tilde{x}_2)$ at $(\tilde{x}_1, \tilde{x}_2)$ in $\tilde{M}_1^{n-1} \times \tilde{M}_2^1$, we also know that $(\tilde{x}_1, \tilde{M}_2^1)$ is mapped by $f \circ p \circ \tilde{q}$ into the complex 2-dimensional plane orthogonal to C^{n-1}; this is the product structure for $\tilde{M}$.

It is not difficult to derive the product structure of M^n from that of $\tilde{M}^n$ which is given above. q.e.d.

Remark. In the real case [3], the condition that the rank of the Gauss map be ≤ 1 is equivalent to that the manifold be flat. However, in the complex case, our condition (1.4) does not imply that M^n is flat. To be more precise, if M^n is a flat Kählerian hypersurface of C^{n+1}, then M^n is a C^n in C^{n+1}.

For higher codimension, the result corresponding to our theorem in this paper can also be obtained, and the proof is a slight modification of the one given here.

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