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SOME METRIC PROPERTIES OF ARITHMETIC QUOTIENTS OF SYMMETRIC SPACES AND AN EXTENSION THEOREM

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This paper has two main objectives. One is to prove:

Theorem A. *Let D be the open unit disc $|z| < 1$ in $\boldsymbol{C}$, and $D^* = D - \{0\}$. Let X be a bounded symmetric domain, and Γ an arithmetically defined torsion-free group of automorphisms of X. Let V^* be the complex analytic compactification of $V = X/\Gamma$ constructed in* [3], a *and* b *positive integers, and $f\colon D^{*a} \times D^b \to V$ a holomorphic map. Then f extends to a holomorphic map of D^{a+b} into V^*.*

In fact, a slightly more general result will be obtained (see Thm. 3.7). Together with some known facts, this implies that if S is an algebraic variety, $h\colon S \to V$ is a holomorphic map, and V is endowed with its natural structure of quasi-projective variety defined in [3, Thm. 3,10], then h is a morphism of algebraic varieties.

The proof of Theorem A makes use of an extension theorem of M. H. Kwack [12], or rather of a slight variant of it [9], and the main point is to check that its assumptions are satisfied in our case. Let d_0 be the Kobayashi invariant pseudo-distance [10] on X; since X is a bounded symmetric domain, it is a distance (cf. § 3.3). Let d_0' be the associated distance on V defined by

$$(1) \qquad d_0'(\pi(x), \pi(y)) = \inf_{\gamma \in \Gamma} d_0(x, y \cdot \gamma) , \qquad (x, y \in X) ,$$

where $\pi\colon X \to V$ is the canonical projection. In view of some distance decreasing properties of f, we have essentially to prove the following result (where Γ may have torsion):

Theorem B. *Let $p, q \in V^* - V$, and let p_n, q_n $(n = 1, 2, \cdots)$ be sequences of points in V converging to p and q respectively. If $d_0'(p_n, q_n) \to 0$, then $p = q$.*

Theorem B will be derived in § 3.5 from properties of Siegel sets and arithmetic groups, whose discussion is the other purpose of this paper. Since they have some independent interest, they will be proved in greater generality and in a stronger form than is needed in § 3.5. Let then Γ be an arithmetic subgroup of a connected semi-simple $\boldsymbol{Q}$-group $\mathscr{G}$, X the symmetric space of maximal compact subgroups of the group G of real points of $\mathscr{G}$, and d_X the

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distance function associated to a G-invariant Riemannian metric on X. Our main result, Theorem 2.3, is:

Theorem C. *Let $\mathfrak{S}$ be a Siegel set in X, and C a finite subset of $\mathcal{G}_{\boldsymbol{Q}}$. Then there exists a constant δ such that $d_X(x \cdot c, x' \cdot c' \cdot \gamma) \geq d_X(x, x') + \delta$ for all $x, x' \in \mathfrak{S}$, $c, c' \in C$ and $\gamma \in \Gamma$.*

If $\pi \colon X \to X/\Gamma$ is the canonical projection, and d' the associated distance function on X/Γ, defined as in (1) above, Theorem C asserts in particular that the difference $d_X(x, x') - d'(\pi(x), \pi(x'))$ is bounded when x and x' vary through $\mathfrak{S}$.

In the case where $G = \boldsymbol{SL}(n, \boldsymbol{R})$, $\Gamma = \boldsymbol{SL}(n, \boldsymbol{Z})$, $C = \{e\}$, and x is fixed, Theorem C reduces to [16, Thm. 4]. The fact that x is also allowed to vary gives a positive answer in general to a question raised in that case at the end of [16, § 4].

Theorems A and 2.5 were proved in 1968, and Theorem A is stated in P. Griffiths' report [7, Thm. 6.6]. Since then, results closely related to Theorems A and B have been proved by P. Kiernan [9] and S. Kobayashi–S. Ochiai [11]. They have influenced the presentation given here, in particular by focussing attention on Theorem B, which had been essentially proved, but not made explicit, originally. The relations between these results are discussed in § 3.9.

Notation. In general, we use that of [5], [6], with one main exception: algebraic groups, which are always defined over $\boldsymbol{R}$ in this paper, are denoted by script letters $\mathcal{G}, \mathcal{H}, \cdots$, while the corresponding Roman capitals $G, H, \cdots$ stand for the group of real points of $\mathcal{G}, \mathcal{H}, \cdots$.

Let X be a differentiable manifold. The tangent space to X at x is denoted $T_x(X)$. Let Y be a differentiable manifold, $\mu \colon X \to Y$ an isomorphism, and g a Riemannian metric on Y. Then $\mu^*(g)$ denotes the induced Riemannian metric on X, i.e.,

$$\mu^*(g)(A, B) = g(d\mu_x(A), d\mu_x(B)) \,, \qquad (A, B \in T_x(X)\,;\, x \in X) \,.$$

If u, v are complex valued functions on a set S and $|u - v|$ is bounded on S, we write $u \approx v$. Assume u and v to have real positive values. We write $u \succ v$ if there exists a constant $c > 0$ such that $u(s) \geq c \cdot v(s)$ for all $s \in S$, and $u \prec v$ (resp. $u \asymp v$) if $v \succ u$ (resp. $u \prec v$ and $v \prec u$).

If a, b are elements of a group H, then ${}^a b$ stands for $a \cdot b \cdot a^{-1}$. The value of a rational character α of an algebraic group $\mathcal{G}$ on an element $x \in \mathcal{G}$ is denoted $\alpha(x)$ or x^α.

Throughout this paper, $\mathcal{G}$ is a connected semi-simple $\boldsymbol{R}$-group, X the symmetric space of maximal compact subgroups of G, and K a maximal compact subgroup of G.

1. Right invariant metrics

1.1. Let B be the Killing form on the Lie algebra $L(G)$ of G, θ be the Cartan involution of $L(G)$ with respect to $L(K)$, and

$$(1) \qquad g_0(X, Y) = -B(X, \theta(Y)) , \qquad (X, Y \in L(G)) .$$

Then g_0 is a positive nondegenerate scalar product on $L(G)$, invariant under Ad K. Let dg^2 be the right invariant metric on G which is equal to g_0 on $L(G)$. Let d or d_G be the associated distance function on G, and

$$(2) \qquad |x| = d(e, x) , \qquad (x \in G) .$$

Since d is symmetric, right invariant under G, left invariant under K, and satisfies the triangular inequality, we have

$$(3) \qquad d(x, y) = |y \cdot x^{-1}| ,$$

$$(4) \qquad |x| = |x^{-1}| , \qquad |x \cdot y| \leq |x| + |y| , \qquad (x, y \in G) ,$$

and also, since $x = x \cdot y \cdot y^{-1}$,

$$(5) \qquad |x| \leq |x \cdot y| + |y| , \qquad (x, y \in G) ,$$

from which it follows immediately that if C is a compact subset of G, then

$$(6) \qquad d(u \cdot x, v \cdot y) \approx d(x, y) , \qquad |u \cdot x \cdot v| \approx |x| ,$$

as x, y vary in G and u, v in C.

1.2. Let $\mathfrak{p}$ be the orthogonal complement to $L(K)$ with respect to the Killing form (or to g_0). We have then

$$\theta(A + B) = A - B \qquad (A \in L(K), B \in \mathfrak{p}) .$$

Let $\sigma \colon G \to X = K \backslash G$ be the canonical projection, and $o = \sigma(K)$. For $C \in L(G)$ we write $o \cdot C$ for $d\sigma_e(C)$. The map $C \mapsto o \cdot C$ induces an isomorphism of $\mathfrak{p}$ onto $T_0(X)$, whence a scalar product on $T_0(X)$, defined by the restriction of g_0 to $\mathfrak{p}$, to be denoted also by g_0. For $Z \in L(G)$, we have

$$(1) \quad Z = Z_k + Z_p \qquad (Z_k = (Z + \theta(Z))/2 \in L(K) ; \; Z_p = (Z - \theta(Z))/2 \in \mathfrak{p}) ,$$

$$(2) \qquad g_0(o \cdot Z, o \cdot Z) = g_0(Z_p, Z_p) = B(Z_p, Z_p) = B(Z, Z_p) .$$

Let dx^2 be the G-invariant Riemannian metric on X which is equal to g_0 on $T_0(X)$, and d_X the associated distance function. It is elementary that $d_X(x, y) = d_G(\sigma^{-1}(x), \sigma^{-1}(y))(x, y \in X)$, whence

$$(3) \qquad d_G(x, y) \geq d_X(o \cdot x, o \cdot y) ,$$

$$(4) \qquad d_X(o\cdot x, o\cdot y) \approx d_G(x, y) \qquad (x, y \in G)\ .$$

1.3. By definition, a parabolic subgroup P of G is the group of real points of a parabolic $\boldsymbol{R}$-subgroup $\mathscr{P}$ of $\mathscr{G}$. Let L be a subfield of $\boldsymbol{R}$, and assume $\mathscr{G}$ and $\mathscr{P}$ to be defined over L. Let $\mathscr{S}$ be a maximal L-split torus of the radical of $\mathscr{P}$, and A the connected component of e in S, in the ordinary topology. After conjugation of K by some element of G we may (and shall) assume that $L(A) \subset \mathfrak{p}$.

The exponential map $\exp\colon L(A) \to A$ is an isomorphism of Lie groups, which carries onto one another the invariant metrics defined by g_0 onto $L(A)$ and A. In particular, if Q is a set of rational characters of $\mathscr{S}$ which form a basis of $X(\mathscr{S}) \otimes \boldsymbol{Q}$, then the invariant metric on A may be written

$$(1) \qquad da^2 = \textstyle\sum_{\alpha,\beta\in Q} c_{\alpha\beta}\alpha^{-1}\beta^{-1}d\alpha d\beta\ ,$$

where the $c_{\alpha\beta}$ are constants such that

$$(2) \qquad g_0 = \textstyle\sum c_{\alpha\beta} d\alpha d\beta\ .$$

Let d_A be the distance function on A associated to da^2. In view of (1) there exists a constant $c > 0$ such that

$$(3) \quad c^{-1}\cdot d_A(a, b) \le (\textstyle\sum_{\alpha\in Q} \ln^2(\alpha(a)/\alpha(b)))^{1/2} \le c\cdot d_A(a, b)\ , \qquad (a, b \in A)\ .$$

Moreover, A is a totally geodesic submanifold of G, hence

$$(4) \qquad d_A(a, b) = d_G(a, b) \qquad (a, b \in A)\ .$$

The group $\mathscr{P}$ is semi-direct product of its unipotent radical $\mathscr{U}$ by the centralizer $\mathscr{Z}(\mathscr{S})$ of $\mathscr{S}$. Let $\mathscr{M}$ be the intersections of the characters χ^2, where χ runs through $X(\mathscr{Z}(\mathscr{S}))$. Then $\mathscr{Z}(\mathscr{S}) = \mathscr{M}\cdot\mathscr{S}$, the intersection $\mathscr{M} \cap \mathscr{S}$ is finite, and $Z(S)$ is the direct product of M and A.

1.4. Lemma. (i) *The Lie algebra $L(M)$ of M is stable under θ and orthogonal to $L(A)$ with respect to B and g_0.* (ii) *We have*

$$(1) \qquad g_0(C, C) = (1/2)g_0(o\cdot C, o\cdot C)\ , \qquad (C \in L(U))\ .$$

Let Φ be the set of roots of $\mathscr{G}$ with respect to $\mathscr{S}$. There exists an ordering on $X(\mathscr{S})$ such that the weights of $\mathscr{S}$ in $L(\mathscr{U})$ are the positive elements of Φ [6, § 3]. The restrictions to $L(A)$ of the differentials of the roots are the roots of $L(G)$ with respect to $L(A)$, in the sense of the theory of Lie algebras. For $\alpha \in \Phi$, let

$$(2) \qquad \mathfrak{g}_\alpha = \{C \in L(G) \mid [X, C] = d\alpha(X)\cdot C, X \in L(A)\}\ .$$

We have the decompositions

$$(3) \qquad L(G) = L(Z(S)) \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \ , \qquad L(U) = \sum_{\alpha > 0} \mathfrak{g}_\alpha \ .$$

Let us put $\mathfrak{g}_0 = L(Z(S))$. We have then

$$(4) \qquad B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \ , \qquad (\alpha, \beta \in \Phi \cup \{0\}; \alpha + \beta \neq 0) \ ,$$

which implies that the restriction of B to $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} (\alpha \in \Phi \cup \{0\})$ is nondegenerate. Let $C \in L(M)$. By definition of M, the trace of $\operatorname{ad} C$ in $\mathfrak{g}_\alpha (\alpha \in \Phi)$ is zero, whence $B(L(M), L(A)) = 0$. Since the restrictions of B to $L(A)$ and $L(\mathscr{Z}(S))$ are nondegenerate, it follows that $L(M)$ is the orthogonal complement of $L(A)$ in $\mathfrak{g}_0$, hence $L(M)$ is stable under θ.

The automorphism θ is $-Id.$ on $L(A)$, hence

$$(5) \qquad \theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha} \qquad (\alpha \in \Phi \cup \{0\}) \ ,$$

and consequently, using (4),

$$(6) \qquad B(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}, \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}) = g_0(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}, \mathfrak{g}_\beta + \mathfrak{g}_{-\beta}) = 0 \qquad (\alpha, \beta > 0; \alpha \neq \beta) \ ,$$

$$(7) \qquad B(\mathfrak{g}_0, L(U)) = g_0(\mathfrak{g}_0, L(U)) = 0 \ .$$

Let now $C = \sum C_\alpha \, (C \in \mathfrak{g}_\alpha, \alpha > 0)$ be an element of $L(U)$. The C_α are mutually orthogonal, and so are the $C_{\alpha,p} = (1/2)(C_\alpha - \theta(C_\alpha)) \in \mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}$ by (6). To prove (ii), it suffices therefore to consider the case where $C = C_\alpha$ for some $\alpha > 0$. We have then, by (3) and § 1.2 (2):

$$g_0(o \cdot C, o \cdot C) = g_0(C, C_p) = (1/2) g_0(C, C - \theta(C)) \ ,$$
$$g_0(o \cdot C, o \cdot C) = (1/2) g_0(C, C) + (1/2) B(C, C) = (1/2) \cdot g_0(C, C) \ ,$$

which proves (ii).

1.5. We already noticed that $P = M \cdot A \cdot U$. More precisely, the map $A \times M \times U \to P$ defined by the product is an isomorphism of analytic manifolds. For $p \in P$, we shall denote by $a(p), m(p), u(p)$ the elements of A, M, U such that $p = a(p) \cdot m(p) \cdot u(p)$.

It is known that $G = K \cdot P = K \cdot M \cdot A \cdot U$. If

$$x = k \cdot a \cdot m \cdot u \ , \qquad (x \in G, k \in K, a \in A, m \in M, u \in U) \ ,$$

then a and u are uniquely determined by x, and are analytic functions of x. They will often be denoted $a(x)$ and $u(x)$. The elements k and m are determined up to the product by an element of $K \cap M$. The group $K \cap M$ is maximal compact in M or P. Let $Z = (K \cap M) \backslash M = (K \cap M^0) \backslash M^0$, and let $\tau \colon M \to Z$ be the canonical projection. It is known that the map $(a, m, u) \mapsto o \cdot a \cdot m \cdot u$ $(a \in A, m \in M, u \in U)$ induces an isomorphism of analytic manifolds

$$(1) \qquad \mu\colon Y = A \times Z \times U \cong X .$$

The group P operates on Y by

$$(2) \quad (a, z, u)\cdot p = (a\cdot a(p), z\cdot m(p), a(p)^{-1}\cdot m(p)^{-1}\cdot u\cdot m(p)\cdot a(p)\cdot u(p)) ,$$

and we have

$$(3) \qquad \mu(y\cdot p) = \mu(y)\cdot p \qquad (y \in Y;\ p \in P) .$$

Let us identify $L(M)_{\mathfrak{p}} = L(M) \cap \mathfrak{p}$ with the tangent space to Z at the origin, and let dz^2 be the M-invariant Riemannian metric defined at the origin by g_0. Let further du^2 be the right-invariant Riemannian metric on U which is equal to the restriction of g_0 on $L(U)$. If φ is an automorphism of U, then $g' = \varphi^*(du^2)$ is also right-invariant and we have

$$(4) \quad g'(C, C') = g_0(d\varphi_e(C\cdot u^{-1}))d\varphi_e(C'\cdot u^{-1})) , \qquad (u \in U; C, C' \in T_u(U)) .$$

1.6. Proposition. *We keep the previous notation. Let $dy^2 = \mu^*(dx^2)$.*

(i) *For any $z \in Z, u \in U$, the metric induced by dy^2 on $A \times \{z\} \times \{u\}$ is da^2.*

(ii) *Let $y = (a, z, u) \in Y$ and $m \in \tau^{-1}(z)$. The tangent spaces at y to the submanifolds $A \times \{z\} \times \{u\}, \{a\} \times Z \times \{u\}$ and $\{a\} \times \{z\} \times U$ are orthogonal, and we have*

$$(1) \qquad (dy^2)_y = (da^2)_a + (dz^2)_z + (1/2)\,((\mathrm{Int}\, am)^*(du^2))_u .$$

It is well-known that the map $C \mapsto o\cdot\exp C$ induces an isomorphism of $L(A)$ onto a closed and flat totally geodesic submanifold of X, whence (i).

For part (ii), let $C \in T_y(Y)$. Write it in the form

$$(2) \quad C = C_1\cdot a + C_2\cdot m + C_3\cdot u \qquad (C_1 \in L(A), C_2 \in L(M)_{\mathfrak{p}}, C_3 \in L(U)) .$$

It is clear that we have

$$(3) \qquad d\mu(C) = o\cdot(C_1 + C_2 + \mathrm{Ad}\, am(C_3))\cdot amu .$$

Let $\langle C, C\rangle$ denote the value of dy^2 on C, and let D be the projection of $C_1 + C_2 + \mathrm{Ad}\, am(C_3)$ in $\mathfrak{p}$. By (3) and § 1.2 (2), we have

$$(4) \qquad \langle C, C\rangle = g_0(D, D) .$$

Since A and M normalize U, we have $\mathrm{Ad}\, am(C_3) \in L(U)$. The elements C_1 and C_2 are in $\mathfrak{p}$. By Lemma 1.4 and § 1.4 (7), they are orthogonal to each other and to $L(U)$, which implies the first assertion of (ii). Moreover, by § 1.4 (1), we have

$$\langle C, C\rangle = g_0(C_1, C_1) + g_0(C_2, C_2) + \tfrac{1}{2} g_0(\mathrm{Ad}\, am(C_3), \mathrm{Ad}\, am(C_3)) ,$$

which, in view of § 1.5 (4), is just another way to write (1).

1.7. Corollary. *Let d_A, d_Z, d_X, d_Y be the distance functions associated to the Riemannian metrics da^2, dz^2, dx^2 and dy^2, and $y = (a, z, u), y' = (a', z', u')$ be two points of Y. Then*

$$(1) \qquad d_X(\mu(y), \mu(y')) \geq \max(d_A(a, a'), d_Z(z, z')) ,$$

$$(2) \qquad d_G(g, h) \geq d_A(a(g), a(h)) = d_G(a(g), a(h)) , \qquad (g, h \in G) .$$

(1) follows from 1.6 by an obvious computation. (2) is a consequence of (1) and § 1.2 (3), § 1.3 (4).

1.8. The space Z is the Riemannian direct product of the symmetric spaces of maximal compact subgroups of the simple noncompact factors of M by a flat space (which has strictly positive dimension if and only if the center of M is not compact). Let F be a direct factor of Z in this decomposition, and F' the remaining factor. Then d_Z majorizes its restrictions $d_F, d_{F'}$ to F and F', which are distance functions associated to invariant Riemannian metrics. Let $\nu: X \to F$ be the composition of $\mu^{-1}: X \to Y$ by the projections $Y \to Z \to F$. It follows from Proposition 1.6 that we have

$$(1) \qquad d_F(\nu(o \cdot x), \nu(o \cdot y)) \leq d_X(o \cdot x, o \cdot y)) \qquad (x, y \in G) .$$

2. Siegel sets and invariant distances

2.1. From now on, $\mathscr{G}$ is defined over $\boldsymbol{Q}$, and $\mathscr{P}$ is a minimal parabolic $\boldsymbol{Q}$-subgroup of $\mathscr{G}$. We keep the notation of § 1.5 (with $L = \boldsymbol{Q}$). Moreover, Φ is the set of roots of $\mathscr{G}$ with respect to $\mathscr{S}$, and Δ the set of simple roots for the ordering associated to $\mathscr{P}$ [5, § 11].

We recall that a Siegel set $\mathfrak{S}$ or $\mathfrak{S}_{t,\omega}$ (with respect to K, P, S, as will always be understood) is a set of the form $\mathfrak{S} = K \cdot A_t \cdot \omega$ where ω is a relatively compact subset of $M \cdot U$ and

$$A_t = \{a \in A \mid \alpha(a) \leq t, (a \in \Delta)\}$$

[5, § 12]. For $x \in \mathfrak{S}$, the decomposition of § 1.5 will sometimes be written

$$x = k_x \cdot a(x) \cdot m_x \qquad (k_x \in K, a_x \in A, m_x \in \omega) .$$

2.2. Lemma. *We keep the previous notation. The differences*

$$d_G(x \cdot u, x' \cdot u') - d_G(a(x) \cdot u, a(x') \cdot u') , \qquad d_G(x, x') - d_A(a(x), a(x'))$$

are bounded in absolute value as x, x' range through $\mathfrak{S}$ and u, u' through G.

We have

$$d_G(x \cdot u, x' \cdot u') = d_G(k_x \cdot {}^{a(x)}m_x \cdot a(x) \cdot u, k_{x'} \cdot {}^{a(x')}m_{x'} \cdot a(x') \cdot u') .$$

The elements $k_x, k_{x'}$ run through a compact set. By a fundamental property of Siegel sets [5, Lemma 12.2], so do ${}^{a(x)}m_x$ and ${}^{a(x')}m_{x'}$, hence (§ 1.1 (6), § 1.3 (4))

$$d_G(x\cdot u, x'\cdot u') \approx d_G(a(x)\cdot u, a(x')\cdot u') , \qquad (x, x' \in \mathfrak{S};\ u, u' \in G) ,$$
$$d_G(x, x') \approx d_A(a(x), a(x')) \qquad (x, x' \in \mathfrak{S}) .$$

By definition, a Siegel set in X is the projection $\sigma(\mathfrak{S}) = o\cdot\mathfrak{S}$ of a Siegel set $\mathfrak{S}$ in G. Hence Theorem 2.3 below is Theorem C of the introduction.

2.3. Theorem. *Let $\mathfrak{S}$ be a Siegel set in G (with respect to K, P, S), C a finite subset of $\mathscr{G}_{\boldsymbol{Q}}$, and Γ an arithmetic subgroup of G. Then there exists a constant δ such that*

$$(1) \qquad d_X(o\cdot x\cdot c, o\cdot x'\cdot c'\cdot\gamma) \geq d_X(x, x') + \delta ,$$

for all $x, x' \in \mathfrak{S}$, $c, c' \in C$ and $\gamma \in \Gamma$.

In view of Lemma 2.2, § 1.1 (3) and § 1.2 (4), our assertion is equivalent to the existence of a constant δ' such that

$$(2) \qquad |a(x')\cdot c'\cdot\gamma\cdot c^{-1}\cdot a(x)^{-1}| \geq |a(x')\cdot a(x)^{-1}| + \delta' ,$$

for all $x, x' \in \mathfrak{S}$, $c, c' \in C$, $\gamma \in \Gamma$.

Using the Bruhat decomposition in $G_{\boldsymbol{Q}}$, we can write

$$(3) \qquad c'\cdot\gamma\cdot c^{-1} = u\cdot w\cdot t\cdot v \qquad (u \in U_w, v \in U, w \in \mathscr{N}(S)_{\boldsymbol{Q}}, t \in S)$$

(see [5, § 11.4; 6, § 5]) where U_w is a certain subgroup of U, and w runs through a set of representatives of $\mathscr{N}(S)/\mathscr{Z}(S)$ in $\mathscr{N}(S)_{\boldsymbol{Q}}$, chosen once and for all. Let

$$(4) \qquad z = a(x')\cdot c'\cdot\gamma\cdot c^{-1}\cdot a(x)^{-1} , \qquad q = w^{-1}\cdot z .$$

We have $|z| \approx |q|$, (§ 1.1), and $|q| \geq |a(q)|$, (Corollary 1.7). Therefore (2) will be proved if we show the existence of a constant δ'' such that

$$(5) \qquad |a(q)| \geq |a(x')\cdot a(x)^{-1}| + \delta'' ,$$

for all $x, x' \in \mathfrak{S}$, $c, c' \in C$ and $\gamma \in \Gamma$.

We note first that $q = w^{-1}\cdot a(x')\cdot u\cdot w\cdot t\cdot v\cdot a(x)^{-1}$, whence

$$(6) \qquad q = {}^{w^{-1}\cdot a(x')}u\cdot({}^{w^{-1}}a(x'))\cdot t\cdot a(x)^{-1}\cdot {}^{a(x)}v .$$

For $\alpha \in \Delta$, let (π_α, V_α) be a strongly rational representation of G whose highest weight λ_α is orthogonal to $\Delta - \{\alpha\}$ (see [6, § 12]). Fix on $V_{\alpha,\boldsymbol{R}}$ a euclidean norm $\|\ \|$ invariant under K, and with respect to which S is represented by self-adjoint operators (see e.g. [5, § 9]). Let e_0 be a unit vector in the (unique) line stable under P. We have $\pi(g)e_0 = \pm e_0$ for $g \in M \cdot U$, whence

$$(7) \qquad \|\pi_\alpha(x)\cdot e_0\| = a(x)^{\lambda_\alpha} \qquad (x \in G) .$$

By construction of U_w, the element $w^{-1}\cdot a(x')\cdot u\cdot a(x')^{-1}\cdot w$ belongs to the unipotent radical U^- of the group P^- opposed to P and containing $\mathscr{Z}(S)$. Now, if $g \in U^-$, then $\pi_\alpha(g)\cdot e_0 = e_0$ modulo the sum of the eigenspaces of S corresponding to lower weights, i.e., modulo the orthogonal complement of $\boldsymbol{R}\cdot e_0$. Therefore

$$(8) \qquad \|\pi_\alpha(g)\cdot\lambda\cdot e_0\| \geq |\lambda| \qquad (\lambda \in \boldsymbol{R};\ g \in U^-) .$$

We have then, using (6) and (7):

$$(9) \qquad \begin{aligned} a(q)^{\lambda_\alpha} = \|\pi(q)\cdot e_0\| &\geq \|\pi_\alpha({}^{w^{-1}}a(x')\cdot t\cdot a(x)^{-1})\cdot e_0\| \\ &= a(x')^{w(\lambda_\alpha)}\cdot t^{\lambda_\alpha}\cdot a(x)^{-\lambda_\alpha} . \end{aligned}$$

There is a matrix realization of $\mathscr{G}$ over $\boldsymbol{Q}$ in which Γ is represented by integral matrices [5, Cor. 7.13]. The elements of $C\cdot\Gamma\cdot C \cup C^{-1}\cdot\Gamma\cdot C^{-1}$ are then rational matrices whose entries have bounded denominators. This implies that $t^{\lambda_\alpha} \succ 1$ (see the proof of Cor. 15.3 in [5]), whence

$$(10) \qquad a(q)^{\lambda_\alpha} \succ a(x')^{w(\lambda_\alpha)}\cdot a(x)^{-\lambda_\alpha} .$$

$w(\lambda_\alpha)$ is a weight of π_α, therefore [6, § 12]

$$w(\lambda_\alpha) = \lambda_\alpha - \sum_{\beta\in\Delta} c_\beta\cdot\beta \qquad (c_\beta \in \boldsymbol{Z}, c_\beta \geq 0) .$$

(10) can then be written

$$(11) \qquad a(q)^{\lambda_\alpha} \succ a(x')^{\lambda_\alpha}\cdot a(x)^{-\lambda_\alpha}\cdot a(x)^{-\Sigma c_\beta\beta} .$$

Since $c_\beta \geq 0$ and $a(x)^\beta \leq t$, the last factor is $\succ 1$, and we have proved the existence of a constant $\delta_1 > 0$ such that

$$(12) \qquad a(q)^{\lambda_\alpha} \geq \delta_1\cdot a(x')^{\lambda_\alpha}\cdot a(x)^{-\lambda_\alpha} ,$$

for all $x, x' \in \mathfrak{S}$, $c, c' \in C$, $\gamma \in \Gamma$ and $\alpha \in \Delta$, with q defined by (4).

For an element $a \in A$, let $n(a)$ be the positive square root of

$$(13) \qquad n(a)^2 = \sum_{\alpha\in\Delta} \ln^2 a^{\lambda_\alpha} .$$

In order to prove (5), it suffices, by § 1.3 (3), to show the existence of a constant δ_2 such that

$$(14) \qquad n(a(q)) - n(a(x')\cdot a(x)^{-1}) \geq \delta_2 ,$$

for all $x, x' \in \mathfrak{S}, c, c' \in C, \gamma \in \Gamma$. We have

$$\begin{aligned}(15) \qquad & n(a(q))^2 - n(a(x')\cdot a(x)^{-1})^2 \\ & = \sum_{\alpha \in \Delta} (\ln a(q)^{\lambda_\alpha} + \ln (a(x')^{\lambda_\alpha}\cdot a(x)^{-\lambda_\alpha}) \\ & \qquad \cdot (\ln a(q)^{\lambda_\alpha} - \ln (a(x')^{\lambda_\alpha}\cdot a(x)^{-\lambda_\alpha}))\ .\end{aligned}$$

If $n(a(q)) = n(a(x')\cdot a(x)^{-1}) = 0$, there is nothing to prove. If not, it is clear that

$$(\ln a(q)^{\lambda_\alpha} + \ln (a(x')^{\lambda_\alpha}\cdot a(x)^{-\lambda_\alpha})/(n(a(q)) + n(a(x')\cdot a(x)^{-1})$$

is ≤ 1 in absolute value. On the other hand, (12) implies that

$$\ln a(q)^{\lambda_\alpha} - \ln (a(x')^{\lambda_\alpha}\cdot a(x)^{-\lambda_\alpha}) = \ln \lambda_\alpha(a(q)\cdot a(x')^{-1}\cdot a(x))$$

is bounded from below. It follows then that $n(a(q)) - n(a(x')\cdot a(x)^{-1})$ is bounded from below, which proves (14), and ends the proof of the theorem.

2.4. Let $J \subset \Delta$. As usual, $\mathscr{P}_J$ denotes the standard parabolic subgroup generated by U and the centralizer of $\mathscr{S}_J$, where $\mathscr{S}_J$ is the identity component of $\bigcap_{\alpha \in J} \ker \alpha$. The group $\mathscr{P}_J$ is the semi-direct product over $\boldsymbol{Q}$ of its unipotent radical $\mathscr{U}_J$ by $\mathscr{Z}(\mathscr{S}_J)$. An element $g \in \mathscr{P}_J$ can be written uniquely as $g = r\cdot u$ ($r \in \mathscr{Z}(\mathscr{S}_J), u \in \mathscr{U}_J$). The element r will be called the *reductive part of* g.

A sequence of elements $x_n \in \mathfrak{S}$ is said to be *of type* J if $a(x_n)^\alpha$ converges for all $\alpha \in \Delta$ and if $\lim (a(x_n)^\alpha \neq 0$ if and only if $\alpha \in J$.

2.5. Theorem. *Let $J, J' \subset \Delta$. Let $\{x_n\}, \{x'_n\}$ $(n = 1, 2, \cdots)$ be sequences of elements in $\mathfrak{S}$, of types J and J' respectively. Assume there exist an element $c \in G_{\boldsymbol{Q}}$ and a sequence of elements $\gamma_n \in \Gamma$ such that $d_G(x_n, x'_n\cdot c\cdot\gamma_n)$ remains bounded as $n \to \infty$. Then*

$$(1) \qquad a(x_n)^\alpha \asymp a(x'_n)^\alpha\ , \qquad (\alpha \in \Delta, n \geq 1)\ ,$$

in particular $J = J'$, and there exists $n_0 \geq 1$ such that $c\cdot\gamma_n \in P_J$ for $n \geq n_0$. Moreover, the set of reductive parts of the elements $c\cdot\gamma_n (n \geq n_0)$ is finite.

By Theorem 2.3, $d_G(x_n x'_n)$ is bounded as $n \to \infty$. Then so is $d_A(a(x_n), a(x'_n))$ by Corollary 1.7. In view of § 1.3 (3), this implies that $\ln a(x_n)^\alpha\cdot a(x'_n)^{-\alpha}$ is bounded in absolute value for every $\alpha \in \Delta$, whence (1) and the equality $J = J'$.

We now revert to the proof of Theorem 2.3, let $x = x_n, x' = x'_n, \gamma = \gamma_n$ and write z_n, q_n, w_n instead of z, w, q. By assumption and Lemma 2.2, z_n is bounded; hence so are q_n and, by Corollary 1.7, $a(q_n)$. In view of (1), § 2.3 (11) yields

$$(2) \qquad a(x_n)^{-\Sigma c_\beta\cdot\beta} \prec 1\ , \qquad (n \geq 1)\ .$$

Assume that $c_\beta \neq 0$ for some $\beta \in \Delta$. By standard properties of weights [6, § 12.14, Prop. 12.16] we have then $c_\alpha \neq 0$. Since $a(x_n)^\beta \leq t$ for all n's and

$\beta \in \Delta$, the relation (2) then forces $\lim \alpha(x_n)^\alpha$ to be $\neq 0$, i.e., $\alpha \in J$. Otherwise said, if $\alpha \notin J$, then $w_n(\lambda_\alpha) = \lambda_\alpha$ for n big enough. But then we have $w_n \in P_J$, hence $c \cdot \gamma_n \in P_J$ for those values of n (see § 2.3 (3)), which proves the second assertion. Let now $n \geq n_0$. Then we can write

$$(3) \qquad c \cdot \gamma_n = r_n \cdot u_n \ , \qquad (r_n \in \mathscr{Z}(S_J), u_n \in U_J) \ .$$

There exists a matrix realization $\mathscr{G} \subset \boldsymbol{GL}(n, \boldsymbol{C})$ of $\mathscr{G}$ over $\boldsymbol{Q}$ such that Γ is represented by elements of $\boldsymbol{GL}(n, \boldsymbol{Z})$. Then $\{c \cdot \gamma_n\}$ consists of rational matrices with bounded denominators. Since the decomposition (3) is over $\boldsymbol{Q}$, there exists then also a rational number f such that $f \cdot r_n \in \boldsymbol{M}(n \cdot \boldsymbol{Z})$ for $n \geq n_0$. In particular the r_n's form a discrete set. In order to show that this set is finite, it therefore remains to show that $\{r_n\}_{n \geq n_0}$ is relatively compact.

Let A_J be the identity component of S_J, in the ordinary topology, and M_J the analogue of the group M in § 1.3 (for $P = P_J$). Then

$$Z(A_J) = Z(S_J) = M_J \times A_J \ , \qquad A = (M_J \cap A) \times A_J \ .$$

Write accordingly

$$\begin{aligned} a(x_n) &= a_{1n} \cdot a_{2n} \ , \quad a(x'_n) = a'_{1n}, a'_{2n} \ , \quad (a_{1n}, a'_{1n} \in M_J \, ; \, a_{2n}, a'_{2n} \in A_J) \ , \\ r_n &= r_{1n} \cdot r_{2n} \ , \qquad\qquad\qquad\qquad\qquad (r_{1n} \in M_J, r_{2n} \in A_J) \ . \end{aligned}$$

For $n \geq n_0$, we have $z_n = a(x'_n) \cdot r_n \cdot u_n \cdot a(x_n)^{-1} \in P_J$, and the reductive part of z_n is

$$a(x'_n) \cdot r_n \cdot a(x_n)^{-1} = (a'_{1n} \cdot r_{1n} \cdot a_{1n}^{-1}) \cdot (a'_{2n} \cdot a_{2n}^{-1} \cdot r_{2n}) \ .$$

z_n is bounded ($n \geq n_0$) hence so are its reductive part and the components

$$a'_{1n} \cdot r_{1n} \cdot a_{1n}^{-1} \in M_J \ , \qquad a'_{2n} \cdot a_{2n}^{-1} \cdot r_{2n} \in A_J$$

of the latter. Since any $\alpha \in J$ is trivial on A_J, we have

$$a(x_n)^\alpha = a_{1n}^\alpha \ , \quad a(x'_n)^\alpha = a'^\alpha_{1n} \quad (\alpha \in J, n \geq n_0) \ ,$$

hence $a_{1n}^\alpha, a'^\alpha_{1n} \asymp 1$ for $\alpha \in J$. But J is a set of coordinates on $M \cap A_J$; therefore a_{1n} and a'_{1n} are bounded, and then so is r_{1n}. The elements a_{1n} and a'_{1n} being bounded, we have

$$a_{2n}^\alpha \asymp a(x_n)^\alpha \ , \quad a'^\alpha_{2n} \asymp a(x'_n)^\alpha \ , \quad (\alpha \in \Delta, n \geq n_0) \ ,$$

and therefore, by (1),

$$a_{2n}^\alpha \asymp a'^\alpha_{2n} \qquad (\alpha \in \Delta, n \geq n_0) \ ,$$

which implies that $a'_{2n} \cdot a_{2n}^{-1}$ runs through a relatively compact subset of A_J. It follows then that r_{2n} is bounded, hence r_n is bounded, which ends the proof of the theorem.

Remark. In view of § 1.2 (4), Theorem 2.5 and its proof remain valid if $d_G(x_n, x'_n \cdot c \cdot \gamma_n)$ is replaced by $d_X(o \cdot x_n, o \cdot x'_n \cdot c \cdot \gamma_n)$. In the case where $G = \boldsymbol{SL}(n, \boldsymbol{R}), \Gamma = \boldsymbol{SL}(n, \boldsymbol{Z}), c = e$, this theorem, thus formulated, yields [16, Thm. 1, p. 19].

3. An extension theorem

From now on, and up to § 3.10 *inclusive, X is assumed to be a bounded symmetric domain.*

3.1. We recall that a complex analytic space E is *hyperbolic* if its Kobayashi pseudo-distance d_0 is a distance [10]. The open unit disc D and the punctured unit disc $D^* = D - \{0\}$ are hyperbolic [10, Chap. IV, § 4].

If E and E' are hyperbolic, then $E \times E'$ is so, and for $x, y \in E, x', y' \in E'$ we have

$$(1) \quad \max(d_0(x, y), d_0(x', y')) \leq d_0((x, x'), (y, y')) \leq d_0(x, y) + d_0(x', y')$$

[10, Chap. IV, Prop. 2.6 and § 4].

Let E be a complex manifold endowed with a hermitian metric whose holomorphic curvature is bounded from above by a strictly negative constant, and d be the associated distance. Then [10, Chap. IV, Thm. 4.11] there exists a constant $c > 0$ such that

$$(2) \qquad d(x, y) \leq c \cdot d_0(x, y) \ , \qquad (x, y \in E) \ .$$

In particular, E is hyperbolic, and d_0 is a complete metric if d is so. We recall that a holomorphic mapping always decreases d_0.

3.2. Assume now X to be irreducible. We claim that there exist constants $c, c' > 0$ such that

$$(1) \qquad c \cdot d_X(x, y) \leq d_0(x, y) \leq c' \cdot d_X(x, y) \ , \qquad (x, y \in X) \ .$$

Any two invariant Riemannian metrics on X are proportional, hence dx^2 is associated to a hermitian metric with holomorphic curvature bounded from above by a strictly negative constant, and § 3.1 (2) yields the first inequality in (1). X contains a totally geodesic polydisc $F = D^l$ (where l is the rank of X). Given $x, y \in X$ there exists an automorphism of X which brings x, y in F. In order to prove the second inequality in (2), we may therefore assume $x, y \in F$. The restriction of dx^2 to F is invariant under Aut F, hence majorizes a multiple of the Poincaré metric of F. But the inclusion map $F \to X$ is holomorphic, hence decreases the Kobayashi distance. Therefore it suffices to prove the second inequality of (1) in the case of a polydisc. But then it follows from [10, Example 1, p. 47].

3.3. Let $X = X_1 \times \cdots \times X_m$ be the canonical decomposition of X as a product of irreducible bounded symmetric domains. This decomposition is also a Riemannian product with respect to any invariant Riemannian metric. In particular if d_i is the restriction of d_X to X_i, then d_X is majorized by the sum of the d_i's. It follows further from § 3.1 (1) and § 3.2 (1) that there exist constants $c, c' > 0$ such that

$$(1) \qquad c \cdot \max_i (d_i(x_i, y_i)) \leq d_0(x, y) \leq c' \cdot \textstyle\sum_i d_i(x_i, y_i) \ ,$$

for all $x = (x_i), y = (y_i), (x_i, y_i \in X_i, 1 \leq i \leq s)$.

3.4. Assume now $\mathscr{G}$ to be simple over $\boldsymbol{Q}$. Let Γ be an arithmetic subgroup of $\mathscr{G}$, $V = X/\Gamma$, V^* be the complex analytic compactification of V constructed in [3], and $\pi: X \to V$ be the canonical projection. We shall now use the results and notation of [3], and recall here briefly those which are most pertinent for the sequel. The numbering of the elements of Δ will be the canoical one [3, § 1.2, Prop. 2.9].

V^* is by definition the quotient by Γ of a space X^* on which $G_{\boldsymbol{Q}}$ operates. X^* is the union of X, which is open in X^*, and of so-called rational boundary components. Those are the transforms under $G_{\boldsymbol{Q}}$ of the standard boundary components F_b $(1 \leq b \leq s = rk_{\boldsymbol{Q}}\mathscr{G})$. For each boundary component F there is a canonical holomorphic projection $\sigma_F: X \to F$, which is constructed as the map ν of § 1.8. In particular, § 1.8 (1) holds true for σ_F.

Let $\{x_n\}_{n \geq 1}$ be a sequence of elements in $\mathfrak{S}$, $d \in G_{\boldsymbol{Q}}$, and F be a rational boundary component. Then $x_n \cdot d$ tends to a point $u \in F$ in the topology of X^*, if b is the greatest index i such that $\lim a(x_n)^{\alpha_i} = 0$, $F_b \cdot d = F$ and $\sigma_F(o \cdot x_n \cdot d) \to u$.

Conversely, let $\{p_n\}$ be a sequence of elements of V which tends to an element $p \in V^* - V$. Let $b \in \{1, \cdots, s\}$ and $c \in G_{\boldsymbol{Q}}$ be such that $p \in \pi(F_b \cdot c)$, and $u \in F = F_b \cdot c$ be such that $\pi(u) = p$. Then, after having replaced $\{p_n\}$ by an infinite subsequence, we may assume that there exist x_n and d as before, with $\pi(x_n \cdot d) = p_n$ for all n's and $F_b \cdot d = F$.

This follows from the description of the topology of X^* or V^* in terms of "truncated Siegel sets" [3, § 4.12, Lemma 4.13], if we take into account the fact that $\sigma_{F_b}(o) = o_b$.

3.5. *Proof of Theorem* B. We first reduce the proof to the case where $\mathscr{G}$ is $\boldsymbol{Q}$-simple. We may assume $\mathscr{G}$ to be adjoint type [3, Lemma 11.5]. It is then the direct product over $\boldsymbol{Q}$ of $\boldsymbol{Q}$-simple groups $\mathscr{G}_i$ $(1 \leq i \leq m)$. The symmetric space X_i of maximal compact subgroups of G_i is a bounded symmetric domain, and X is (complex analytically) the product of the X_i's. Let $\Gamma_i = \Gamma \cap G_i$, and Γ' be the product of the Γ_i. It is a normal subgroup of Γ, which is arithmetic [5, § 7], hence of finite index in Γ. The canonical projection $V' = X/\Gamma' \to V$ extends to a holomorphic map $V'^* \to V^*$ with finite fibers, which sends boundary components onto boundary components. From this it is elementary that it suffices to prove Theorem B for V' and V'^*. But V' is the product of the quotients $V_i = X_i/\Gamma_i$, and then V'^* is the product of the complexifications V_i^* of the V_i's, whence the reduction to the $\boldsymbol{Q}$-simple case.

Let now p_n, q_n and p, q be as in the statement of Theorem B. Let F and F' be rational boundary components of X^* whose projections contain p and q respectively, and $u \in F, u' \in F'$ be such that $\pi(u) = p, \pi(u') = q$. We have to show the existence of $\gamma \in \Gamma$ such that $u' = u \cdot \gamma$. Let $b, b' \in \{1, \cdots, s\}$ and $c, c' \in G_{\boldsymbol{Q}}$ be such that $F = F_b \cdot c, F' = F_{b'} \cdot c'$. In view of § 3.4 we may assume (after having taken subsequences and renumbered), the existence of elements $x_n, x'_n \in \mathfrak{S}$ and $d, d' \in G_{\boldsymbol{Q}}$ such that b (resp. b') is the greatest index i for which $a(x_n)^{\alpha_i}$ (resp. $a(x'_n)^{\alpha_i}$) tends to zero, $F_b \cdot d = F, F_{b'} \cdot d' = F'$ and $\nu_F(o \cdot x_n \cdot d) \to u$ (resp. $\nu_{F'}(o \cdot x'_n \cdot d') \to u'$).

Let d'_V be the distance function on X/Γ defined by

$$d'_V(\pi(x), \pi(y)) = \operatorname*{Inf}_{\gamma \in \Gamma} d_X(x, y \cdot \gamma) , \qquad (x, y \in X) .$$

By § 3.3 and the assumption of Theorem B, $d'_V(p_n, q_n) \to 0$; therefore we can find $\gamma_n \in \Gamma$ such that

$$d'_X(o \cdot x_n \cdot d, o \cdot x'_n \cdot d' \cdot \gamma_n) \to 0 .$$

By Theorem 2.5, we have then

$$a(x_n)^\alpha \asymp a(x'_n)^\alpha \qquad (\alpha \in \Delta) ,$$

which implies in particular that $b = b'$. Let $J = \Delta - \{b\}$. The group P_J is the normalizer of F_b. After having gone over to subsequences, we may assume $a(x_n)^\alpha$ and $a(x'_n)^\alpha$ to converge for all $\alpha \in \Delta$; x_n and x'_n are then of type J' for some $J' \supset J$; by Theorem 2.5, there exists then n_0 such that

$$(1) \qquad d' \cdot \gamma_n \cdot d^{-1} \in P_J \qquad (n \geq n_0) .$$

We have therefore

$$(2) \qquad F' \cdot \gamma_n = F_b \cdot d' \cdot \gamma_n \cdot d^{-1} \cdot d = F_b \cdot d = F , \qquad (n \geq n_0) ,$$

whence also

$$(3) \qquad \gamma_{n_0}^{-1} \cdot \gamma_n \in d^{-1} \cdot P_J \cdot d = \mathcal{N}(F) \qquad (n \geq n_0) .$$

By construction

$$(4) \qquad \lim \sigma_F(o \cdot x_n \cdot d) = u , \qquad \lim \sigma_{F'}(o \cdot x'_n \cdot d') = u' .$$

Since $F' \cdot \gamma_{n_0} = F$, the second relation implies

$$(5) \qquad \lim \sigma_F(o \cdot x'_n \cdot d' \cdot \gamma_{n_0}) = u' \cdot \gamma_{n_0} .$$

As pointed out at the end of § 3.4, σ_F is distance decreasing, F being endowed with a suitable invariant distance function d_F. We have then

$$(6) \qquad \lim d_F(\sigma_F(o\cdot x_n\cdot d), \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_n)) = 0 \ .$$

Together with (4), this yields

$$(7) \qquad \lim \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_n) = u \ .$$

We have $o\cdot x'_n\cdot d\cdot\gamma_n \in F$ and

$$o\cdot x'_n\cdot d\cdot\gamma_n = o\cdot x'_n\cdot d'\cdot\gamma_{n_0}\cdot\gamma_{n_0}^{-1}\cdot\gamma_n \ .$$

The map ν commutes with the action of $\mathscr{N}(F)$ on X and F. Since $\gamma_{n_0}^{-1}\cdot\gamma_n \in \mathscr{N}(F)$ we have

$$(8) \qquad \sigma_F(o\cdot x_n\cdot d'\cdot\gamma_n) = \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_{n_0})\cdot\gamma_{n_0}^{-1}\cdot\gamma_n \ .$$

Let $\mathscr{Z}(F) = \{g \in \mathscr{N}(F) \mid f\cdot g = f, (f \in F)\}$ be the centralizer of F, and $G(F) = \mathscr{N}(F)/\mathscr{Z}(F)$. Since F is a rational boundary component, the image of $\mathscr{N}(F) \cap \Gamma$ in $G(F)$ is a discrete subgroup (in fact of arithmetic type, see [3, Thm. 3.7]). In particular, it acts in a properly discontinuous manner on F. In view of (4) and (8), there exists then $\gamma \in \mathscr{N}(F) \cap \Gamma$ such that

$$(9) \qquad \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_{n_0})\cdot\gamma = \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_{n_0})\gamma_{n_0}^{-1}\cdot\gamma_n = \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_n)$$

for infinitely many n's. We have then, using (5) and (7):

$$u'\cdot\gamma_{n_0}\cdot\gamma = \lim \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_{n_0})\cdot\gamma = \lim \sigma_F(o\cdot x'_n\cdot d'\cdot\gamma_n) = u \ ,$$

hence $u \in u'\cdot\Gamma$ and $p = \pi(u) = \pi(u') = q$.

3.6. Let Z be a complex space. A holomorphic map $f\colon Z \to V$ is said to be *locally liftable* if for every $z \in Z$ there exist a neighborhood U_z of z in Z and a holomorphic map f_z of U_z into X such that the restriction of f to U_z is equal to $\pi\circ f_z$. If Γ is torsion-free, then it operates freely on X, the projection π is a covering map, and every holomorphic map of Z into V is locally liftable. Since a product of open discs is a hyperbolic space, the following theorem contains Theorem A of the introduction as a special case.

3.7. Theorem. *Let Z be a normal hyperbolic space, a a positive integer and $f\colon D^{*a} \times Z \to V$ a locally liftable holomorphic map. Then f extends to a holomorphic map of $D^a \times Z$ into V^*.*

The space $E = D^{*a} \times Z$ is a normal hyperbolic analytic space, since both factors are so. Let d_E be the hyperbolic metric on E. Then [10, Prop. 6.1, p. 104]

$$(1) \qquad d'_0(f(u), f(v)) \le d_E(u, v) \ , \qquad (u, v \in E) \ .$$

Since $D^a \times Z$ is normal, it suffices, by Riemann's extension theorem [1, Thm. 44.42, p. 420], to show that f extends to a continuous map of $D^a \times Z$ into V^*.

First let Z be reduced to a point. In view of (1) and § 3.5, our assertion is essentially contained in [9, Thm. 2]. For the sake of completenss, we sketch a proof. First let $a = 1$. Let C_n be a sequence of concentric circles in D^*, with center at the origin, whose radii r_n tend to zero. Then the hyperbolic length $L(C_n) = 2\pi(\ln(1/r_n))^{-1}$ (see [10, p. 81]) tends to zero. By (1), the length of $f(C_n)$ also tends to zero, and § 3.5 then shows that, for some infinite subsequence, the images $f(C_n)$ converge to a point of V^*. Hence M. H. Kwack's theorem [12, Thm. 3], [10, Thm. 3.1] obtains and yields our assertion.

Let $a \geq 2$, and $u = (u_i) \in D^a - D^{*a}$. Let r be a strictly positive number $< \max(|u_i|, 1 - |u_i|)$ if $u_i \neq 0$ $(1 \leq i \leq a)$. Let $\lambda = (\lambda_i)$ be a sequence of complex numbers of modulus one. Then $f_{u,\lambda}\colon z \mapsto (u_i + \lambda_i \cdot z)$ is a holomorphic embedding of $D_r^* = \{z \in \boldsymbol{C} \mid 0 < |z| < r\}$ into D^{*a}; by the above, $f \circ f_{u,\lambda}$ extends to a holomorphic mapping of $D_r = \{z \in \boldsymbol{C} \mid 0 \leq |z| < r\}$ into V^*. Let $v_{u,\lambda}$ be its value at the origin. In fact, $v_{u,\lambda}$ is independent of λ. This follows from Theorem 1 in [9], or also from § 3.1, § 3.5 and the fact that if $\lambda' = (\lambda_i')$ is another sequence of complex numbers of modulus one, and if $z_n \in D_r^*$ tends to 0, then the hyperbolic distance of $\lambda_i z_n$ and $\lambda_i' \cdot z_n$ also tends to zero. We then define an extension f' of f to D^a by putting $f'(u) = f_{u,\lambda}(0)$ for $u \in D^a - D^{*a}$. Let now z_n be a sequence of elements in D^{*a} which converges to u. By Theorem 1 of [9] applied to the sequence of maps f_{u,λ_n} where $\lambda_n = (z_{ni}/|z_{ni}|)$, we have

$$\lim f(z_n) = \lim f_{u,\lambda_n}(z_n) = \lim f_{u,\lambda_n}(0) = f'(u) ,$$

whence the continuity of f'.

Let now Z be not reduced to a point. For each $z \in Z$, there is a continuous extension $f_z\colon D^a \times \{z\} \to V^*$ of the restriction of f to $D^{*a} \times \{z\}$. Let then $f'\colon D^a \times Z \to V^*$ be the map whose restriction to $D^a \times \{z\}$ is equal to f_z for every $z \in Z$. It extends f, and there remains to show that it is continuous. To prove the continuity of f' it suffices to show that if $(y, z) \in D^a \times Z$ and (y_n, z_n) is a sequence of elements in E which converges to (y, z), then $f((y_n, z_n)) \to f_z(y, z)$. By § 3.1,

$$d_E((y_n, z_n), (y_n, z)) \leq d_Z(z_n, z) ,$$

hence the left hand side tends to zero. By (1) and § 3.5, it follows (after having taken subsequences convergent in V^*) that

$$\lim_{n\to\infty} f((y_n, z_n)) = \lim_{n\to\infty} f((y_n, z)) .$$

Since the right hand side is $f_z((y, z))$ by definition, this proves our contention.

3.8. Remark. If Γ has torsion, d_0' is not necessarily the hyperbolic distance on V, but majorizes it [10, p. 103]. Thus, strictly speaking, V is not necessarily hyperbolically imbedded in V^* in the sense of [9], [11], and the theorems of [9] do not apply directly to our situation. However, this is a harmless point: since we have the "hyperbolic embedding" condition for the modified pseudo-distance d_0', and the distance decreasing property § 3.7 (1), the arguments of [9] apply without change to our case (see [11] for similar remarks).

3.9. Let us denote by V^{**} the set V^* endowed with the topology defined by Piateckii-Sapiro [12], using "cylindrical sets" in realizations of X as "Siegel domains of the third kind". The identity map $\iota: V^* \to V^{**}$ is continuous [2]. Since V^* is compact and Hausdorff, ι is a homeomorphism if and only if V^{**} is Hausdorff. In [12], it is asserted that this follows from the results announced in [4], but the assertion is not really convincing to the author of the present paper.

Theorem B is obviously equivalent to the following assertion:

($*$) *Given $p \in V^* - V$ and a neighborhood U_1 of p in V^*, there exists a neighborhood U_2 of p in U_1 such that*

$$d_0'(U_2 \cap V, V - (U_1 \cap V)) > 0 .$$

This statement is proved in [11] for V^{**}. From this, the authors deduce Theorem 3.7 for $E = D^* \times D^b$ by using Theorem 6.1 of [10]. However, in the latter, all spaces under consideration are of course Hausdorff. Similarly [9] proves Theorem A for V^{**}, using ($*$) above, and Theorem 1 of [9], where again the spaces are Hausdorff. It is also pointed out there that the proof of Theorem 2 is also valid for V^*, provided ($*$) holds true in V^*.

All these distinctions will be happily superfluous and the situation more satisfactory once it is shown that V^{**} is Hausdorff. The author believes it follows from his joint work with *J–P*. Serre on corners, but prefers not to commit himself firmly on this point until everything is fully written up. The proof of ($*$) in [11], unlike the one of Theorem B here, does not involve any reduction theory; this is maybe an indication that indeed some rather strong results in reduction theory are hidden behind the equivalence of the two topologies.

In the following theorem, which was pointed out by P. Deligne, V is endowed with its canonical structure of quasi-projective variety [3, § 10].

3.10. Theorem. *Let S be a complex algebraic variety, f a holomorphic map of S, viewed as an analytic space, into V. Then f is a morphism of algebraic varieties.*

This assertion is local in the Zariski topology of S, so we may assume S to be affine, and in particular to be quasi-projective. Furthermore, it suffices to show that the restriction of f to an open Zariski-dense subset is algebraic. So we may assume S to be smooth. By Hironaka's desingularization theorem [8], S may be embedded as a Zariski-open subset in a smooth projective variety $\bar{S}$

in such a way that $\bar{S} - S$ consists of smooth divisors with normal crossings; more precisely, around any point of $\bar{S} - S$ there are coordinates with respect to which S is of the form $D^{*a} \times D^b$ $(a + b = n)$. By Theorem A, f extends to a holomorphic map f' of $\bar{S}$ into V^*. Since $\bar{S}$ and V^* are projective varieties, the map f' is a morphism of algebraic varieties by Chow's theorem [13, Prop. 15], whence the theorem.

3.11. Let us now drop the assumption that X carries an invariant complex structure. Theorem B is then also true for the embedding of $V = X/\Gamma$ into any Satake compactification [4, § 1.4]. The proof is essentially the same and relies on suitable analogues of the facts recalled in § 3.4. However, since there is no adequate reference for them in the literature, we shall not try to make this more precise.

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