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THE HOLONOMY ALGEBRA OF IMMERSED MANIFOLDS OF CODIMENSION TWO

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1. Introduction

In [4] S. Kobayashi has proved that the holonomy algebra of a compact riemannian manifold immersed in euclidean space of one dimension greater is the whole orthogonal algebra. The purpose of this paper is to generalize this result to the case of codimention two and, to a certain extent, the non-compact case. Our technique gives a simple proof of Kobayashi's theorem. The extensions to codimension two are as follows.

Theorem 1. *Let M be a riemannian manifold of dimension D isometrically immersed in a euclidean space R^{D+2} of dimension $D+2$. Then there are the following possibilities for $r(m)$, the Lie algebra generated by the curvature transformations at a point $m \in M$:*

(a) *The relative curvature space $k(m)$ at m decomposes into an orthogonal direct sum, $k(m) = V + W$, and $r(m) = o(V) + o(W)$, the direct sum of the orthogonal algebras based on V and W. (V or W may be of dimension zero or one, so that $r(m)$ is itself an orthogonal algebra.) Or:*

(b) *There is a complex structure on $k(m)$, $r(m)$ is the unitary algebra of that structure, and the second fundamental forms are all of signature zero, unless $\dim k(m) = 4$.*

In particular, if there is a point m at which some second fundamental form is nondegenerate, that is, $k(m) = M_m$, then the global holonomy algebra h_m at m is one of the possibilities listed.

Theorem 2. *Let M be a compact riemannian manifold of dimension $D \neq 4$ isometrically immersed in R^{D+2}. Then $h_m = o(V) + o(W)$, where $M_m = V + W$ is an orthogonal direct sum.*

The results in this paper are algebraic, except for a minor point used in passing from Theorem 1 to Theorem 2. They have been used as a starting point by Stephanie B. Alexander [1] to show that under the hypothesis of Theorem 2 the immersion is usually the product of two hypersurface immersions.

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2. Notation

We assume throughout that the structures we deal with are C^∞. The tangent space of a manifold M at m will be denoted M_m. The riemannian inner product is $\langle\, ,\rangle$. Euclidean space of dimension F with the ordinary flat riemannian structure is denoted R^F. The Grassmann product of vectors will be denoted by juxtaposition, without wedges, and, accordingly, the space of Grassmann bivectors over a vector space V will be denoted V^2. When V has an inner product, the orthogonal algebra of V, that is, the skew-symmetric endomorphisms of V, denoted $o(V)$, has a natural identification with V^2 induced by the inner product $\langle\, ,\rangle$, which can be expressed by the formula

$$xy(z) = \langle x, y\rangle y - \langle y, z\rangle x, \tag{1}$$

where $x, y, z \in V$. The corresponding formula for bracket in V^2 is

$$[xy, zw] = \langle x, z\rangle yw + \langle y, w\rangle xz - \langle x, w\rangle yz - \langle y, z\rangle xw. \tag{2}$$

3. Curvature transformations of immersed manifolds

Let M be a connected D-dimensional riemannian manifold isometrically immersed in R^{D+E}. Then for a choice of orthonormal basis and normal basis at $m \in M$ the classical formulas for the curvature tensor are

$$R_{hijk} = \sum_a (S_{ahj}S_{aik} - S_{ahk}S_{aij}), \tag{3}$$

where S_{ahi} are the components of the second fundamental forms, $h, i, j, k = 1, \cdots, D$, $a = 1, \cdots, E$. The second fundamental forms may be interpreted as E symmetric endomorphisms of M_m, $S_1, \cdots, S_E$. If the normal basis is changed, then the second fundamental forms undergo a corresponding (linear orthogonal) change.

We may interpret the curvature tensor as a symmetric linear mapping of M_m^2 into itself, in which case the expression in terms of the second fundamental forms is quite simple (see [3, p, 195]):

$$R(xy) = \sum_a (S_a x)(S_a y). \tag{4}$$

For a symmetric endomorphism S we denote $D(S)$ the subspace which is simultaneously the range of S, the orthogonal complement of the kernel of S, and the span of the nonnullity eigenvectors of S. The *relative curvature subspace* $k(m)$ at m is $D(S_1) + D(S_2) + \cdots + D(S_E)$. By a proper choice of normal basis we can make $k(m) = D(S_1)$. The orthogonal complement of $k(m)$ is annihilated by all second fundamental forms and is called the *relative nullity subspace*, denoted $\mathscr{R}(m)$. The dimension of $\mathscr{R}(m)$ is called the *index of relative nullity* $\nu(m)$ *at* m, so that ν is an integer-valued function on M.

4. The holonomy algebra

The group consisting of parallel translations of M_m around loops based at m is called the holonomy group at m. Since M is connected the holonomy groups at different points are isomorphic. The holonomy group at m is a Lie group and we shall denote its Lie algebra h_m. It is well known that h_m contains all of the curvature transformations at m, that is, the range of the mapping given by (4). In fact, by [2] and [5] h_m is spanned by the parallel translates to m of the curvature transformations at all of the points of M. Denote by $r(m)$ the Lie algebra generated by the curvature transformations at m; thus $r(m)$ is a subalgebra of h_m and it is clear that $r(m) \subset k(m)^2$ for every m.

5. Kobayashi's theorem

If M is a compact immersed hypersurface, that is, $E = 1$, then $h_m = o(M_m)$. It follows, from standard facts relating orientability of M to the existence of orientation reversing parallel translations, that the holonomy group of M is the orthogonal group if M is nonorientable, and is the special orthogonal group if M is orientable. To prove $h_m = M_m^2$, we note that if m is at maximal distance in R^{D+1} from the origin, then the second fundamental form S of the outward pointing normal at m is positive definite. Thus the curvature tensor $R = S \cdot S \cdot$ is nonsingular, so its range is all of M_m^2.

6. Generators of an orthogonal algebra

The range of the curvature mapping R is a subspace of $k(m)^2$. In this section we investigate some circumstances under which a subspace of $k(m)^2$ will generate all of $k(m)^2$. Abstracting, we suppose that U is a real vector space with a positive definite inner product $\langle\, , \rangle$ and that V and W are nonzero subspaces such that $U = V + W$. Then $U^2 = V^2 + VW + W^2$. If $V \cap W = 0$, then VW may be identified with the tensor product $V \otimes W$.

Lemma 1. *U^2 is generated by VW.*

Proof. Let $v_1, v_2 \in V$, $w \in W$ such that $\langle w, w\rangle = 1$. Then, by (2),

$$[v_1 w, v_2 w] = v_1 v_2 + \text{terms in } VW .$$

Thus V^2 is contained in the algebra generated by VW. Similarly W^2 is also in that algebra, hence all of U^2.

Remark. Lemma 1 applies, with the same proof, to the case of a complex vector space with a nondegenerate symmetric bilinear form in the case where neither V nor W is isotropic.

Lemma 2. *If $v \neq 0$, then vU generates U^2.*

Lemma 3. *If $v \in V$, $v \neq 0$, $w \in W$, $w \neq 0$, and $V \cap W \neq 0$, then $vV + wW$ generates U^2.*

Proof. By Lemma 2 the algebra generated by $vV + wW$ contains $V^2 + W^2$. Hence if $u \in V \cap W$, $u \neq 0$, it also contains $u(V + W) = uU$, which generates U^2.

Lemma 4. *If $v \in V$, $w \in W$, and $vw \neq 0$, then $vw + V^2 + W^2$ generates U^2.*

Proof. Let V_1 be spanned by V and w, W_1 by W and v. Then $vV_1 + wW_1 \subset$ span of $vw + V^2 + W^2$, so the result follows from Lemma 3.

Lemma 5. *If $V = W^\perp$ and not both V and W have dimension two, then $X = V^2 + W^2$ is a maximal proper subalgebra of U^2.*

Lemma 5 follows from a well known theorem about irreducible symmetric spaces.

Lemma 6. *If $V \cap W = 0$, $\dim V > 2$, $\dim W > 1$, and $V \neq W^\perp$, then $X = V^2 + W^2$ generates U^2.*

Proofs of Lemmas 5 and 6. We first show that X and any other element of U^2 generates U^2.

If $\dim V = 1$ with V spanned by v, then any element not in X is congruent to some $vw \bmod X$, where $w \in W$. Then $wU \subset$ span of $vw + X$, which generates U^2 by Lemma 2.

We have covered the cases where one of the two subspaces has dimension one, and in the other cases possible under the hypotheses of either lemma, one of the subspaces has dimension greater than two, so we now assume $\dim V > 2$. Then for an orthonormal basis $v_1, v_2, \cdots$ of V and any element $\sum v_i w_i$ not in X, where $w_i \in W$, we may assume $w_2 \neq 0$, in which case, by (2),

$$[v_1 v_2, \textstyle\sum v_i w_i] \equiv v_2 w_1 - v_1 w_2 \mod X,$$
$$[v_1 v_3, v_1 w_2 - v_2 w_1] \equiv v_3 w_2 \mod X.$$

So X and $\sum v_i w_i$ generate U^2 by Lemma 4. This completes the proof of Lemma 5.

To complete the proof of Lemma 6 we only need to show that X is not a subalgebra if $V \neq W^\perp$. Let $v_1 \in V$, $w_1 \in W$ such that $\langle v_1, w_1 \rangle = 1$, and let $v_2 \in V$, $w_2 \in W$ such that $v_1 v_2 w_1 w_2 \neq 0$. Then $[v_1 v_2, w_1 w_2] = v_2 w_2 + \cdots \notin X$.

Example. It is clear that in Lemma 6 the restriction $\dim W > 1$ is necessary. The following example shows that the remaining restrictions in Lemmas 5 and 6 are also necessary. Let U be 4-dimensional with orthonormal basis x, y, z, w. Then xy, zw, $xz + yw$, $xw - yz$ span a subalgebra, the unitary algebra $u(2)$ of the complex structure J such that $Jx = y$, $Jz = w$. Then $u(2)$ properly contains a subalgebra of the form X with $V = W^\perp$, where V is spanned by x, y, and W by z, w. It also contains a subspace of the form X with $V \neq W^\perp$, where V is spanned by x, y and W by $x - w$, $y + z$, but $u(2) \neq U^2$. It is not hard to show that all counterexamples to the $\dim V = \dim W = 2$ cases of Lemmas 5 and 6 are of this type.

7. The signature nonzero case

Under the hypothesis of Theorem 1, let us also assume that for some choice of normal frame at m some second fundamental form S_1 has nonzero signature. We let $U = k(m)$ and henceforth consider only the restrictions $S_a \mid U$, but we shall omit "$\mid U$" from our notation. As we rotate the normal frame from the chosen position to its opposite, S_1 passes to $-S_1$, and hence undergoes a change in signature. Thus for another choice of normal frame we must have that S_1 is singular (on U), and it is this frame with which we now work. Let $V = D(S_1)$ and $W = D(S_2)$, so $U = V + W$ and $V \neq U$.

Lemma 7. *The range of R contains $(S_1W^\perp)V + (S_2V^\perp)W$. If $V \cap W = 0$, then the range of R equals $V^2 + W^2$.*

Proof, If $x \in W^\perp$, then $S_2x = 0$ and $R(xy) = (S_1x)(S_1y)$. Thus the range of R contains $(S_1W^\perp)V$, and, similarly, it contains $(S_2V^\perp)W$. If $V \cap W = 0$, then $S_1W^\perp = V$ and $S_2V^\perp = W$. But the range of R is always contained in $V^2 + W^2$.

Lemma 8. *The subalgebra generated by $V^2 + W^2$ is $r(m)$.*

Proof. If $W = U$, then, by Lemma 7, $r(m)$ contains wU for some $w \neq 0$; so by Lemma 2 we have $r(m) = U^2 = V^2 + W^2$. Otherwise $S_1W^\perp \neq 0$; so $r(m)$ contains $V^2 + W^2$ again by Lemma 2.

Theorem 1(a′). *If some second fundamental form at m has nonzero signature, then there are the following possibilities for $r(m)$:*

(a) *$r(m) = k(m)^2$, or*

(b) *$r(m) = V^2 + W^2$, where V and W are nonzero, orthogonal to each other, and $k(m) = V + W$, or*

(c) *$r(m) = u(2)$, the unitary algebra of some complex structure on $k(m)$, and* $\dim k(m) = 4$.

Proof. This follows by combining Lemmas 8 and 6 and, in the case $\dim V = \dim W = 2$, using the fact pointed out in the example.

Remark. In case (b), if neither of V or W has dimension one, the frame for M_m and the normal frame may be chosen so that the matrices of S_1, S_2 have the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where A and B are nonsingular diagonal matrices. This choice of normal frame is unique up to sign and order. Since S_1 and S_2 commute, the normal curvature vanishes.

In case (a), and also case (b) if one of V or W has dimension one, the normal curvature may or may not vanish. If the normal curvature vanishes, then case (c) cannot occur.

8. The signature zero case

The possibility not included in the analysis in §7 is that all second fundamental forms at m are nonsingular on $U = k(m)$ and hence their signatures are all zero. Again using the convention that we omit noting restrictions to U, choose any normal basis and let $A = S_2S_1^{-1}$. Then $A - \mu I = (-\mu S_1 + S_2)S_1^{-1}$ is nonsingular for all real μ; so A has no real eigenvalues.

Denote the complexifications of U and $r(m)$ by U_c and $r(m)_c$. We extend the inner product to U_c to be complex bilinear, and it follows that $r(m)_c$ is the algebra generated by all $xy + (Ax)(Ay) = (S_1S_1^{-1}x)(S_1S_1^{-1}y) + (S_2S_1^{-1}x)(S_2S_1^{-1}y)$, where $x, y \in U_c$. We identify $o(U_c)$ and $(U_c)^2$ by the formula (1) as before. For any complex number q denote the null space of $(A - qI)^h$ by $V(q, h)$. In particular, $V(q, 0) = 0$, $V(q, 1)$ is the space of eigenvectors of A having eigenvalue q, and U_c is the direct sum of the subspaces $V(q) = V(q, D)$. Since A is real, $V(\bar{q}, h) = \overline{V(q, h)}$. We call $V(q)$ the q-eigenspace of A. Denoto AA the extension of A to U_c^2, that is, $AA(xy) = (Ax)(Ay)$.

Lemma 9. *For every pair of complex numbers q, r, the subspace $V(q)V(r)$ of U_c^2 is contained in the (qr)-eigenspace of AA.*

Proof. We prove by induction on (h, j) that $V(q, h)V(r, j)$ is contained in the (qr)-eigenspace of AA. Since this is clear for $h = j = 0$ we proceed with the inductive step. For convenience let $V(q, -1) = 0$. Suppose that $x \in V(q, h)$ and $y \in V(r, j)$. Then $Ax = qx + z$ and $Ay = ry + w$, where $z \in V(q, h - 1)$ and $w \in V(r, j - 1)$. Thus

$$(AA - qrI)xy = qrxy + qxw + rzy + zw - qrxy = qxw + rzy + zw .$$

By the inductive assumption,

$$xw, zy, zw \in V(q, h - 1)V(r, j) + V(q, h)V(r, j-1)$$

are annihilated by some power of $AA - qrI$; hence so also is xy.

Lemma 10. *Let $V = \sum_{\mathrm{Im}\, q>0} V(q)$, $W = \bar{V}$. Then VW is contained in the range of $AA + I$ and $U_c = V + W$.*

Proof. From Lemma 9 we have that $AA + I$ is nonsingular on its invariant subspace VW, since $qr \neq -1$ if $\mathrm{Im}\, q > 0$ and $\mathrm{Im}\, r < 0$.

Theorem 1 (b′). *If all second fundamental forms at m have the same rank, and hence zero signature, then $r(m)$ is either $k(m)^2$ or the unitary algebra of an isometric complex structure on $k(m)$.*

Proof. By the remark after Lemma 1, VW generates U_c^2 unless V and W are both isotropic. Hence we assume that V is isotropic, and hence also W is isotropic, since $W = \bar{V}$. The condition for this to happen is that there be an isometric complex structure J on U having V and W as its eigenspaces, say V consists of all $x - iJx$, where $x \in U$. Moreover, it then follows that VW is the complexification of the unitary algebra $u(J)$ of J. Indeed, $u(J)$ consists of elements of U^2 which commute with J and it is easily shown that the real and imaginary parts of $(x - iJx)(y + iJy)$, that is, $xy + JxJy$ and $xJy + yJx$, commute with J, so VW is contained in $u(J)_c$. However, the dimensions of $u(J)_c$ and VW over C are the same; so $VW = u(J)_c$.

It is well known that $SO(2n)/U(n)$ is an irreducible symmetric space, so $u(J)$ is a maximal subalgebra of U^2. Thus if the range of R contains more than VW, then $r(m) = U^2$. This completes the proofs of Theorems 1 and 1(b′).

Remark. If we utilize Lemma 9 and its proof further we can conclude the following when $r(m) = u(J)$:

(1) If $\dim U > 4$, then $V = V(i, 1)$ and $A = J$.

(2) If $\dim U = 4$, then besides the possibility $A = J$ we can also have $V = V(q, 1) + V(-1/q, 1)$ for some q, or $V = V(i, 2) \neq V(i, 1)$.

In the case where $A = J$ we can choose a frame of U and a normal frame such that matrices of S_1, S_2, J on U are

$$\begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix}, \quad \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

respectively, where Q is positive definite and diagonal.

9. The proof of Theorem 2

As in the proof of Kobayashi's Theorem, if we choose a point m at maximal distance from the origin in R^{D+2}, we will have a positive definite second fundamental form at m. This clearly eliminates the zero signature case and also gives us $k(m) = M_m$. Thus h_m contains a subalgebra $r(m) = V^2 + W^2$, where $V + W = M_m$ and $V = W^{\perp}$, and Lemma 5 completes the proof.

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