

COCYCLE SUPERRIGIDITY AND BOUNDED COHOMOLOGY FOR NEGATIVELY CURVED SPACES

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1. Introduction

In this paper and its companion [31], we introduce new techniques and results in an attempt to extend rigidity theory beyond the scope of linear groups. Amongst our main tools is the bounded cohomology theory recently developed by Burger and Monod [9], [30]. This theory had previously been used for rigidity by Bestvina–Fujiwara, Burger, Iozzi and Monod [2], [7], [8], [9], [22], [30], building on invariants in bounded cohomology with trivial coefficients. Our approach relies in an essential way on bounded cohomology with non-trivial coefficients. It involves, among other things, general cohomology vanishing and non-vanishing results which seem of independent interest. We also make use of a measurable boundary construction introduced by Burger–Monod (*loc. cit.*, later improved by Kaimanovich [25]), which enables us to apply boundary theory for general locally compact groups.

Margulis’ fundamental superrigidity theorem may be viewed as a result describing the finite dimensional representation theory of higher rank lattices. Although its original proof was measure theoretic, later remarkable developments of geometric rigidity (cf. [19] and the references therein) were able to approach it as well, at least in the Archimedean co-compact cases. Meanwhile, a rather parallel direction to the geometric one, both in methods and in applications, arose with the appearance of Zimmer’s cocycle superrigidity theorem. At the conceptual level, that result, which proved very powerful in various applications, placed the emphasis on the ambient group itself rather than the lattice.

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Consequently, unlike with Margulis' superrigidity, one may try to state cocycle superrigidity theorems (with the appropriate hypotheses) for general groups – including discrete ones – without referring to an “ambient group”. In return, many of the applications of Zimmer's theorem, which are known for higher rank groups, become available in a wider setting. This is the purpose of our first main result, and the framework in which this work takes its place. We first describe the general setting and notation adopted throughout this paper.

Let G_1, \dots, G_n with $n \geq 2$ be locally compact groups, and $G = G_1 \times \dots \times G_n$ be their direct product. Recall that a lattice $\Gamma < G$ is said to be an irreducible lattice if the projection of Γ to every G_j is dense. This is equivalent to the ergodicity of the sub-product $G'_j := \prod_{i \neq j} G_i$ on G/Γ . Thus, more generally, a measurable G -action on a standard measure space Ω is called an irreducible action, if G'_j acts ergodically on Ω for every j .

Such groups G , which admit a “non-trivial” commutativity property, are our substitute for higher rank semi-simple algebraic groups in the framework of general locally compact groups. On the target side, we shall consider groups acting isometrically on generalized negatively curved spaces; more precisely:

Definition 1.1 (Generalized negative curvature and elementarity). For the purpose of this paper we call a geodesic metric space X a generalized negatively curved (GNC) space, if it is amongst the following four classes of spaces:

- (i) proper CAT(−1) spaces;
- (ii) Gromov-hyperbolic graphs of bounded valency;
- (iii) Gromov-hyperbolic proper spaces on which $\text{Isom}(X)$ acts co-compactly;
- (iv) simplicial trees (not necessarily locally finite).

Let X be a GNC space, G be a locally compact group, Ω be a Borel G -space, and $\alpha : G \times \Omega \rightarrow \text{Isom}(X)$ be a cocycle. We call α -elementary if there is a measurable map f on Ω ranging in the space of bounded subsets of X , or in points or couples of ∂X , that is α -equivariant (or α -invariant), i.e., $f(gz) = \alpha(g, z)f(z)$.

As is easily observed (cf. Section 3), Definition 1.1 of cocycle elementarity is a natural extension of that of elementary homomorphisms into $\text{Isom}(X)$ (i.e., elementary actions on X). Also note that in the CAT(−1) case, the space of bounded subsets in the definition (which is

adapted to the general setting of GNC spaces), can be replaced by X itself by using the circumcentre operation.

Theorem 1.2 (Cocycle superrigidity). *Let $G = G_1 \times \cdots \times G_n$ be any locally compact σ -compact group with an irreducible measure preserving action on a standard probability space Ω . Let X be a proper GNC space, $H < \text{Isom}(X)$ a closed subgroup and $\alpha : G \times \Omega \rightarrow H$ a non-elementary measurable cocycle.*

Then, there is a closed subgroup $H' < H$ and a normal compact subgroup $K \triangleleft H'$ such that α is cohomologous to a cocycle $\alpha' : G \times \Omega \rightarrow H'$ whose composition with the natural map $H' \rightarrow H'/K$ yields a continuous homomorphism $G \rightarrow H'/K$, which factors through some G_j .

In other words, loosely speaking: *Modulo a compact normal subgroup, any non-elementary cocycle is cohomologous to a continuous homomorphism.* Notice that if X is CAT(-1), then the existence of H' and K simply amounts to saying that α is equivalent to a continuous homomorphism of G upon possibly restricting to an invariant convex subspace (of the K -fixed points) of X .

In the particular case where the G_j 's are linear algebraic groups, Theorem 1.2 (and Theorem 1.3) generalizes results of Burger–Mozes [10], Spatzier–Zimmer [41] and Adams [1]; see Remark 3.14 below for the case of higher rank simple groups. Concerning the elementarity condition of the theorem, we remark the following:

I. If X has at most exponential growth (as in [1]) e.g. if it is a Riemannian manifold of bounded geometry, or in cases (ii) and (iii) of Definition 1.1 – then non-elementarity of α is automatically granted if its Mackey range is non-amenable.

It follows that if moreover G is Kazhdan, then the final conclusion, of the theorem holds true without any (elementarity) restriction on the cocycle α .

II. When X is a symmetric space and $H = \text{Isom}(X)$, α is non-elementary if and only if it is not cohomologous to a cocycle into an amenable subgroup of H .

Remark. One of the classical applications of cocycle superrigidity lies in orbit equivalence, and one can indeed readily deduce some results in that direction from Theorem 1.2. However, we propose in [31] stronger orbit equivalence superrigidity statements which go beyond applications of the above theorem. In fact, the *only* result of the present

paper that is needed for [31] is the cohomological non-vanishing deduced in Section 7.

Following “Zimmer’s philosophy”, it is quite standard by now that cocycle superrigidity implies homomorphism superrigidity. One can indeed readily deduce from Theorem 1.2 that any non-elementary Γ -action on X extends to a continuous G -action upon possibly factoring X by a compact group. However, we shall establish a considerably more general homomorphism superrigidity theorem, which may be viewed as a new “non-linear” extension of the latter, capitalizing on the additional strength of cocycle superrigidity. To motivate this result, we first remark the following (see Sections 3.3 and 3.4 for more details):

- It seems to be the first example of a superrigidity result for homomorphisms (with locally compact target), which can only be proved using cocycle superrigidity techniques, unlike classical cases where superrigidity for homomorphisms is easier to establish and does not require cocycles.
- We show that it applies to uncountably many locally compact (non-discrete) target groups, which do not arise as isometry groups of CAT(0) spaces as in previous works.
- It implies that in superrigidity results for isometric actions on spaces admitting a discrete co-compact action, the (local) negative curvature property of the quotient space can be significantly relaxed to a property of its homotopy type only.

Theorem 1.3 (Homomorphism superrigidity). *Let J be an arbitrary locally compact second countable group, containing a closed subgroup $H < J$ of finite invariant co-volume which admits a proper action on a GNC space X of at most exponential growth. Let $\Gamma < G = \prod G_i$ be an irreducible lattice in a product of any locally compact σ -compact groups and $f : \Gamma \rightarrow J$ a homomorphism such that $L = \overline{f(\Gamma)}$ is non-amenable.*

Then, there is a compact normal subgroup $M \triangleleft L$ such that the induced homomorphism $\Gamma \rightarrow L/M$ extends continuously to G , factoring through some G_i .

Moreover, although without the growth assumption on X the assertion may fail, the conclusion still holds if instead H acts non-elementarily on X and $L \supseteq H$.

When X is a tree, perhaps of infinite valency, properness of the H -action is meant with respect to its action on the edges. Observe that the

case $J = H = L$ yields a superrigidity theorem for non-elementary Γ -actions on negatively curved spaces X (which itself generalizes Shalom’s results [40] on the rigidity of irreducible lattices). An exemplary non-classical case to which this theorem applies is that of the non-uniform Kac–Moody lattices Γ studied by Rémy [34], [36], [37] (see also Corollary 3.17). In fact, Theorem 1.3 is merely a concrete instance of the more general homomorphism superrigidity Theorem 3.11 below. Here is an illustration of the last item preceding Theorem 1.3, which is of interest even for lattices in semisimple Lie groups (generalizing [10]):

Corollary. *Let H be either a Gromov-hyperbolic group or a discrete subgroup of a rank one Lie group. Assume for simplicity that H is non-amenable and that Γ is Kazhdan (e.g., several Kac–Moody groups, compare Corollary 3.17). Let Z be any compact connected metric space whose only required property is that its fundamental group be isomorphic to H .*

Then, for any isometric Γ -action on the universal covering \tilde{Z} there is some compact isometry group M normalized by Γ , such that the induced Γ -action on \tilde{Z}/M extends continuously to G , factoring through some G_i .

Although our approach is quite general, some technicalities arise if the properness (i.e., local compactness) assumption on the space X is removed. In this case, it is not even obvious which notion of measurability of a cocycle into $\text{Isom}(X)$ one should consider (the latter group need not be locally compact). However, we do handle the most significant example of such a space, a tree, whose importance in group theory comes via Bass–Serre theory, by showing (Theorem 4.1) that: *Any non-elementary action of an irreducible lattice on any simplicial tree extends continuously to the ambient group, after possibly passing to an invariant subtree.*

Even though our treatment of the tree case is similar to that in the setting of proper GNC spaces, one additional ingredient of our approach may be of independent interest: the introduction of a new topology on the usual ray completion of trees, which coincides with the standard one in the locally finite case, but is weaker and compact in general. See Section 4 for more details.

As mentioned earlier, the proofs of Theorems 1.2 and 1.3 use bounded cohomology in an essential (and novel) way. Suffice it to say at this point that bounded (continuous) cohomology H_b^\bullet (resp. H_{cb}^\bullet), is defined similarly to usual group (continuous) cohomology, but using bounded

cochains. An examination of our proofs then shows that the one factor G_j appearing in the conclusion of Theorem 1.2 or Theorem 1.3 must have the following cohomological property: *There is a unitary G_j -representation (π, \mathcal{H}) without non-zero invariant vectors such that $H_{\text{cb}}^2(G_j, \mathcal{H}) \neq 0$ (Remark 3.12).* Moreover, if G is any locally compact σ -compact group – not necessarily a product – for which there is *no* such representation (π, \mathcal{H}) , then every cocycle α as in Theorem 1.2 must be elementary (Corollary 3.13). This fact, which is of particular interest for discrete G , provides a strong motivation for the following cohomology vanishing result.

Theorem 1.4 (Cohomology vanishing in higher rank). *Let $\Gamma < G = \mathbf{G}(k)$ be a lattice, where k is a local field and \mathbf{G} is a connected almost k -simple algebraic group defined over k with $\text{rank}_k \mathbf{G} \geq 2$. Let E be an isometric Banach Γ -module with separable dual E^* (e.g., a separable unitary Γ -module). Then*

$$\dim H_{\text{b}}^2(\Gamma, E^*) = \begin{cases} \dim E^{*\Gamma} & \text{if } k = \mathbf{R} \text{ and } \pi_1(G) \text{ is infinite} \\ 0 & \text{in all other cases,} \end{cases}$$

where $E^{*\Gamma} \subseteq E^*$ is the subspace of Γ -invariant vectors.

In particular, $H_{\text{b}}^2(\Gamma, -)$ vanishes for every unitary Γ -representation without non-zero invariant vectors. An analogous statement holds for G in place of Γ .

The separability assumption is necessary. Previously, it was shown by Burger–Monod [8], [9] that for Γ and G as above, the natural comparison map to usual second cohomology is injective for unitary coefficients; the product case is treated in [9].

Consequently, the above results show that *bounded cohomology provides an abstract group property which accounts for, and generalizes, most previously known rigidity results for higher rank groups in the context of negatively curved targets.*

Of course, it follows from the discussion above that groups which do act non-elementarily on negatively curved spaces necessarily fail to have the vanishing property of Theorem 1.4. Moreover, if the action is proper, the vanishing fails already for the regular representation (see Corollaries 7.6 and 7.8); in turn, we deduce that there are uncountably many other representations with non-vanishing H_{b}^2 , see Proposition 7.1. This non-vanishing property $H_{\text{b}}^2(\Gamma, \ell^2(\Gamma)) \neq 0$ is a key ingredient in our work on Orbit Equivalence [31]. Thus, the results of this paper also

provide a large class of groups to which the rigidity theorems of [31] apply.

The non-vanishing cohomology of groups acting on GNC spaces turns out to suffice for rigidity of actions – say, the case $J = H = L$ of Theorem 1.3. The non-vanishing is however not sufficient for the cocycle superrigidity Theorem 1.2 or for Theorem 1.3 in its full generality. For these, we need the following more geometric result, which is central to our approach and whose proof occupies all of Section 5:

Theorem 1.5 (Nowhere vanishing L^2 cocycle). *Let X be a proper CAT(−1) space, ∂X be its boundary and $H = \text{Isom}(X)$. There is a weakly continuous H -equivariant alternating bounded cocycle*

$$\omega : \partial X \times \partial X \times \partial X \longrightarrow \bigoplus_{n=1}^{\infty} L^2(H)$$

whose restriction to the distinct triples $\partial^3 X$ vanishes nowhere.

Remark. An extension to the Gromov-hyperbolic case has been obtained in joint work with Mineyev [29] after the completion of the first version of this paper; this accounts for the generality of Theorems 1.2 and 1.3. Compare also Section 7.

It is essential for the applications to rigidity that the above cocycle ω vanishes nowhere on $\partial^3 X$. In contrast, it is easy to deduce from Theorem 1.4 that if X is a Bruhat–Tits building of rank ≥ 2 (or any higher rank irreducible symmetric space of non-compact type that is not Hermitian) and ∂X is its geometric boundary, then: Any measurable $\text{Isom}(X)$ -equivariant bounded alternating cocycle on $(\partial X)^3$, ranging in any separable dual Banach $\text{Isom}(X)$ -representation, vanishes essentially everywhere on $(\partial X)^3$ (Corollary 6.5).

The proof of Theorem 1.5 involves a precise cohomological-analytic form of the “thin triangles” principle, an approach inspired by Gromov [18, 7.E₁] (going back to Sela [38]). Furthermore, even though it is not needed for the above results, some additional functoriality allows us to rephrase Theorem 1.5 in such a way that we obtain a cohomological invariant of the sole space X which detects the (non-)elementarity of any isometric group action on X (see Section 7).

Finally, the above results promote the philosophy that bounded cohomology with L^2 -coefficients is the appropriate framework to encode negative curvature. There has been substantial work in this direction for

trivial coefficients (often using quasimorphisms); in particular Brooks' construction [5] for hyperbolic surfaces has been considerably generalized to the Gromov-hyperbolic setting (see e.g., [2], [14]). However, these invariants, though sometimes sufficient to exclude actions, are in general not able to detect actions on negatively curved spaces. For instance, irreducible lattices in products such as $\mathrm{SL}_2(\mathbf{Q}_p) \times \mathrm{SL}_2(\mathbf{Q}_q)$ or $\mathrm{SO}(n, 1) \times \mathrm{SO}(n, 1)$ (with $n \geq 3$) have vanishing H_b^2 for trivial coefficients [9]. However, the invariants that we construct encode the fact that such lattices can act on negatively curved spaces, and can only do so via the factors of the ambient group. As another example, $\Gamma = \mathbf{SL}_2(\mathbf{Z}[\sqrt{2}])$, which does not have quasi-morphisms (and is not "negatively curved"), carries an invariant in $H_b^2(\Gamma, L^2(\mathbf{SL}_2(\mathbf{R})))$ characterizing its only two non-elementary actions on the hyperbolic plane.

There is, however, a price to pay for this conceptual switch from trivial to non-trivial coefficients. Contrary to the case of quasimorphisms whose non-vanishing as cohomology classes are typically easy to establish directly, for non-trivial coefficients it is in general difficult to arrive at the same conclusion. This accounts for the use of boundary theory and the functorial approach to bounded cohomology.

2. Cohomological tools

We summarize below those essential facts about continuous bounded cohomology that we shall use throughout the paper. For a more detailed account and proofs of the statements that we merely quote, we refer the reader to [9], [30].

Let G be a group. A Banach G -module (π, E) is a Banach space E with an isometric linear G -representation π . If G is a topological group, the module is called continuous if the map $G \times E \rightarrow E$ is continuous. A coefficient G -module is the contragradient of a separable continuous Banach G -module; thus, unitary representations on separable Hilbert spaces are examples of coefficient modules.

For a locally compact group G and a Banach G -module E , define the continuous bounded cohomology $H_{\mathrm{cb}}^\bullet(G, E)$ to be the cohomology of the complex

$$0 \longrightarrow C_b(G, E)^G \xrightarrow{d} C_b(G^2, E)^G \xrightarrow{d} C_b(G^3, E)^G \xrightarrow{d} \dots$$

of continuous bounded maps that are invariant for the diagonal regular representation λ_π (i.e., equivariant maps $G^{\bullet+1} \rightarrow E$). Here and below, d

denotes the usual Alexander–Spanier coboundary map. For the relation with other definitions of bounded cohomology compare [30, No. 7.4.7]. The bounded cohomology $H_b^\bullet(\Gamma, -)$ of an abstract group Γ is defined by endowing Γ with the discrete topology.

2.1. Strong boundaries. Let G be a locally compact group. It is shown in [9], [30] that one can compute $H_{cb}^\bullet(G, -)$ using cochains on standard Borel G -spaces B with a quasi-invariant probability measure, provided the action on B is amenable in Zimmer’s sense [42].

Theorem 2.1. *Let G be a locally compact σ -compact group, B an amenable G -space and E a coefficient G -module. Then, the cohomology of the complex*

$$0 \longrightarrow L_{w*}^\infty(B, E)^G \longrightarrow L_{w*}^\infty(B^2, E)^G \longrightarrow L_{w*}^\infty(B^3, E)^G \longrightarrow \dots$$

is canonically isomorphic to $H_{cb}^\bullet(G, E)$. The corresponding statement holds for the sub-complex of alternating cochains.

(Originally stated for second countable groups, this extends immediately to the σ -compact case as explained in [30], p. 51 and 152.)

As a first instance, one can take $B = G/N$ for a closed amenable subgroup $N < G$. If N is normal, we recover the following general principle (special cases of which were previously observed in [17], [23], [24], [33]).

Corollary 2.2. *Let G be a locally compact σ -compact group, $N \triangleleft G$ an amenable closed normal subgroup and E a coefficient G -module. Then, the maps $G \rightarrow G/N$ and $E^N \rightarrow E$ induce for all $n \geq 0$ isomorphisms*

$$H_{cb}^n(G/N, E^N) \cong H_{cb}^n(G, E^N) \cong H_{cb}^n(G, E).$$

However, Theorem 2.1 has much stronger consequences due to the existence of more interesting amenable spaces. In order to state this, we need the following.

Definition 2.3. A standard Borel G -space B with a quasi-invariant probability measure is a strong boundary (for G) if

- (i) the G -action on B is amenable,
- (ii) for any separable coefficient G -module E , any measurable G -equivariant map $B \times B \rightarrow E$ is essentially constant.

(The separability assumption implies that strong, weak and weak-* measurability coincide.)

Condition (ii) above is a very strong ergodicity property, as the following illustrates.

Proposition 2.4. *Let B be a strong boundary for G and Ω a standard Borel G -space B with an invariant probability measure. Then, the diagonal G -action on $\Omega \times B \times B$ is ergodic.*

Proof. We have to show that every G -invariant function $f \in L^\infty(\Omega \times B \times B)$ is essentially constant. By the theorems of Fubini–Lebesgue and Dunford–Pettis, we may view f as an equivariant weak-* measurable map $B \times B \rightarrow L^\infty(\Omega)$. Since we have an invariant probability measure on Ω , there is an equivariant embedding of coefficient modules $L^\infty(\Omega) \rightarrow L^2(\Omega)$. The latter being separable, we conclude that f is essentially constant. q.e.d.

It is of fundamental importance that such a strong boundary exists in great generality, as shown in [9, Theorem 6] for compactly generated groups and then in [25] for σ -compact groups:

Theorem 2.5. *Every locally compact σ -compact group admits a strong boundary.*

Strong boundaries yield particularly concrete realizations for the bounded cohomology in degree two; indeed, Theorem 2.1 implies immediately in the following corollary.

Corollary 2.6. *Let G be a locally compact σ -compact group and E a separable coefficient G -module. If B is a strong boundary for G , then there is a canonical isomorphism*

$$H_{\text{cb}}^2(G, E) \cong ZL_{\text{alt}}^\infty(B^3, E)^G,$$

where the right-hand side is the space of G -invariant alternating measurable essentially bounded cocycles on B^3 .

Here is a demonstration of this principle.

Corollary 2.7. *Let G be a locally compact σ -compact group and $(\pi_n, \mathcal{H}_n)_{n=1}^\infty$ a family of continuous unitary G -representations in separable Hilbert spaces \mathcal{H}_n . Then*

$$H_{\text{cb}}^2(G, \bigoplus_{n=1}^\infty \mathcal{H}_n) = 0 \iff \forall n \geq 1 : H_{\text{cb}}^2(G, \mathcal{H}_n) = 0.$$

(The direct sums are always understood to be Hilbertian sums; the proof can also be adapted to the case of direct integrals.)

Proof of the Corollary. Let B be a strong boundary as granted by Theorem 2.5. Write $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ and assume that $H_{cb}^2(G, \mathcal{H})$ does not vanish. Write $p^n : \mathcal{H} \rightarrow \mathcal{H}_n$ for the orthogonal projection and let $\kappa \neq 0$ be a class in $H_{cb}^2(G, \mathcal{H})$. By Corollary 2.6, we may represent this class by a G -invariant alternating measurable essentially bounded cocycle $\omega : B^3 \rightarrow \mathcal{H}$ and $p_*^n(\kappa)$ is represented by $p^n \circ \omega$ (by naturality of the isomorphism of Corollary 2.6). Suppose now for a contradiction that $p_*^n(\kappa) = 0$ for all n . Then, by Corollary 2.6, the cocycle $p^n \circ \omega$ vanishes for all n . This entails $\omega = 0$, a contradiction. The converse is immediate. q.e.d.

We turn to another important application of strong boundaries, addressed in [9], [30]. If Γ is a lattice in a locally compact group G and E a separable coefficient Γ -module, the L^2 induction module is defined to be the Banach G -module $L^2\mathbf{I}_{\Gamma}^G E = L^{[2]}(G, E)^{\Gamma}$, where the right-hand side denotes the functions whose norm is square-summable on $\Gamma \backslash G$; the G -representation is given by right translation and turns $L^2\mathbf{I}_{\Gamma}^G E$ into a coefficient G -module, which is separable provided G is second countable. If now B is any amenable G -space, one defines the L^2 induction map

$$\mathbf{i} : H_{cb}^n(\Gamma, E) \longrightarrow H_{cb}^n(G, L^2\mathbf{I}_{\Gamma}^G E)$$

by associating to a Γ -invariant bounded map $\omega : B^{n+1} \rightarrow E$ the map $\mathbf{i}\omega : B^{n+1} \rightarrow L^2\mathbf{I}_{\Gamma}^G E$ defined by $\mathbf{i}\omega(x_0, \dots, x_n)(g) = \omega(gx_0, \dots, gx_n)$. We used here Theorem 2.1 for both G and Γ since B is also amenable for the latter.

The trick to replace ω by $\mathbf{i}\omega$ can be applied to the second condition in Definition 2.3 to deduce.

Lemma 2.8. *Let B be a strong boundary for G and let $\Gamma < G$ be a lattice. Then, B is also a strong boundary for Γ .*

Proof. One applies a standard induction argument as in [30, No. 11.1.10], except that some extra care is needed since if G is not second countable, the induction module is not separable and hence, one cannot apply the property (ii) of Definition 2.3. However, the G -action on B factors through a second countable quotient G/K of G by a compact subgroup $K \triangleleft G$ as in [30, p. 152]. Therefore, one can argue with a separable induction module of Γ -invariant maps on G/K and conclude as in the given reference. q.e.d.

Applying the same caveat for second countability, one has (compare [30, No. 11.1.5]).

Corollary 2.9. *Let G be a locally compact σ -compact group, $\Gamma < G$ a lattice and E a separable dual Banach Γ -module. Then, the L^2 induction*

$$i : H_{cb}^2(\Gamma, E) \longrightarrow H_{cb}^2(G, L^2 I_\Gamma^G E)$$

is injective.

Another tool that we shall borrow from [9], [30] (with [25]) is the following product formula; again, one can reduce to the second countable case upon taking a quotient by a compact subgroup.

Theorem 2.10. *Let G_1, \dots, G_n be locally compact σ -compact groups and let $G = \prod_{j=1}^n G_j$, $G'_j = \prod_{i \neq j} G_i$. Let (π, E) be a separable coefficient G -module. Then, we have a canonical isomorphism*

$$H_{cb}^2(G, E) \cong \bigoplus_{j=1}^n H_{cb}^2(G_j, E^{G'_j}).$$

For a direct way to deduce this result from Theorem 2.5 and Corollary 2.6, see [32]. Using cohomological induction, one derives the following consequence [9], [30].

Corollary 2.11. *Let G_1, \dots, G_n be locally compact σ -compact groups and let $\Gamma < G = G_1 \times \dots \times G_n$ be an irreducible lattice. For any separable coefficient Γ -module (π, E) , there is an isomorphism*

$$H_b^2(\Gamma, E) \cong \bigoplus_{j=1}^n H_{cb}^2(G_j, E_j),$$

wherein E_j denotes the maximal Γ -submodule of E such that the restriction $\pi|_{E_j}$ extends continuously to G , factoring through $G \twoheadrightarrow G_j$.

2.2. Cochains on compact spaces. Let G be a Baire topological group (e.g., locally compact) with a continuous action on a compact metrizable space Z and let (π, E) be a separable coefficient G -module.

Since weak-* compact sets are bounded (see e.g., Theorem 1 in [11, II Section 3]), the sup-norm turns the space $C_{w*}(Z, E)$ of weak-* continuous functions into a separable Banach G -module. Denote by J^\bullet the cohomology of the complex

$$0 \longrightarrow C_{w*}(Z, E)^G \xrightarrow{d} C_{w*}(Z^2, E)^G \xrightarrow{d} C_{w*}(Z^3, E)^G \xrightarrow{d} \dots$$

Let \mathcal{P} denote the simplex of Borel regular probability measures endowed with the weak- $*$ topology. Although the following consequence of the functorial characterization of H_{cb}^\bullet is strictly speaking not necessary for the proof of Theorem 1.2, it will be useful in a related context (e.g., Theorem 7.2) and puts our construction of a geometric cocycle in Section 5 into a natural context.

Proposition 2.12. *There is a natural map $j : J^\bullet \rightarrow H_{cb}^\bullet(G, E)$. Moreover, if $\varrho : H \rightarrow G$ is any continuous homomorphism from a locally compact second countable group H to G , S an amenable regular H -space and $\varphi : S \rightarrow \mathcal{P}(Z)$ a measurable ϱ -equivariant map, then the map*

$$\begin{aligned} \varphi^* : C_{w*}(Z^{n+1}, E) &\longrightarrow L_{w*}^\infty(S^{n+1}, E), \\ \varphi^* f(s_0, \dots, s_n) &= \varphi(s_0) \otimes \dots \otimes \varphi(s_n)(f) \end{aligned}$$

induces the map $\varrho^ \circ j : J^\bullet \rightarrow H_{cb}^\bullet(H, E)$ up to the canonical isometric isomorphisms.*

In their appendix to [9], Burger–Iozzi prove this statement for Borel cocycles and locally compact groups, which prompts an important issue since there is no *a priori* measure class on Z ; their statement thus addresses the recurrent problem of regularity for boundary maps. We do not encounter that problem here.

Proof of Proposition 2.12. The generality considered here fits into the general framework of [30] and the proposition is an immediate application of the cohomological machinery proposed there. More precisely, Proposition 8.4.2 of [30] applies *verbatim*. Indeed, its various assumptions hold in view of 7.2.4, 7.2.6, 7.5.3 and 8.4.1 therein. The only point to check is that the G -complex $C_{w*}(Z^{n+1}, E)$ is continuous and admits a homotopy; the latter is obtained by evaluation at any given point of Z and the former is Lemma 2.13. q.e.d.

Lemma 2.13. *The regular G -representation λ_π on $V = C_{w*}(Z^{n+1}, E)$ is continuous.*

Remark 2.14. This states that all orbit maps $g \mapsto \lambda_\pi(g)f \in V$ are norm continuous, even though we shall typically apply this to maps f that are not norm continuous, see Remark 5.19.

Proof of Lemma 2.13. It is enough to prove that for every $f \in V$, the associated orbit map $G \rightarrow V$ is Borel [30, No. 1.1.3]. Let \mathcal{T} be the

topology defined on V by the collection of semi-norms $\|\Lambda h\|_\infty$, where Λ ranges over the predual E^\flat . Since Λh is uniformly continuous on Z^{n+1} and E^\flat is continuous, the orbit maps in V are \mathcal{T} -continuous. Thus, it suffices to show that the norm and \mathcal{T} induce the same Borel structure on V , i.e., that any norm open $A \subseteq V$ is \mathcal{T} -Borel. Lindelöf's theorem (Proposition 1 (i) in [3, IX, Appendice 1]) implies that A is a countable union of open balls, hence also of closed balls. For $F \subseteq E^\flat$ countable dense, we have $\bar{B}(r) = \bigcap_{\Lambda \in F} \{h \in V : \|\Lambda h\|_\infty \leq r\}$; thus each closed ball is \mathcal{T} -Borel, and so is A . q.e.d.

We observe that an almost identical argument shows that if E is any continuous separable Banach G -module such that the contragradient E^* is continuous, then the G -action on the space $C_w(Z, E)$ of weakly continuous functions is continuous. The assumption on E^* is satisfied automatically if E^* separable, see [30, No. 1.1.3, 3.3.3].

3. Cocycle and homomorphism superrigidity

The main goal of this section is to establish Theorem 1.2, using the results of the (independent) Section 5, and for the hyperbolic setting, the construction of [29]. We refer to [4] for a detailed background reference on CAT(-1) spaces and [16] for hyperbolic spaces; we shall go further into relevant notations in Section 5.

Let X be a proper GNC space, $H_0 = \text{Isom}(X)$ its group of isometries endowed with the (locally compact second countable) compact-open topology, $\bar{X} = X \sqcup \partial X$ the ray compactification. We denote by $\partial^n X \subseteq (\partial X)^n$ the space of distinct n tuples and by $D_2(\partial X)$ the space of unordered pairs of distinct points. Let $B(X)$ be the space of closed bounded subsets of X endowed with the Borel structure of the Hausdorff metric and with the natural H_0 -action.

3.1. Preliminaries on elementarity. Recall that an isometric action of a group Γ on X is called elementary if Γ has bounded orbits or fixes an element in $\partial X \sqcup D_2(\partial X)$. The natural generalization to elementary cocycles is defined in Definition 1.1 of Section 1, a notion, which is obviously preserved under equivalence of cocycles. That it indeed extends the definition of elementary homomorphism can be seen in the following observation, whose proof is left to the reader.

Lemma 3.1. *Assume that α is given by a group homomorphism $\rho : \Gamma \rightarrow H_0 = \text{Isom}(X)$ and Γ is ergodic on Ω . Then, α is elementary if*

and only if the Γ -action on X given by ϱ is elementary. Furthermore, if ϱ is as above and $\Gamma < G$ is a lattice, then ϱ is elementary if and only if the cocycle α induced to $G \times G/\Gamma \rightarrow \text{Isom}(X)$ (via some choice of a fundamental domain) is elementary.

For any locally compact space, denote by \mathcal{P} the space of probability measure (in its weak-* topology), \mathcal{P}_k the set of probability measures supported on exactly k points, by $\mathcal{P}_{\leq k}$ those supported on at most k points and similarly $\mathcal{P}_{\geq k}$ on at least k points.

Lemma 3.2. *Let G be any group, Ω a measure G -space, X a proper GNC space and $\alpha : G \times \Omega \rightarrow \text{Isom}(X)$ a cocycle. If either*

- (i) *there is a measurable α -equivariant map $\Omega \rightarrow \mathcal{P}(\partial X)$, or*
- (ii) *α is equivalent to a cocycle ranging in a compact subgroup,*

then α is elementary.

Proof. There is an H_0 -equivariant measurable centre of mass (or barycentre map $CM : \mathcal{P}_{\geq 3}(\partial X) \rightarrow B(X)$, see [1]. Assuming that we have a map $\varphi : \Omega \rightarrow \mathcal{P}(\partial X)$ as in (i), define the H_0 -equivariant map

$$\mathcal{P}(\partial X) = \mathcal{P}_{\geq 3}(\partial X) \sqcup \mathcal{P}_{\leq 2}(\partial X) \longrightarrow B(X) \sqcup D_2(X) \sqcup \partial X$$

using CM and the support map; we get a map as in Definition 1.1. In case (ii), there is by assumption a compact subgroup $M < H_0$ and a measurable map $\varphi : \Omega \rightarrow H_0$ such that

$$\varphi(gz)\alpha(g, z)\varphi(z)^{-1} \in M \quad \forall g \in G \forall z \in \Omega.$$

Let $A \subseteq X$ be the closure of any M -orbit; since M is compact, A is an (M -invariant) element of $B(X)$. Now, the map $\psi : \Omega \rightarrow B(X)$ defined by $\psi(z) = \varphi(z)^{-1}A$ is α -equivariant. q.e.d.

3.2. Proof of Theorem 1.2. Applying Theorem 2.5 to G_i , we get strong boundaries (B_i, β_i) for G_i . Then, $(B, \beta) = B_1 \times \dots \times B_n$ is a strong boundary for G . Endow now $\Omega \times B$ with the diagonal G -action and consider the cocycle $\tilde{\alpha} : G \times \Omega \times B \rightarrow H$ obtained from α by omitting the last variable. The following is a general version of a standard ingredient in rigidity arguments.

Proposition 3.3. *There exists an $\tilde{\alpha}$ -equivariant measurable map $f : \Omega \times B \rightarrow \partial X$.*

Proof. Since the action on $\Omega \times B$ is amenable, there is an $\tilde{\alpha}$ -equivariant measurable Furstenberg map $f_F : \Omega \times B \rightarrow \mathcal{P}(\partial X)$ (as in [42, 4.3.9], see [42, p. 103]).

Lemma 3.4. *We have $f_F(\Omega \times B) \subseteq \mathcal{P}_{\leq 2}(\partial X)$ after discarding a null-set.*

Proof of Lemma 3.4. Assume for a contradiction that $f_F(\Omega \times B)$ is not in $\mathcal{P}_{\leq 2}(\partial X)$. Since G is ergodic on Ω , we have $f_F(\Omega \times B) \subseteq \mathcal{P}_{\geq 3}(\partial X)$. Since G is ergodic on $\Omega \times B$ (Proposition 2.4) and since the H -action on $\mathcal{P}_{\geq 3}(\partial X)$ is tame with compact stabilizers [1, Corollary 5.3], the map f_F yields an $\tilde{\alpha}$ -equivariant map $\tilde{f} : \Omega \times B \rightarrow H/K$ for some compact subgroup $K < H$. Since G is also ergodic on $\Omega \times B \times B$ (Proposition 2.4 again), we are now in position to apply Lemma 5.2.10 in [42] to $\alpha|_{G \times \Omega}$ and deduce that α is equivalent to a cocycle ranging in a compact subgroup of H . By point (ii) in Lemma 3.2, α is elementary, a contradiction. q.e.d.

Lemma 3.5. *The map f_F ranges in the Dirac masses after discarding a null-set.*

Proof of Lemma 3.5. Suppose this fails, so that by ergodicity and Lemma 3.4 we may assume that f_F ranges in $D_2 = D_2(\partial X)$. Let δ be some continuous $\text{Isom}(X)$ -invariant map $D_2 \times D_2 \rightarrow \mathbf{R}_+$ such that $\delta(q, q') = 0$ if and only if $q \cap q' \neq \emptyset$ ($q, q' \in D_2$). Such a map exists by virtue of the mere topological properties of the $\text{Isom}(X)$ -action on ∂X (Lemma 23 in [29]); however, in the $\text{CAT}(-1)$ case, there are very simple concrete constructions of such functions δ : Consider e.g., for $\xi, \xi', \eta, \eta' \in \partial X$, all distinct the quantity

$$(1) \quad \delta(\{\xi, \xi'\}, \{\eta, \eta'\}) = \left(1 + d([\xi, \eta], [\xi', \eta']) + d([\xi, \eta'], [\xi', \eta])\right)^{-1},$$

where $[\cdot, \cdot]$ denotes the unique geodesic between two distinct points in ∂X and the “distance” d between two geodesics is just the infimum of distances of their points. Observe that (1) is well-defined and extends continuously to a map $D_2 \times D_2 \rightarrow \mathbf{R}_+$ as sought (this is a crude analogue of symplectic cross-ratios).

Going back to the general case, we note that the sets $S(q, t) = \{q' \in D_2 : \delta(q, q') < t\}$ are neighbourhoods of $q \in D_2$ when $t > 0$. By ergodicity of $\Omega \times B \times B$, there is $r \geq 0$ such that $\delta(f_F(z, x), f_F(z, x')) = r$ for almost every (z, x, x') in $\Omega \times B \times B$. If r were positive, consider for $z \in \Omega$ the image measure $\beta_z = f_F(z, \cdot)_* \beta$ on D_2 . For almost every $(z, x) \in \Omega \times B$, there is $0 < t < r$ such that $\beta_z S(f_F(z, x), t) = 0$; this is impossible since $\beta_z \neq 0$.

Thus, $r = 0$ and therefore, $f_F(z, x) \cap f_F(z, x') \neq \emptyset$ almost everywhere; let $a \in \{1, 2\}$ be the essentially constant number of points in this intersection. If $a = 2$, then f_F does not depend on B and therefore, the resulting map $\Omega \rightarrow D_2$ makes α -elementary, contradiction. Thus, $a = 1$. By Fubini Lebesgue, we see that for almost every fixed (z, x) either $f_F(z, x) \cap f_F(z, x')$ consists of the same point for almost all x' or $f_F(z, x) \cup f_F(z, x') \cup f_F(z, x'')$ contains exactly three points for almost all x', x'' since $f_F(z, x') \cap f_F(z, x'') \neq \emptyset$; in particular this triple is determined by z . By ergodicity on $\Omega \times B$, one of the cases occurs almost surely. In the first case, we get a α -equivariant map $\Omega \rightarrow \partial X$, again contradicting non-elementarity. In the second case, we get a map into triples of points, which in view of point (i) in Lemma 3.2 is also a contradiction. q.e.d.

Writing now f for the resulting $\tilde{\alpha}$ -equivariant measurable map, $f : \Omega \times B \rightarrow \partial X$ finishes the proof of Proposition 3.3. q.e.d.

Let $H_0 = \text{Isom}(X)$, $\mathcal{H} = \bigoplus_{n=1}^{\infty} L^2(H_0)$ and $\omega : (\partial X)^3 \rightarrow \mathcal{H}$ be as in Theorem 1.5 (or [29] for hyperbolic cases). Define the G -representation $\mathcal{V} = L^2(\Omega, \mathcal{H})$ by the α -twisted representation. We may assume \mathcal{V} separable. Define $\eta : B^3 \rightarrow \mathcal{V}$ by

$$\eta(x_0, x_1, x_2)(z) = \omega(f(z, x_0), f(z, x_1), f(z, x_2))$$

so that η is a G -equivariant measurable essentially bounded alternating cocycle.

Lemma 3.6. *The cocycle η yields a non-trivial class in $H_{\text{cb}}^2(G, \mathcal{V})$.*

Proof of Lemma 3.6. In view of Corollary 2.6, it is enough to show $\eta \neq 0$. Otherwise, by Fubini–Lebesgue and the non-vanishing statement of Theorem 1.5 and [29], for almost every $z \in \Omega$, the image measure $f(z, \cdot)_* \beta$ would be supported on a set of at most two points in ∂X . Passing to the uniformly distributed probability on this set, we obtain a measurable α -equivariant measurable map $\psi : \Omega \rightarrow \mathcal{P}_{\leq 2}(\partial X)$. Thus, α is elementary, contradiction. q.e.d.

Applying now Theorem 2.10, we deduce upon possibly reordering the factors G_j that $H_{\text{cb}}^2(G_1, \mathcal{V}^{G_1'})$ is non-trivial; in particular $\mathcal{V}^{G_1'} \neq 0$. Hence, we have a non-zero measurable map

$$F : \Omega \longrightarrow \mathcal{H} = \bigoplus_{n=1}^{\infty} L^2(H_0)$$

which is $\alpha|_{G'_1}$ -equivariant with respect to the diagonal regular H_0 -representation on the Hilbertian sum on the right-hand side. Since G'_1 acts ergodically on Ω , we can apply cocycle reduction [42, 5.2.11] to $\alpha|_{G'_1}$ and deduce that α is equivalent to a cocycle ϑ such that $\vartheta(G'_1 \times \Omega) \subseteq L$ for the stabilizer $L < H$ of a non-zero element in \mathcal{H} . In view of the definition of \mathcal{H} , the group L is compact.

Our next purpose is to find a minimal such group L . Let Θ be the set of pairs (η, M) consisting of a compact subgroup $M < L$ and a cocycle $\eta \sim \alpha$ such that $\eta(G'_1 \times \Omega) \subseteq M$. Endow Θ with the pre-order given by inclusion of these groups M .

Lemma 3.7. *The set Θ contains a minimal element (α', K) .*

Proof Lemma 3.7. By Zorn's lemma, there is a maximal chain $\Sigma \subseteq \Theta$. Consider $\vartheta|_{G'_1}$ as a cocycle

$$G'_1 \times \Omega \longrightarrow N = \prod_{(\eta, M) \in \Sigma} L/M$$

via the diagonal L -action on N . Since $\eta \sim \alpha \sim \vartheta$, each η comes with a map $\varphi : \Omega \rightarrow L$ such that $\varphi(g'z)\eta(g', z) = \vartheta(g', z)\varphi(z)$ for $g' \in G'_1$. The resulting product map $\Phi : \Omega \rightarrow N$ is thus $\vartheta|_{G'_1}$ -equivariant. The L -action on N is tame (technically, the compactness of L implies tameness if N is separable; but since H is second countable, there is no loss of generality in assuming that); thus, we deduce by cocycle reduction that ϑ is cohomologous to some α_1 such that $\alpha_1|_{G'_1}$ ranges in the stabilizer $K_1 < L$ of some element $\nu = (\ell_{(\eta, M)}M)_{(\eta, M) \in \Sigma}$ of N . Since $(\alpha_1, K_1) \in \Theta$, it remains only to show that some L -conjugate $K = \ell^{-1}K_1\ell$ is contained in $M_\infty = \bigcap_{(\eta, M) \in \Sigma} M$ and to take for α' the corresponding conjugate of α_1 . View ν as a net in L indexed by the inverse order of Θ . Since L is compact, there is a directed set I and a converging subnet $(\ell_i)_{i \in I}$ of ν ; let $\ell \in L$ be the limit. By definition, $K_1 < \ell_i M_i \ell_i^{-1}$ for all $i \in I$. It follows that for any $i_0 \in I$, one has $\ell_i^{-1} K_1 \ell_i < M_{i_0}$ for i large enough. Thus, $\ell^{-1} K_1 \ell$ is in each M_{i_0} and hence in M_∞ . q.e.d.

We now define $H' < H$ to be the normalizer of K in H .

Proposition 3.8. *After discarding a null-set, α' ranges in H' .*

Proof Proposition 3.8. Since $\alpha'|_{G'_1 \times \Omega}$ ranges in K , we only need to show that $\alpha'|_{G_1 \times \Omega}$ ranges in H' . We shall consider the α' -twisted G -action on the set of maps $\psi : \Omega \rightarrow H/K$ defined by $(g\psi)(z) = \alpha'(g^{-1}, z)^{-1}\psi(g^{-1}z)$ for $g \in G$ and $z \in \Omega$. The map $\psi : \Omega \rightarrow H/K$ taking constant value $K \in H/K$ is $\alpha'|_{G'_1}$ -equivariant. Note that an α' -equivariant map may be viewed as a fixed element for the α' -twisted action on maps $\Omega \rightarrow H/K$. Thus, by the commutativity of G_1 and G'_1 , it follows that the set of G'_1 fixed maps is invariant under G_1 , namely, $g_1\psi$ is also $\alpha'|_{G'_1}$ -equivariant. In view of the G'_1 -ergodicity on Ω , we may apply cocycle reduction to $\alpha'|_{G'_1} : G'_1 \times \Omega \rightarrow K$. Thus, we deduce that $g_1\psi$ ranges in a single K -orbit Kh_0K in H/K and that α' is equivalent to a cocycle mapping $G'_1 \times \Omega$ to the stabilizer in K of $h_0K \in H/K$; that is, to $K \cap h_0Kh_0^{-1}$. By the minimality established in Lemma 3.7, we must have $h_0Kh_0^{-1} = K$. Hence, $h_0 \in H'$ and thus $(g_1\psi)(z) = \alpha'(g_1^{-1}, z)^{-1}K$ is in H'/K for all $g_1 \in G_1$ and almost every $z \in \Omega$. q.e.d.

We may now consider the cocycle $\bar{\alpha} : G \times \Omega \rightarrow H'/K$ obtained by composing α' with the map $H' \rightarrow H'/K$. Then, $\bar{\alpha}|_{G'_1}$ is trivial and for $g' \in G'_1, g_1 \in G_1, z \in \Omega$, we have

$$\begin{aligned} \bar{\alpha}(g_1, g'z) &= \bar{\alpha}(g_1, g'z)\bar{\alpha}(g', z) = \bar{\alpha}(g_1g', z) \\ &= \bar{\alpha}(g'g_1, z) = \bar{\alpha}(g', g_1z)\bar{\alpha}(g_1, z) = \bar{\alpha}(g_1, z). \end{aligned}$$

Since G'_1 is ergodic on Ω , we conclude that $\bar{\alpha}$ does not depend on Ω and thus it is indeed a (measurable hence continuous) homomorphism from G to H'/K factoring through G_1 . This concludes the proof of Theorem 1.2. q.e.d.

3.3. Proof of Theorem 1.3 and additional statements. The following is a natural generalization of the notion of (non-) elementary subgroups of $\text{Isom}(X)$ which will be useful for the proof of Theorem 1.3.

Definition 3.9. Let J be an arbitrary locally compact second countable group admitting a closed subgroup of finite invariant co-volume $H < J$ such that H is a closed subgroup of isometries of a proper GNC space X . An arbitrary subgroup $\Lambda < J$ is called H -elementary if the natural cocycle $\eta : \Lambda \times J/H \rightarrow H$ is elementary (according to Definition 1.1). A homomorphism $f : \Gamma \rightarrow J$ is said to be H -elementary if $\Lambda = f(\Gamma)$ is H -elementary.

Example 3.10. If $\overline{f(\Gamma)}$ contains H and H is non-elementary, then f is not H -elementary.

Indeed, we may assume, upon replacing J with $L = \overline{\Lambda}$ for $\Lambda = f(\Gamma)$ and since $H < L$, that $J = L$. Thus, we need to show that a dense subgroup $\Lambda < J$ is not H -elementary when H is non-elementary. Indeed, suppose it were, and let F be a measurable map from J/H to bounded sets in X or finite sets in ∂X such that F is η -invariant for the Λ -action on J/H . By measurability of F and density of Λ , F is also η -invariant for the J -action on J/H . However, by Proposition 4.2.19 in [42], this implies that Λ is elementary, in contradiction to our assumption.

Keep the notation of Definition 3.9. We shall prove the following general result, and then show how to deduce Theorem 1.3 from it.

Theorem 3.11. *Let $\Gamma < G = \prod G_i$ be an irreducible lattice in a product of any locally compact σ -compact groups and let $f : \Gamma \rightarrow J$ be a homomorphism that is not H -elementary.*

Then, there is a compact normal subgroup $M \triangleleft L = \overline{f(\Gamma)}$ such that the induced homomorphism $\Gamma \rightarrow L/M$ extends continuously to G , factoring through some G_i .

Remark 3.12. Notice that the factor G_j appearing in the conclusion of Theorem 1.2 or Theorem 3.11 must have the following cohomological property: *there is a unitary G_j -representation (π, \mathcal{H}) without non-zero invariant vectors, such that $H_{\text{cb}}^2(G_j, \mathcal{H}) \neq 0$.* Indeed, keep the notation of the preceding section and of Theorem 1.2; recall that we choose $j = 1$. We may take for \mathcal{H} the G_1 -representation \mathcal{V}^{G_1} , which we claim has no non-zero invariant vectors. Indeed, we would otherwise have a non-zero G -invariant element of \mathcal{V} , which means an α -equivariant map $\Omega \rightarrow \mathcal{H}$. By cocycle reduction, this contradicts non-elementarity.

In fact, our proof of Theorem 1.2 also shows.

Corollary 3.13. *Let G be any locally compact σ -group, with the property that $H_{\text{cb}}^2(G, \mathcal{H}) = 0$ for every unitary representation \mathcal{H} without non-zero G -invariant vectors.*

Then, for any measure preserving G -action on a standard probability space Ω and every proper GNC space X , every cocycle $\alpha : G \times \Omega \rightarrow \text{Isom}(X)$ is elementary.

Recall that if G also satisfies the abstract cohomological vanishing property (T), then “elementary” can be replaced by “bounded” in many cases (see point I following 1.2).

Proof of Corollary 3.13. Suppose for a contradiction that α is non-elementary. Observe that up to Lemma 3.6 (included) we did not use at all the product structure of G or the irreducibility assumption; thus we have $H_{cb}^2(G, \mathcal{V}) \neq 0$. The assumption on G yields $\mathcal{V}^G \neq 0$; this entails a contradiction as in Remark 3.12. q.e.d.

Remark 3.14. Juxtaposing Corollary 3.13 with Theorem 1.4, one recovers in particular Adams’ cocycle reduction for higher rank simple groups [1]. Recall that this result has been generalized by Gao [15] to the non-proper case, which we do not address here.

Proof of Theorem 3.11. Keep the notation of the theorem and of Definition 3.9. The key point is to have a replacement for Theorem 1.5; in fact, we can adapt the arguments used in Section 2 to show in the following lemma.

Lemma 3.15. *Let $\Lambda < J$ be any countable subgroup that is not H -elementary. Then, $H_b^2(\Lambda, L^2(J))$ does not vanish.*

Proof of the lemma. Let $\Omega = J/H$ and consider the cocycle $\eta : \Lambda \times \Omega \rightarrow H$ of Definition 3.9. The proof given above for Corollary 3.13 applies *verbatim* here and shows that we have $H_b^2(\Lambda, \mathcal{V}) \neq 0$ for the η -twisted representation $\mathcal{V} = L^2(\Omega, L^2(H))$. But in the present case, it is easy to see that \mathcal{V} is no other than $L^2(J)$ restricted to Λ . q.e.d.

To finish the proof of Theorem 3.11, let $f : \Gamma \rightarrow J$ be a homomorphism that is not H -elementary. By Lemma 3.15, $H_b^2(f(\Gamma), L^2(J))$ is non-zero, which by [30, 12.4.2] implies that $H_b^2(\Gamma, \pi) \neq 0$ where (π, V_π) is the unitary Γ -representation obtained by pulling back to Γ the restriction of $L^2(J)$ to $f(\Gamma)$. We can now continue with the arguments of the proof of Theorem 1.2, starting right after Lemma 3.6, but now with the cocycle $\alpha_f : G \times G/\Gamma \rightarrow L = f(\Gamma)$ induced by the choice of some Borel section $G/\Gamma \rightarrow G$. The equivalence between the cocycle superrigidity for this α_f and for f is fairly standard [42]. More directly, it follows from $H_b^2(\Gamma, \pi) \neq 0$ and the product superrigidity for bounded cohomology of irreducible lattices [9] that there is some non-zero Γ -invariant submodule of V_π on which the Γ -representation extends continuously to G , factoring through some G_i . It is shown in the proof of Theorem 0.3 of [40] exactly how this information can be used to deduce the required conclusion of Theorem 3.11.

Finally, one technicality that may arise here is that $\Omega = G/\Gamma$ could be a non-standard probability space if G were not second countable.

However, since G is σ -compact, there is a compact normal subgroup $C \triangleleft G$ such that G/C is second countable, see Satz 6 in [26]; one can moreover, choose C such that $C \cap \Gamma = 1$, so that Γ is a lattice in G/C . We can now run the whole argument for $\Gamma < G/C$ and deduce the statement for G itself. This concludes the proof of Theorem 3.11. q.e.d.

Proof of Theorem 1.3. Keep the notation of Theorem 1.3 and set $\Lambda = f(\Gamma)$. As in the proof of Theorem 3.11, all we have to show is that the space $H_b^2(\Lambda, L^2(J))$ does not vanish. For the first assertion, we need to show that if L is non-amenable, then (under the growth assumption) Λ is not H -elementary. In the notation of Definition 3.9, we have in the following lemma.

Lemma 3.16. *If $\eta : \Lambda \times J/H \rightarrow H$ is elementary, then the Mackey range of η is amenable.*

Proof of the lemma. It is a result of Adams (see 6.8 in [1]) that the H -action on ∂X , hence also on $D_2(\partial X)$, is universally amenable. On the one hand, the existence of an η -equivariant map $J/H \rightarrow \partial X \sqcup D_2(\partial X)$ forces the Mackey range of η to be amenable. On the other, an equivariant map to $B(X)$ would by cocycle reduction make η -cohomologous to a cocycle into a compact group, in which case the Mackey range is again amenable. q.e.d.

However, in our setting the Mackey range of η is simply the H -action on J/L . But if the latter action is amenable, then it follows by Corollary 4.3.7 of [42] that L is amenable since H has finite co-volume in J ; a contradiction.

For the additional assertion where $H \subseteq L$, suppose first that X is proper. Then, in view of Example 3.10, we are in a particular case of Theorem 3.11. The only non-proper case is when X is a tree of infinite valence. But then, the arguments of Section 4 allow us to follow the same scheme starting with $H_{cb}^2(H, L^2(H)) \neq 0$ (compare Remark 7.12). q.e.d.

Justification of point I following Theorem 1.2. For the first claim, apply Lemma 3.16 to the cocycle $\alpha : G \times \Omega \rightarrow H$.

If now, moreover, G is Kazhdan, then it follows from an argument of Spatzier and Zimmer [41] (see 4.4 in [1]) that the amenability of the Mackey range makes α -cohomologous to a cocycle into a compact group K anyway; then, the theorem holds trivially. q.e.d.

Justification of point II following Theorem 1.2. Assume that X is a symmetric space (of rank one by the $\text{CAT}(-1)$ assumption) and that α is elementary. As mentioned above, an α -equivariant map to X makes α -cohomologous to a cocycle into a compact group. On the other hand, since in the present case the $\text{Isom}(X)$ -action is transitive on ∂X as well as on $D_2(\partial X)$, an equivariant map to $\partial X \sqcup D_2(\partial X)$ would make α cohomologous to a cocycle into the stabilizer of a point (or couple) at infinity, which is an amenable group (here, it is even a compact extension of a soluble group).

For the converse, we may assume that α ranges in an amenable subgroup $A < \text{Isom}(X)$. Thus, there is an A -invariant probability measure on ∂X . Upon possibly applying a centre of mass, we get an A -invariant point in $\overline{X} \sqcup D_2(\partial X)$. Considering it as a constant function, we see that α is elementary. q.e.d.

As an illustration of Theorem 1.3, we propose the following result for Kac–Moody groups (for background, see [34], [36], [35], [37]).

Corollary 3.17. *Let Γ be a Kac–Moody group over \mathbf{F}_q and assume that the entries of its Cartan matrix are finite.*

Then, for q large enough, any Γ -action by isometries on any bounded geometry Riemannian manifold with sectional curvature ≤ -1 has a global fixed point.

Proof. Assume Γ acts on a bounded geometry Riemannian manifold M with sectional curvature ≤ -1 . Passing possibly to a totally geodesic submanifold M' , we may reduce to the case where there is no non-empty convex Γ -invariant subset in M' (except M' itself). Let Δ_+, Δ_- be the twin buildings associated to Γ and G_\pm the closure of the image of Γ in $\text{Aut}(\Delta_\pm)$. Rémy shows [34], [35] that for q large enough (e.g., larger than the order of the generalized Cartan matrix) the diagonal map $\Gamma \rightarrow G = G_+ \times G_-$ realizes the quotient $\overline{\Gamma}$ of Γ by its (finite) centre as a (non-uniform) lattice in G . This lattice is irreducible by definition of G_\pm . By our minimality assumption, the finite centre acts trivially on M' , so we may abuse notation in replacing $\overline{\Gamma}$ with Γ . On the other hand, owing to the assumption on the entries of the Cartan matrix, the groups G_\pm (and hence G) are Kazhdan for q large enough by a result of Dymara–Januszkiewicz [12] (see also [13] and [35]). Therefore, applying Theorem 1.2 or Theorem 1.3, we find that the Γ -action on M' with sectional curvature ≤ -1 must extend continuously to G and

factor through, say, G_+ . Since M' is a manifold, $\text{Isom}(M')$ has no small subgroups. Therefore, since G_+ is totally disconnected, we deduce that its image in $\text{Isom}(M')$ is discrete. On the other hand, Rémy has proved (private communication) that G_+ is topologically simple, therefore the image of G_+ in $\text{Isom}(M')$ is trivial (and thus M' is a point). q.e.d.

3.4. Additional examples for Theorem 3.11. The idea is to construct discrete groups H of isometries of GNC spaces, but such that the isometry group J of their Cayley graph (with respect to an appropriate finite generating set) is non-discrete. Since J , taken with the compact-open topology, is a locally compact second countable group containing H discretely and co-compactly, Theorem 3.11 readily applies to it. We shall construct uncountably many such groups J , arising, among other constructions, from uncountably many different freely decomposable groups H . We remark that as far as totally disconnected groups J are concerned, this construction is actually the most general one, in the sense that every such group arises as the isometry group of some graph – its (generalized) Cayley graph.

To find such Cayley graphs, we use groups H which are free products of two groups A, B to be specified later. Assume that A and B are generated by finite subsets S, T respectively and denote by $\text{Cay}_S(A)$ and $\text{Cay}_T(B)$ the corresponding Cayley graphs. Let $K_A < \text{Isom}(\text{Cay}_S(A))$, $K_B < \text{Isom}(\text{Cay}_T(B))$ be the subgroups which fix the identity element.

Consider now the group $H = A * B$ and its generating set $U = T \cup S$. Observe that the resulting Cayley graph $\text{Cay}_U(H)$ admits the (compact) subgroup $K_A^{\mathbf{N}} = \prod_{\mathbf{N}} K_A$ in the stabilizer of the identity element. Indeed, for every sequence $\Psi = (\psi_1, \psi_2 \dots) \in K_A^{\mathbf{N}}$, $\Psi(a_1 b_1 a_2 b_2 \dots) = (\psi_1(a_1) b_1 \psi_2(a_2) b_2 \dots)$ defines such an isometry. Similarly, $\text{Isom}(H)$ contains $K_B^{\mathbf{N}}$. We note that although $K_A^{\mathbf{N}}$ and $K_B^{\mathbf{N}}$ commute, once one of them is non-trivial, it does not commute with, nor is normalized by, the left action of $H = A * B$ on itself. Therefore, as soon as the isometry group of one of $\text{Cay}_S(A)$ or $\text{Cay}_T(B)$ contains at least one non-trivial isometry fixing the identity, the group $J = \text{Isom}(\text{Cay}_U(H))$ becomes a locally compact group which is not discrete – nor a compact extension of such. We also remark that J may still be much larger than the group generated by $H, K_A^{\mathbf{N}}$ and $K_B^{\mathbf{N}}$.

There are various examples of groups A which admit a non-trivial isometry fixing the identity, the simplest of which being the infinite cyclic group \mathbf{Z} , or more generally, any free group. This is useful to

us because it is easy to see that if B is either any discrete group of isometries of X in the classes (i), (ii) or (iii) of Definition 1.1, or admits a non-elementary action on a tree that is proper on the edges, then $H = F * B$ is in the same class for any free group F . Our foregoing discussion now applies to H and $J = \text{Isom}(H)$. Note that in this way, we can find such J which contain the isometry group of a tree or, more generally, by taking groups A acting simply transitively on the vertices of appropriate buildings, one can find examples where J contains $\mathbf{PGL}_n(F_p(\!(t)\!))$. In fact, for any group B , e.g., an irreducible lattice Γ in a semisimple Lie group G , $F * B$ has obviously an appropriate action on a tree, which shows both that the Theorems apply to uncountably many groups J , and that the non-amenability assumption on $\overline{f(\Gamma)}$ is not sufficient in general for the conclusion of Theorems 1.3 and 4.1 to hold. Other interesting examples to keep in mind are when A and/or B are taken to be $\mathbf{SL}_n(\mathbf{Z})$ with respect to the unit elementary matrices. Here, the symmetric group S_n acts isometrically, fixing the identity, on the associated Cayley graph, thereby generating a copy of $S_n^{\mathbf{N}}$ in $J = \text{Isom}(A * B)$.

4. Groups acting on trees

The main goal of this section is to prove the following.

Theorem 4.1. *Let G_1, \dots, G_n be locally compact σ -compact groups and let $\Gamma < G = G_1 \times \dots \times G_n$ be an irreducible lattice acting non-elementarily on a simplicial tree T .*

Then, there is an invariant subtree on which the Γ -action extends continuously to a G -action, factoring through some G_j .

We adopt Serre’s notation [39], in which a graph $T = (V, E)$ is given by a vertex set V , a set of edges E , an involution $e \mapsto \bar{e} \neq e$ of E and source/terminus maps $s, t : E \rightarrow E$ such that $s(\bar{e}) = t(e)$. We define the set \underline{E} of unoriented edges by $\underline{E} = \{\{e, \bar{e}\} : e \in E\}$. We denote the metric realization of T by $|T|$, while V and E are endowed with the discrete topology. A non-empty graph T is called a tree if $|T|$ is connected and simply connected.

4.1. Compactifying trees. Notice that we can assume T countable in Theorem 4.1 upon passing to the invariant subtree spanned by a Γ -orbit.

From now on, let $T = (V, E)$ be a countable tree.

Recall that to the (discrete) space V one associates a complete metrizable space $\bar{T} = V \sqcup \partial T$ where ∂T is the set of classes of cofinal rays

endowed with the cone topology and V is open dense in \overline{T} . (One can also consider $\overline{|T|} = |T| \sqcup \partial T$, in which case, this is a particular case of the bordification of complete CAT(-1) spaces [4, II.8].) We call this topology on \overline{T} the strong topology.

To any edge $e \in E$ corresponds its shadow, or half-space, $S_e \subseteq V$ defined to be the set of vertices

$$S_e = \{v \in V : d(t(e), v) < d(s(e), v)\}.$$

The extended shadow \overline{S}_e is the closure of S_e in \overline{T} ; let Σ be the set of shadows and $\overline{\Sigma}$ of extended shadows. Let σ be the topology on \overline{T} generated by $\overline{\Sigma}$ (i.e., the smallest topology with $\sigma \supseteq \overline{\Sigma}$) and denote by \overline{T}^σ the resulting topological space; let V^σ be the set V endowed with the induced topology, which is generated by Σ .

This new topology on the same underlying spaces is clearly weaker than the strong topology. Moreover in the following proposition.

Proposition 4.2. *The space \overline{T}^σ is compact.*

Since σ is second countable by definition, it follows that \overline{T}^σ is compact metrizable. Moreover, the $\text{Aut}(T)$ -action on the set \overline{T} is by homeomorphisms of \overline{T}^σ since it preserves $\overline{\Sigma}$.

Proof of Proposition 4.2. Let \mathcal{M} be the set of finitely additive probability measures on the Boolean algebra generated by $\overline{\Sigma}$. Thus, \mathcal{M} is compact for the weak-* topology, and moreover, the subset $\mathcal{U} \subseteq \mathcal{M}$ of $\mu \in \mathcal{M}$ taking only 0-1 values is closed in \mathcal{M} . Since the map taking $x \in \overline{T}^\sigma$ to the Dirac mass δ_x is by definition a continuous homeomorphism onto its image, it is enough to show that it is onto \mathcal{U} . Pick $\mu \in \mathcal{U}$ and observe that the set \mathcal{O} of edges with $\mu(\overline{S}_e) = 1$ is an orientation of T since for all $e \in E$, we have $\overline{T} = \overline{S}_e \sqcup \overline{S}_{\bar{e}}$. For every $v \in V$, there is at most one edge e with $s(e) = v$ and $e \in \mathcal{O}$ since all such shadows are disjoint. If there is always one such edge, then using connectedness one verifies that \mathcal{O} determines a point x in ∂T , and moreover, $\mu = \delta_x$ because $\{x\} = \bigcap_{e \in \mathcal{O}} \overline{S}_e$. If on the other hand, there is v such that no edge e with $s(e) = v$ is in \mathcal{O} , then one checks similarly that this v is unique and again $\mu = \delta_v$, because $\{v\} = \bigcap_{e \in \mathcal{O}} \overline{S}_e$. q.e.d.

Remark 4.3. The above argument showing that “every ultrafilter is principal” will be needed again in Proposition 4.5. As far as compactness only is concerned, one can also check directly that any cover of \overline{T}^σ by elements of $\overline{\Sigma}$ admits a finite subcover, and then conclude with Alexander’s lemma.

Here are some elementary properties of the topology σ .

Proposition 4.4.

- (i) *the tautological continuous bijection $\overline{T}^\sigma \rightarrow \overline{T}$ is a homeomorphism if and only if T is locally finite;*
- (ii) *the two Borel structures coincide;*
- (iii) *the two topologies coincide on ∂T ;*
- (iv) *the two topologies coincide on any locally finite subtree;*
- (v) *the closure in V^σ of the “unit sphere” U around a vertex v of infinite valence is $U \cup \{v\}$;*
- (vi) *suppose T has no leaves. Then, the closure of ∂T in \overline{T}^σ consists of ∂T together with the vertices of infinite valence;*
- (vii) *the space \overline{T}^σ is totally disconnected; if every vertex has infinite valence, then \overline{T}^σ is a Cantor space;*

(It is in view of (iii) that we did not introduce a separate notation for ∂T viewed in \overline{T}^σ .)

Proof. Point (iii) follows from the definition of the cone topology, so to prove (ii), it remains to check that each $v \in V$ is Borel for σ , which follows from $\{v\} = \bigcap_{t(e)=v} \overline{S}_e$. To prove (vi), observe first that every vertex of finite valence is isolated because again of the expression $\bigcap_{t(e)=v} \overline{S}_e$ (which settles (iv) as well). Let now $v \in V$ be of infinite valence and fix an enumeration (without repetitions) $(e_n)_{n=0}^\infty$ of the edges with $s(e_n) = v$. Then, each \overline{S}_{e_n} contains some $x_n \in \partial X$ because there are no leaves; by Proposition 4.2, this sequence must have a cluster point $y \in \overline{T}^\sigma$ and $y = v$ since \overline{S}_{e_n} is a partition of $\overline{T}^\sigma \setminus \{v\}$. Point (v) follows from a similar argument. As for (vii), observe that any \overline{S}_e is closed since its complement is $\overline{S}_{\bar{e}}$; under the valence assumption, isolated points in V^σ are ruled out by (v) and ∂X has none, anyway, if the valence is always at least three. Point (i) follows from the others. q.e.d.

Proposition 4.5 will serve a purpose analogous to the centre of mass argument in the locally compact case; we use as before the notation D_2 for unordered pairs of distinct elements.

Proposition 4.5. *There is a measurable $\text{Aut}(T)$ -equivariant map*

$$\mathcal{P}(\overline{T}^\sigma) \longrightarrow \underline{E} \sqcup \overline{T} \sqcup D_2(\partial T).$$

Proof. For every probability measure $\mu \in \mathcal{P}(\overline{T}^\sigma)$, define $E_\mu \subseteq E$ by

$$E_\mu = \left\{ e \in E : \mu(\overline{S}_e) = \mu(\overline{S}_{\bar{e}}) \right\}$$

and write \mathcal{P}' for the set of $\mu \in \mathcal{P}(\overline{T}^\sigma)$ with $E_\mu = \emptyset$ and \mathcal{P}'' for those with $E_\mu \neq \emptyset$. We have an equivariant map $F : \mathcal{P}' \rightarrow \mathcal{U}$ defined by

$$F\mu(\overline{S}_e) = 1 \iff \mu(\overline{S}_e) > 1/2$$

for $\mathcal{U} \cong \overline{T}^\sigma$ as in the proof of Proposition 4.2. Let, on the other hand, $\mu \in \mathcal{P}''$. Since $\mu(\overline{T}^\sigma) = 1$, every vertex of the subgraph $T_\mu < T$ spanned by E_μ has valence at most two. Since for any edge e between $e_1, e_2 \in E_\mu$ each of the two shadows $\overline{S}_e, \overline{S}_{\bar{e}}$ contains one of the four $\overline{S}_{e_i}, \overline{S}_{\bar{e}_i}$, we deduce $e \in E_\mu$ and hence the latter is connected. Therefore, E_μ yields either (i) a segment of even length, (ii) a segment of odd length, (iii) a ray or (iv) a geodesic. In case (i), we associate the middle vertex to μ ; in case (ii), the middle unoriented edge; in case (iii), the class of the ray in ∂T and for (iv), we take the element of $D_2(\partial T)$ corresponding to the geodesic. Putting everything together, we have an equivariant map as in the statement and it is Borel with respect to the weak-* topology on $\mathcal{P}(\overline{T}^\sigma)$ and the common Borel structure on the right-hand side. q.e.d.

4.2. The cocycle ω . The construction given below can be considered as a baby-case of the cocycle that we construct in Section 5 for $\text{CAT}(-1)$ spaces.

We write $E \curlywedge E$ for the set of pairs of successive non-backtracking edges, that is,

$$E \curlywedge E = \left\{ (e, e') \in E \times E : t(e) = s(e'), \bar{e} \neq e' \right\}.$$

Thus, $\ell^2(E \curlywedge E)$ is a subrepresentation of the continuous unitary representation of $\text{Aut}(T)$ on $\ell^2(E \times E)$.

Proposition 4.6. *Let $T = (V, E)$ be a countable tree. There is a weakly continuous $\text{Aut}(T)$ -equivariant alternating bounded cocycle*

$$\omega : \partial T \times \partial T \times \partial T \longrightarrow \ell^2(E \curlywedge E)$$

whose restriction to the distinct triples $\partial^3 T$ vanishes nowhere.

Proof. We define a map $\alpha : \overline{T}^2 \rightarrow C(E \curlywedge E)$ to the space of functions on $E \curlywedge E$ by

$$\alpha(x, y)(e, e') = \begin{cases} 1 & \text{if } x \in \overline{S}_{\bar{e}} \text{ and } y \in \overline{S}_{e'}, \\ -1 & \text{if } y \in \overline{S}_{\bar{e}} \text{ and } x \in \overline{S}_{e'}, \\ 0 & \text{otherwise.} \end{cases}$$

We see that α is $\text{Aut}(T)$ -equivariant and alternating; in particular, $\alpha(x, x)$ vanishes for all $x \in \overline{T}$. Moreover, $\alpha(x, y)$ vanishes if x, y are adjacent vertices.

More precisely, $\alpha(x, y)$ assigns 1 to the pairs of consecutive edges contained in a geodesic from x to y , assigns 1 to the pairs of consecutive edges contained in the reverse geodesic and 0 to all others. We define now

$$\omega = da : \overline{T} \times \overline{T} \times \overline{T} \longrightarrow C(E \curlywedge E),$$

that is, we set $\omega(x_0, x_1, x_2) = \alpha(x_1, x_2) - \alpha(x_0, x_2) + \alpha(x_0, x_1)$. In particular, ω is an alternating $\text{Aut}(T)$ -cocycle. We claim that ω (contrary to α) ranges in $\ell^2(E \curlywedge E)$. As a matter of fact, ω ranges even in the finitely supported functions on $E \curlywedge E$.

Lemma 4.7. *For all $x_0, x_1, x_2 \in \overline{T}$, the support of $\omega(x_0, x_1, x_2)$ contains at most six pairs of edges. If the x_i are distinct and all in ∂T , then the support of $\omega(x_0, x_1, x_2)$ contains exactly six pairs of edges.*

Proof. Easy verification. In the generic case, the x_i span a (possibly infinite) tripod in T . Denoting by $v \in V$ the forking point of this tripod and by e_i the edge pointing to x_i with $s(e_i) = v$, we find that $\omega(x_0, x_1, x_2)$ is 1 on the pairs (\bar{e}_1, e_2) , (\bar{e}_2, e_0) and (\bar{e}_0, e_1) ; it is -1 on the pairs (\bar{e}_2, e_1) , (\bar{e}_0, e_2) and (\bar{e}_1, e_0) ; it is zero on all other pairs. q.e.d.

Since the function $\omega(x_0, x_1, x_2)$ is bounded on $E \curlywedge E$ independently of x_i , we conclude from this lemma that ω ranges indeed in $\ell^2(E \curlywedge E)$ and is bounded. The continuity being immediate, this concludes the proof of Proposition 4.6 upon restricting ω to ∂X . q.e.d.

At this point, we observe that we could have defined ω directly in terms of the description given in the proof of Lemma 4.7, which would have left us with the task to verify $d\omega = 0$. However, we believe that the above presentation sheds more light on the general construction given in Section 5.

Remark 4.8. Since our cocycle ω is actually defined on the compact space \overline{T}^σ , we can apply Proposition 2.12 and deduce that ω defines canonically a class $\kappa = j[\omega]$ in $H_b^2(\text{Aut}(T), \ell^2(E \curvearrowright E))$, where it is understood that we endow $\text{Aut}(T)$ with the discrete topology.

4.3. Superrigidity for actions on T . Recall that a group action on a tree T is called elementary if it fixes an element of $\overline{T} \sqcup D_2(\overline{T})$. This is clearly equivalent to fixing an element of $\underline{E} \sqcup \overline{T} \sqcup D_2(\partial T)$.

Retain the notation of Theorem 4.1. By Theorem 2.5, we have strong boundaries (B_i, β_i) for G_i . Consider the strong boundary $(B, \beta) = B_1 \times \cdots \times B_n$ for G ; by Lemma 2.8, the space B is also a strong boundary for Γ . Since \overline{T}^σ is a compact metrizable Γ -space by Proposition 4.2, there is a Γ -equivariant measurable Furstenberg map [42, 4.3.9] $f_F : B \rightarrow \mathcal{P}(\overline{T}^\sigma)$. Let

$$f : B \longrightarrow \underline{E} \sqcup \overline{T} \sqcup D_2(\partial T)$$

be the measurable G -map obtained by composing f_F with the map of Proposition 4.5. Write $D_2 = D_2(\partial T)$.

Lemma 4.9. *After discarding a null-set, f ranges in ∂T .*

Proof. By Γ -ergodicity on B , we have to rule out the following possibilities: (i) f ranges essentially in \underline{E} ; (ii) f ranges essentially in V ; (iii) f ranges essentially in D_2 . In the first two cases, the Γ -ergodicity on B^2 applied to the composition of $f \times f$ with a combinatorial distance function on V^2 or \underline{E}^2 would yield a bounded Γ -invariant subset. Passing to the ‘‘circumcentre’’ (which might be in both cases either an unoriented edge or a vertex), we contradict non-elementarity of Γ . In the third case, we can argue in a way similar to Lemma 3.5: There is some continuous $\text{Aut}(T)$ -invariant map $\delta : D_2 \times D_2 \rightarrow \mathbf{R}_+$ such that $\delta(q, q') = 0$ if and only if $q \cap q' \neq \emptyset$; indeed, the formula (1) of Lemma 3.5 makes sense for T as well. Since Γ is ergodic on $B \times B$, the map $\delta(f, f)$ is essentially constant; let r be the essential value. If $r > 0$, we get a contradiction because one can cover ∂X with countably many Borel sets of the form $\{q' \in D_2 : \delta(q, q') < t\}$ such that no two points q', q'' in them satisfy $h(q', q'') = r$. If $r = 0$, since by non-elementarity f cannot be constant, the combinatorial argument of Lemma 3.5 gives either a single intersection point, which is, therefore, fixed and contradicts non-elementarity, or a triple $A \in \partial^3 T$ such that $f(x) \subseteq A$ for almost every $x \in B$; then, the centre of the ideal tripod spanned by A is a fixed point in V , contradiction. q.e.d.

End of the proof of Theorem 4.1. In view of the above lemma, we can compose $f \times f \times f$ with the cocycle ω of Proposition 4.6 and obtain an alternating Γ -equivariant bounded measurable cocycle $f^*\omega : B^3 \rightarrow \ell^2(E \curlywedge E)$. Since ω does not vanish on $\partial^3 X$, the map $f^*\omega$ vanishes almost nowhere since otherwise $f_*\beta$ would be supported on at most two points; this support being non-empty and Γ -invariant, it would contradict non-elementarity of Γ . By Corollary 2.6, we deduce $H_b^2(\Gamma, \ell^2(E \curlywedge E)) \neq 0$. Corollary 2.11 shows that for some G_i there is a non-trivial Γ -invariant subspace of $\ell^2(E \curlywedge E)$ on which the representation extends continuously to G factoring through $G \twoheadrightarrow G_i$. Now, the argument on page 45 in [40] finishes the proof – the argument given there for $\ell^2(E)$ can be immediately adapted to $\ell^2(E \curlywedge E)$. q.e.d.

5. A geometric cocycle in negative curvature

The goal of this section is to prove Theorem 1.5 from the Introduction by constructing an appropriate weakly continuous H -equivariant alternating bounded cocycle

$$\omega : \partial X \times \partial X \times \partial X \longrightarrow \bigoplus_{n=1}^{\infty} L^2(H),$$

where $H = \text{Isom}(X)$. It is good to keep in mind the analogy to the simpler cocycle constructed above for trees in Section 4.2.

Remark 5.1. In view of Proposition 2.12, the above cocycle defines canonically a class $\kappa = j[\omega]$ in $H_{cb}^2(H, \bigoplus_{n=1}^{\infty} L^2(H))$.

We observe, moreover, that the cocycle of Theorem 1.5 will in particular be Borel for the norm topology and $\omega|_{A^3} \neq 0$ whenever $A \subseteq \partial X$ contains at least three points.

5.1. Notations. For x in a metric space (X, d) and $r > 0$, we write $B(x, r)$ for the corresponding open ball, the closed ball $\bar{B}(x, r)$ is $\{y \in X : d(x, y) \leq r\}$ (thus contains the closure of the former). The r -neighbourhood of $A \subseteq X$ is

$$\mathcal{N}_r(A) = \{y \in X : \exists a \in A \text{ with } d(a, y) < r\}.$$

A map between metric spaces is called an isometry if it preserves the distances; a geodesic is an isometry $\mathbf{R} \rightarrow X$, where \mathbf{R} is endowed with its usual metric. A geodesic segment is an isometry $[a, b] \rightarrow X$ for $a, b \in \mathbf{R}$ and a geodesic ray is an isometry $\mathbf{R}_+ \rightarrow X$. We endow the set

$\mathfrak{G}X$ of geodesics with the topology of uniform convergence on bounded sets. The space $\mathfrak{G}X$ has a continuous \mathbf{R} -action by translations (the geodesic flow) and an involution $g \mapsto \tilde{g}$, $\tilde{g}(t) = g(-t)$.

A metric space is called proper if all closed balls are compact; in this case, the closed bounded sets are exactly the compact sets, and the space is locally compact (for complete geodesic metric spaces, properness is equivalent to local compactness by the Hopf–Rinow theorem [4, No. I.3.7]).

The notation C_0 always denotes the space of real-valued compactly supported continuous functions (hence, of finitely supported functions when the underlying space is a set).

5.2. A measure with control. Let (X, d) be a proper metric space and set $H = \text{Isom}(X)$. Notice that X is separable, that the compact-open topology turns H in a locally compact second countable group and that the H -action on X is proper.

The following proposition will help us in handling spaces without any bounded geometry assumption.

Proposition 5.2. *For every $T > 0$, there is an H -invariant positive Radon measure μ on X such that*

- (i) *the support of μ is X ,*
- (ii) *$\mu(\bar{B}(x, T)) \leq 1$ for all $x \in X$,*
- (iii) *the unitary H -representation on $L^2(X, \mu)$ is contained in a multiple of the regular H -representation.*

Proof. Fix a left Haar measure λ on H . For every $y \in X$, the orbit map $\alpha^y : H \rightarrow X$, $g \mapsto gy$, is proper and hence the image $\alpha_*^y \lambda$ is a positive Radon measure on X . It is H -invariant and supported on the orbit Hy . Let $r(y) = \alpha_*^y \lambda(\bar{B}(y, 2T)) \neq 0$ and $\mu_y = r(y)^{-1} \alpha_*^y \lambda$. If $x \in X$ is such that $\mu_y(\bar{B}(x, T)) \neq 0$, then a translate of x is in $\bar{B}(y, T)$ and thus, a translate of $\bar{B}(x, T)$ in $\bar{B}(y, 2T)$. Thus, $\mu_y(\bar{B}(x, T)) \leq 1$ for all $x \in X$.

Choose now a Radon probability p on X of full support. One checks that the map $y \mapsto \alpha_*^y \lambda$ is continuous for the weak topology of duality with $C_0(X)$. As one can see, e.g., when X is a tree, r need not be continuous; however.

Lemma 5.3. *The map $r : X \rightarrow \mathbf{R}_+^*$ is upper semi-continuous.*

Proof. Let $\varepsilon > 0$, $x \in X$ and $(x_n)_{n \in \mathbf{N}}$ tend to x . We may assume that for all $n \in \mathbf{N}$, $2d(x, x_n) \leq 1/(n + 1)$ holds; hence

$$d(gx, x) \leq d(gx, gx_n) + d(gx_n, x_n) + d(x_n, x) \leq d(gx_n, x_n) + 2d(x_n, x)$$

implies

$$(2) \quad d(gx, x) \leq d(gx_n, x_n) + 1/(n + 1) \quad \forall g \in H, n \in \mathbf{N}.$$

Since

$$A_k = \{g \in H : 2T + 1/(k + 1) < d(gx, x) \leq 2T + 1/k\} \quad (k \geq 1)$$

is a countable Borel partition of a relatively compact set in H , there is n_ε such that $\sum_{k=n_\varepsilon}^\infty \lambda(A_k) < \varepsilon$. We have

$$\begin{aligned} r(x_n) - r(x) &= \lambda\{g \in H : d(gx_n, x_n) \leq 2T\} \\ &\quad - \lambda\{g \in H : d(gx, x) \leq 2T\} \\ &\leq \lambda\{g \in H : d(gx_n, x_n) \leq 2T, d(gx, x) > 2T\}, \end{aligned}$$

which by (2) is

$$\sum_{k=n+1}^\infty \lambda\left(A_k \cap \{g \in H : d(gx_n, x_n) \leq 2T\}\right) \leq \sum_{k=n+1}^\infty \lambda(A_k)$$

and thus is less than ε for all $n \geq n_\varepsilon$. q.e.d.

In particular, r is measurable and, via the Riesz theorem, the formula

$$(f \in C_{00}(X)) \quad \mu(f) = \int_X \mu_y(f) dp(y)$$

defines a positive H -invariant Radon measure μ on X satisfying (ii). For (i), since p has full support, the lower semi-continuity of $y \mapsto r(y)^{-1}$ implies immediately that μ has also full support.

Finally, (iii) follows by Mackey's orbit theorem from the direct integral decomposition

$$L^2(X, \mu) \cong \int_A^\oplus L^2(H/H_y) d\bar{p}(y),$$

where H_y is the stabilizer of y , $A \subseteq X$ is a Borel section of X/H and \bar{p} the measure induced on A . Indeed, since each H_y is compact, $L^2(H/H_y)$ is contained in $L^2(H)$ for all y . q.e.d.

Remark 5.4. In general, we may not strengthen (ii) to a bound on the measure of a ball in terms of its radius, because we do not want to exclude spaces of unbounded geometry. Thus, there might be $T' > 0$ with $\mu(\bar{B}(x, T'))$ unbounded as x ranges over X .

On the other hand, sets which are roughly one dimensional can, of course, be controlled.

Lemma 5.5. *Let T, μ be as in Proposition 5.2 and assume $T > 1/2$. Then*

$$\mu(\mathcal{N}_{T-\frac{1}{2}} g([a, b])) \leq |b - a| + 1$$

for any geodesic segment $g : [a, b] \rightarrow X$.

Proof. It takes at most $|b - a| + 1$ balls of radius T to cover this neighbourhood. q.e.d.

5.3. CAT(0) spaces and projections on geodesics. We start with a discussion valid in the more general CAT(0) setting, having [4] as our background reference. Let (X, d) be a proper CAT(0) space, $H = \text{Isom}(X)$ and ∂X its geometric boundary at infinity endowed with the cone topology [4, II.8] which is compact metrizable. We write $\bar{X} = X \sqcup \partial X$ for the geometric compactification.

Because of the convexity of the metric [4, No. II.2.2], there is for every geodesic $g \in \mathfrak{G}X$ a closest point projection $P_g : X \rightarrow g(\mathbf{R})$; this map is continuous and does not increase distances. Moreover, for every $x \in X$ and $t \in \mathbf{R}$, the Alexandrov angle at $P_g x$ satisfies $\angle_{P_g x}(x, g(t)) \geq \pi/2$ ([4, No. II.2.4]). For $g \in \mathfrak{G}X$ and $x, y \in X$, we define

$$\Delta_g(x, y) = g^{-1}(P_g y) - g^{-1}(P_g x).$$

In other words, this is the distance between $P_g x$ and $P_g y$ with a sign coming from the orientation of g .

Lemma 5.6. *Consider the map $\Delta : \mathfrak{G}X \rightarrow C(X \times X)$.*

- (i) Δ is \mathbf{R} -invariant, H -equivariant and $\Delta_{\bar{g}} = -\Delta_g$;
- (ii) Δ is continuous when $C(X \times X)$ is endowed with the topology of uniform convergence on compact sets, and $|\Delta_g(x, y)| = d(P_g x, P_g y) \leq d(x, y)$;
- (iii) if $x \in g(\mathbf{R})$, then $|\Delta_g(x, y)| = d(x, y)$ implies $y \in g(\mathbf{R})$.

Proof. (i) and (ii) are clear, so assume $x \in g(\mathbf{R})$ and $|\Delta_g(x, y)| = d(x, y)$. The condition on the Alexandrov angle at $P_g y$ applied to $g(t) = x$ yields

$$d(x, y)^2 \geq d(x, P_g y)^2 + d(P_g y, y)^2$$

by comparison with the Euclidean cosine law (see [4, No. II.3.1]). Since $d(x, P_g y) = |\Delta_g(x, y)|$, we deduce $y = P_g y \in g(\mathbf{R})$. q.e.d.

We shall need some information about the dependence of P_g upon g .

Lemma 5.7. *Let $\varrho \geq 0$ and suppose that $g, g' \in \mathfrak{G}X$ satisfy*

$$d(g(t), g'(t)) \leq L^t \quad \forall t \geq 0$$

for some $0 < L < 1$. Then, for every $n \in \mathbf{N}$ and $x \in X$ with $d(x, P_g x) \leq 2\varrho + 4$ and $g^{-1}(P_g x) \geq n + 4\varrho + 10$, we have

$$d(P_g x, P_{g'} x) \leq C_3 L^{n/2}$$

for a constant C_3 depending only on ϱ .

Proof. First, we claim crudely that $g'^{-1}(P_{g'} x) \geq n$. Indeed, let $t = g^{-1}(P_g x) \geq n + 4\varrho + 10$. Then, $d(x, P_{g'} x) \leq d(x, g'(t))$ so that

$$d(P_{g'} x, g'(t)) \leq d(P_{g'} x, x) + d(x, g'(t)) \leq 2d(x, g'(t)).$$

But $d(x, g'(t)) \leq d(x, g(t)) + d(g(t), g'(t)) \leq 2\varrho + 4 + L^n \leq 2\varrho + 5$. Therefore,

$$g'^{-1}(P_{g'} x) \geq t - d(P_{g'} x, g'(t)) \geq t - 4\varrho - 10,$$

proving the claim.

Now, we write $y = P_g P_{g'} x$ and $y' = P_{g'} P_g x$. We have $d(P_g x, y') \leq L^n$ and, by the claim, $d(P_{g'} x, y) \leq L^n$. Therefore,

$$(3) \quad d(x, y') \leq d(x, P_g x) + L^n \leq d(x, y) + L^n \leq d(x, P_{g'} x) + 2L^n.$$

On the other hand, we have as before an estimate for the Alexandrov angle: $\angle_{P_{g'} x}(x, y') \geq \pi/2$. Writing $\ell = d(y', P_{g'} x)$, the comparison with the cosine law yields

$$\ell^2 \leq d(x, y')^2 - d(x, P_{g'} x)^2.$$

Combining this with (3), we obtain $\ell^2 \leq 4L^n(d(x, P_{g'} x) + L^n)$. Further,

$$d(x, P_{g'} x) \leq d(x, y') \leq d(x, P_g x) + L^n \leq 2\varrho + 5.$$

Summing up, we have $\ell^2 \leq 4L^n(2\varrho + 6)$, so that finally

$$d(P_g x, P_{g'} x) \leq d(P_g x, y') + \ell \leq L^n + 2\sqrt{(2\varrho + 6)L^n} L^{n/2}.$$

The lemma follows since $L^n \leq L^{n/2}$. q.e.d.

5.4. CAT(−1) spaces. Hereafter, we suppose, moreover, that (X, d) is CAT(−1); in particular, it is Gromov hyperbolic for a universal hyperbolicity constant [4, No. III.H.1.2] (we shall call universal the constants that are the same for all proper CAT(−1) spaces; such constants can usually be explicitly computed using hyperbolic geometry, but this will not be of interest to us). A geodesic g determines two distinct points $g(+\infty)$ and $g(-\infty)$ in ∂X . CAT(−1) spaces have the visibility property [4, No. II.9.28-32], which is that any pair of distinct points in ∂X are endpoints of some geodesic. Thus, one has a natural H -equivariant homeomorphism

$$(4) \quad \mathfrak{G}X/\mathbf{R} \cong \partial^2 X,$$

where $\partial^2 X$ denotes the pairs of distinct points in ∂X . More generally, $\partial^n X \subseteq (\partial X)^n$ is the subset of n tuples of pairwise distinct points in ∂X ; since it is open in $(\partial X)^n$, the induced topology is locally compact metrizable.

Observe that the involution $g \mapsto \tilde{g}$ of $\mathfrak{G}X$ descends to the natural flip in $\partial^2 X$.

One calls a triple (g_0, g_1, g_2) of geodesics an ideal triangle if $g_i(+\infty) = g_{i+1}(-\infty)$ for all $i \in \mathbf{Z}/3\mathbf{Z}$. The main property of CAT(−1) spaces that we shall use is the following fact.

Lemma 5.8. *There are universal constants $D > 0$ and $0 < L < 1$ such that for every ideal triangle (g_0, g_1, g_2) in any CAT(−1) space (X, d) one can fix numbers s_i, t_i such that for all $i \in \mathbf{Z}/3\mathbf{Z}$, one has $|t_i - s_i| \leq D$ and*

$$(5) \quad d(g_i(t_i + t), g_{i+1}(s_{i+1} - t)) \leq L^t \quad \forall t \geq 0.$$

Proof. This follows from standard comparison arguments with the hyperbolic plane \mathbf{H}^2 . However, due to the importance of this lemma in the sequel, we provide a detailed proof.

Since X is Gromov-hyperbolic, there are universal constants C_1, D_1 such that for every ideal triangle (g_0, g_1, g_2) there are s'_i, t'_i with $|t'_i - s'_i| \leq D_1$ and

$$d(g_i(t'_i + t), g_{i+1}(s'_{i+1} - t)) \leq C_1 \quad \forall t \geq 0, i \in \mathbf{Z}/3\mathbf{Z}.$$

We claim on the other hand that there are universal constants C_2 and $0 < L < 1$ such that for any rays r, r' with $d(r(0), r'(0)) \leq C_1$ and $r(+\infty) = r'(+\infty)$, there is ε with $|\varepsilon| \leq C_1$ and

$$d(r(t + \varepsilon), r'(t)) \leq C_2 L^t \quad \forall t \geq |\varepsilon|.$$

Proof of the claim: the claim is well known and easy to check in \mathbf{H}^2 because the Busemann functions associated to r and r' can only differ by a constant ε bounded by C_1 : after shifting r by this constant to make it synchronous with r' , the statement is a form of the usual exponential convergence in \mathbf{H}^2 . Let C_2, L be the constants obtained in \mathbf{H}^2 . In X , denote for every $T \geq 0$ by σ_T be the geodesic segment from $r'(0)$ to $r(T)$. Considering the comparison triangles in \mathbf{H}^2 for $r(0), r'(0), r(T)$ with arbitrarily large T , we obtain ε with $|\varepsilon| \leq C_1$ and such that

$$\limsup_{T \rightarrow \infty} d(r(t + \varepsilon), \sigma_T(t)) \leq C_2 L^t \quad \forall t \geq |\varepsilon|.$$

However, since $d(r(t), r'(t))$ is bounded (by C_1), even the comparison with \mathbf{R}^2 for the triangle $r(T), r'(T), r'(0)$ yields

$$\lim_{T \rightarrow \infty} d(\sigma_T(t), r'(t)) = 0 \quad \forall t \geq 0,$$

thus proving the claim.

We come back to our ideal triangle and apply the claim to $r(t) = g_i(t'_i + t)$ and $r'(t) = g_{i+1}(s'_i - t)$, so that

$$d(g_i((t'_i + \varepsilon) + t'), g_{i+1}(s'_{i+1} - t')) \leq C_2 L^{t'} \quad \forall t' \geq |\varepsilon|.$$

In order to deduce (5), it suffices now to increase t'_i to t_i and decrease s'_i to s_i by an amount A depending only on ε and $\log C_2 / \log L$. One completes the proof by letting $D = D_1 + 2A$. q.e.d.

5.5. The cocycle ω for CAT(-1) spaces. We fix once and for all $D > 0$ and $0 < L < 1$ as in Lemma 5.8 and some $\varrho \geq 4D + 6$. For every $g \in \mathfrak{GX}$, we define $N_g : X \rightarrow \mathbf{R}_+$ by

$$N_g(x) = \begin{cases} 0 & \text{if } d(P_g x, x) \geq 1/2, \\ 1/2 - d(P_g x, x) & \text{otherwise.} \end{cases}$$

Thus, N_g is continuous, bounded by $1/2$, $N_g = N_{\bar{g}}$ and it depends only on the class of g in \mathfrak{GX}/\mathbf{R} . The map $\mathfrak{GX} \rightarrow C_b(X)$ given by $g \mapsto N_g$ is continuous for the topology of uniform convergence on compact sets.

We define also a map $R \in C_b(X^2)$ by

$$R(x, y) = \begin{cases} 0 & \text{if } |d(x, y) - \varrho| \geq 1, \\ 1 - |d(x, y) - \varrho| & \text{otherwise.} \end{cases}$$

Next, we define for every $g \in \mathfrak{G}X$ a map $C_g \in C_b(X^2)$ by

$$(6) \quad C_g(x, y) = R(x, y) N_g(x) N_g(y) \Delta_g(x, y).$$

One checks that

$$(7) \quad |R(x, y) \Delta_g(x, y)| \leq \varrho.$$

All the above together with Lemma 5.6.

Corollary 5.9. *Consider the map $C : \mathfrak{G}X \rightarrow C_b(X \times X)$.*

- (i) C is \mathbf{R} -invariant, H -equivariant and $C_{\bar{g}} = -C_g$;
- (ii) C is continuous when source and target are endowed with the topology of uniform convergence on compact sets;
- (iii) $\|C_g\|_\infty \leq \varrho/4$.

In view of the H homeomorphism (4) and of the Corollary 5.9, we may define the map

$$\alpha : \partial X \times \partial X \longrightarrow C_b(X^2)$$

as follows. If $\xi_0, \xi_1 \in \partial X$ are distinct, we set $\alpha(\xi_0, \xi_1) = C_g$, where g is any geodesic with $g(-\infty) = \xi_0$ and $g(+\infty) = \xi_1$. If $\xi_0 = \xi_1$, we set $\alpha(\xi_0, \xi_1) = 0$.

Lemma 5.10. *The map α is continuous for the topology of uniform convergence on compact sets.*

Proof. We know already that α is continuous on the open set $\partial^2 X \subseteq (\partial X)^2$. Therefore, it is enough to show that $\alpha(\xi_n, \zeta_n)$ tends to zero whenever $(\xi_n, \zeta_n)_{n \in \mathbf{N}}$ is a sequence of distinct points converging to some $\eta \in \partial X$. Fix a compact set $K \subseteq X^2$. We can pick $g_n \in \mathfrak{G}X$ with $g_n(-\infty) = \xi_n$ and $g_n(+\infty) = \zeta_n$. Let $K_1 \subseteq X$ be projection of K on the first variable. By Proposition 3.21 in [4, No. III.H], the topology of ∂X is the same as the topology induced by any visual metric. The only information that we retain from this is that there is n_K such that all geodesics g_n with $n \geq n_K$ must pass at distance at least $1/2$ from K_1 . Therefore, $N_{g_n}(x) = 0$ for any such n and any $x \in K_1$. In consequence, the formula (6) shows that $\alpha(\xi_n, \zeta_n)$ vanishes on K for all $n \geq n_K$. q.e.d.

We write

$$\omega = d\alpha : \partial X \times \partial X \times \partial X \longrightarrow C_b(X^2)$$

for the coboundary of α . In particular, in view of $C_{\bar{g}} = -C_g$, we have for every ideal triangle (g_0, g_1, g_2) the formula

$$\omega(\xi_0, \xi_1, \xi_2) = C_{g_0} + C_{g_1} + C_{g_2},$$

where $\xi_i = g_i(+\infty)$. The whole point of our construction will be that ω actually ranges in a L^2 -space. But first, we collect some immediate properties of ω .

Corollary 5.11. *The map ω above is a H -equivariant alternating cocycle and is continuous when $C_b(X^2)$ is endowed with the topology of uniform convergence on compact sets.*

We elaborate now on Lemma 5.7.

Proposition 5.12. *Suppose that $g, g' \in \mathfrak{O}X$ satisfy*

$$d(g(t), g'(t)) \leq L^t \quad \forall t \geq 0.$$

Consider the Borel partition $X^2 = \bigsqcup_{m \in \mathbf{Z}} Y_m$, where

$$Y_m = \{(x, y) \in X^2 : P_g x \in g([m, m + 1))\}.$$

Then, for every $m \in \mathbf{N}$ with $m \geq 5\rho + 11$, we have

$$\|(C_g - C_{g'})|_{Y_m}\|_\infty \leq C_4 L^{m/2}$$

for a constant C_4 depending only on ρ .

Proof. Let $(x, y) \in Y_m$; we may assume $d(x, y) \leq \rho + 1$ since otherwise $R(x, y) = 0$ and there is nothing to prove. In view of (6), the function $C_g - C_{g'}$ vanishes at (x, y) unless at least $d(x, P_g x) < 1/2$ or $d(x, P_{g'} x) < 1/2$.

Lemma 5.13. *One has $d(x, P_g x) < 3/2$ and thus, for x , the assumptions of Lemma 5.7 are satisfied for $0 \leq n \leq m - 4\rho - 10$. Moreover, for y , the assumptions of Lemma 5.7 are satisfied for $0 \leq n \leq m - 5\rho - 11$.*

Proof of the lemma. We assumed $d(x, y) \leq \rho + 1$ and $g^{-1}(P_g x) \geq m$; since

$$d(y, P_g y) \leq d(y, x) + d(x, P_g x) + d(P_g x, P_g y) \leq d(x, P_g x) + 2d(x, y)$$

and

$$g^{-1}(P_g y) \geq g^{-1}(P_g x) - d(P_g x, P_g y) \geq g^{-1}(P_g x) - d(x, y),$$

we see that all we have to show is really $d(x, P_g x) < 3/2$, which is void if $d(x, P_g x) < 1/2$; so assume $d(x, P_{g'} x) < 1/2$. Let $t' \in \mathbf{R}$ such that $P_{g'} x = g'(t')$. We are done if $t' \geq 0$ because then,

$$d(x, P_g x) \leq d(x, g(t')) \leq d(x, g'(t')) + 1 < 3/2.$$

Now, assume for a contradiction that $t' < 0$ and let $t \in \mathbf{R}$ such that $P_g x = g(t)$. We have

$$d(x, g(0)) \leq d(x, g'(0)) + 1 \leq d(x, g'(t')) - t' + 1 < 3/2 - t'.$$

On the other hand, since $t \geq m \geq 0$,

$$d(x, P_g x) = d(x, g(t)) \geq d(x, g'(t)) - 1 > d(g'(t'), g'(t)) - 3/2 = t - t' - 3/2$$

But now, $d(x, P_g x) \leq d(x, g(0))$ gives the contradiction $t < 3 < m$. q.e.d.

In consequence, the Lemma 5.7 gives us

$$(8) \quad d(P_g x, P_{g'} x), \quad d(P_g y, P_{g'} y) \leq C'_3 L^{m/2}$$

for a constant $C'_3 = C_3 L^{-(5\varrho+11)/2}$ depending only on ϱ . We have now

$$\begin{aligned} |C_g(x, y) - C_{g'}(x, y)| &\leq |R(x, y)N_g(y)\Delta_g(x, y)| \cdot |N_g(x) - N_{g'}(x)| \\ &\quad + |R(x, y)N_{g'}(x)\Delta_g(x, y)| \cdot |N_g(y) - N_{g'}(y)| \\ &\quad + |R(x, y)N_{g'}(x)N_{g'}(y)| \cdot |\Delta_g(x, y) - \Delta_{g'}(x, y)| \end{aligned}$$

so that with (7)

$$(9) \quad \begin{aligned} &|C_g(x, y) - C_{g'}(x, y)| \\ &\leq \frac{\varrho}{2}|N_g(x) - N_{g'}(x)| + \frac{\varrho}{2}|N_g(y) - N_{g'}(y)| \\ &\quad + \frac{1}{4}|\Delta_g(x, y) - \Delta_{g'}(x, y)|. \end{aligned}$$

For the first summand, we discuss the following cases:

(i) if both $d(x, P_g x)$ and $d(x, P_{g'} x)$ are less than $1/2$, then by (8)

$$|N_g(x) - N_{g'}(x)| = |d(x, P_g x) - d(x, P_{g'} x)| \leq d(P_g x, P_{g'} x) \leq C'_3 L^{m/2};$$

(ii) if $d(x, P_g x) < 1/2$ and $d(x, P_{g'} x) \geq 1/2$, then by (8)

$$d(x, P_g x) \geq d(x, P_{g'} x) - d(P_g x, P_{g'} x) \geq 1/2 - C'_3 L^{m/2},$$

so that $|N_g(x) - N_{g'}(x)| = N_g(x) \leq C'_3 L^{m/2}$;

(iii) If $d(x, P_g x) \geq 1/2$ and $d(x, P_{g'} x) < 1/2$, then we may argue as in (ii).

In any case, the first summand in (9) is bounded by $(\varrho C'_3/2) L^{m/2}$. The second summand is handled in the same way with (8).

The third summand is bounded by

$$\frac{1}{4} |g^{-1}(P_g x) - g'^{-1}(P_{g'} x)| + |g^{-1}(P_g y) - g'^{-1}(P_{g'} y)|.$$

Writing $t = g^{-1}(P_g x) \geq m$, we have

$$\begin{aligned} |g^{-1}(P_g x) - g'^{-1}(P_{g'} x)| &= d(g'(t), P_{g'} x) \\ &\leq d(g'(t), g(t)) + d(P_g x, P_{g'} x) \\ &\leq L^m + C'_3 L^{m/2} \leq (C'_3 + 1) L^{m/2}. \end{aligned}$$

Likewise,

$$|g^{-1}(P_g y) - g'^{-1}(P_{g'} y)| \leq (C'_3 + 1) L^{m/2}$$

and Proposition 5.12 is proved.

q.e.d.

We fix now a measure μ on X as granted by Proposition 5.2 for $T = \varrho + 2$. We denote by \mathcal{H} the Hilbert space

$$\mathcal{H} = L^2(X \times X, \mu^2)$$

endowed with its natural (diagonal) continuous unitary H -representation.

We are now ready to establish the main property of ω . Since μ has full support, one can ask whether a given continuous function on X^2 belongs to \mathcal{H} without bothering about almost everywhere identities.

Theorem 5.14. *The cocycle ω ranges in \mathcal{H} and is bounded*

$$\|\omega(\xi_0, \xi_1, \xi_2)\|_2 \leq K \quad \forall \xi_0, \xi_1, \xi_2 \in \partial X$$

for a constant K depending only on ϱ .

Remark 5.15. As we shall see, ω is weakly continuous and thus automatically bounded. But indeed, the whole point of Theorem 5.14 is that ω ranges in \mathcal{H} . It just so happens that the proof that $\omega(\xi_0, \xi_1, \xi_2)$ is square summable is completely uniform and thus yields some constant K which is universal when one sets e.g., $\varrho = 4D + 6$.

Proof of Theorem 5.14. Pick $\xi_0, \xi_1, \xi_2 \in \partial X$; we may assume that they are distinct. Because of the visibility, we may choose $g_i \in \mathfrak{G}X$ with

$$g_i(+\infty) = g_{i+1}(-\infty) = \xi_i \quad \forall i \in \mathbf{Z}/3\mathbf{Z}.$$

We recall

$$\omega(\xi_0, \xi_1, \xi_2) = C_{g_0} + C_{g_1} + C_{g_2}$$

so that by Corollary 5.9, we have

$$(10) \quad \|\omega(\xi_0, \xi_1, \xi_2)\|_\infty \leq 3\varrho/4.$$

Let D, s_i, t_i, L be as in Lemma 5.8. We set $\eta = 5\varrho + 12$ and define

$$\begin{aligned} Z_i^+ &= \left\{ (x, y) \in X^2 : x \in \mathcal{N}_{\frac{1}{2}} g_i([t_i + \eta, +\infty)), d(x, y) \leq \varrho + 1 \right\}, \\ Z_i^- &= \left\{ (x, y) \in X^2 : x \in \mathcal{N}_{\frac{1}{2}} g_i((-\infty, s_i - \eta]), d(x, y) \leq \varrho + 1 \right\}, \\ W_i &= \left\{ (x, y) \in X^2 : x \in \mathcal{N}_{\frac{1}{2}} g_i([s_i - \eta, t_i + \eta]), d(x, y) \leq \varrho + 1 \right\}. \end{aligned}$$

Observe that by (6)

$$(11) \quad \text{supp}(C_{g_i}) \subseteq W_i \cup Z_i^+ \cup Z_i^- \quad \forall i \in \mathbf{Z}/3\mathbf{Z}.$$

We claim, moreover:

Lemma 5.16.

$$\text{supp}(C_{g_i}) \cap Z_{i+1}^+ = \text{supp}(C_{g_i}) \cap Z_{i-1}^- = \emptyset \quad \forall i \in \mathbf{Z}/3\mathbf{Z}.$$

Proof of the lemma. Let for a contradiction $(x, y) \in \text{supp}(C_{g_i}) \cap Z_{i+1}^+$. Set $x' = P_{g_i} x$ and $x'' = P_{g_{i+1}} x$, so that $x'' = g_{i+1}(t_{i+1} + \delta)$ for some $\delta > \eta - 1$. Since $|t_i - s_i| \leq D$, one of the following two cases must occur: either

- (i) there is $t \geq 0$ such that $d(x', z) \leq D/2$ for $z = g_i(t_i + t)$, or
- (ii) there is $t' \geq 0$ such that $d(x', z') \leq D/2$ for $z' = g_i(s_i - t')$.

In the first case, write

$$\begin{aligned} &|t_{i+1} - s_{i+1} + \delta + t| \\ &= d(x'', g_{i+1}(s_{i+1} - t)) \\ &\leq d(x'', x) + d(x, x') + d(x', z) + d(z, g_{i+1}(s_{i+1} - t)). \end{aligned}$$

If we use

$$(12) \quad d(x, x'), d(x, x'') < 1/2$$

and Lemma 5.8, this gives

$$|t_{i+1} - s_{i+1} + \delta + t| \leq 1/2 + 1/2 + D/2 + 1 = D/2 + 2.$$

But in view of $|t_{i+1} - s_{i+1}| \leq D$, the left-hand side is bounded below by

$$\delta + t - D \geq \delta - D \geq \eta - D - 1 \geq \varrho - D,$$

and this contradicts $\varrho \geq 4D + 6$.

In the second case, write

$$\begin{aligned} & |t_{i-1} - s_{i-1} + t' + \delta| \\ &= d(g_{i-1}(t_{i-1} + t'), g_{i-1}(s_{i-1} - \delta)) \\ &\leq d(g_{i-1}(t_{i-1} + t', z') + d(z', x') + d(x', x) + d(x, x'') \\ &\quad + d(x'', g_{i-1}(s_{i-1} - \delta))). \end{aligned}$$

Applying Lemma 5.8 to the first and last term (where $g_{(i-1)-1} = g_{i+1}$) and using (12), we get

$$|t_{i-1} - s_{i-1} + t' + \delta| \leq 1 + D/2 + 1/2 + 1/2 + 1 = D/2 + 3.$$

But this time, $|t_{i-1} - s_{i-1}| \leq D$ shows that the left-hand side is at least $t' + \delta - D \geq \delta - D$, again a contradiction. The proof for Z_{i-1}^- is completely symmetric. q.e.d.

We use now (11) to write

$$\begin{aligned} (13) \quad & \|\omega(\xi_0, \xi_1, \xi_2)\|_2 \\ & \leq \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \|\omega(\xi_0, \xi_1, \xi_2)|_{W_i}\|_2 \\ & \quad + \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \|\omega(\xi_0, \xi_1, \xi_2)|_{Z_i^+}\|_2 + \sum_{i \in \mathbf{Z}/3\mathbf{Z}} \|\omega(\xi_0, \xi_1, \xi_2)|_{Z_i^-}\|_2. \end{aligned}$$

Lemma 5.17. *Let $g : [a, b] \rightarrow X$ be a geodesic segment and $A \subseteq X^2$ a Borel set contained in*

$$(14) \quad \left\{ (x, y) \in X^2 : x \in \mathcal{N}_{\frac{1}{2}} g([a, b]), d(x, y) \leq \varrho + 1 \right\}.$$

Then, $\mu^2(A) \leq (|b - a| + 1)^2$.

Proof of the lemma. The set (14) is contained in

$$\mathcal{N}_{\frac{1}{2}} g([a, b]) \times \mathcal{N}_{\varrho+1+\frac{1}{2}} g([a, b]).$$

Since $\varrho + 1 + 1/2 = T - 1/2$, the estimate follows from Lemma 5.5. q.e.d.

In view of this lemma, we can use (10) to get

$$\|\omega(\xi_0, \xi_1, \xi_2)|_{W_i}\|_2 \leq \frac{3\varrho}{4} \sqrt{\mu^2(W_i)} \leq 3\varrho(D + 2\eta + 1)/4 =: K_1.$$

On the other hand, the Lemma 5.16 implies

$$\|\omega(\xi_0, \xi_1, \xi_2)|_{Z_i^+}\|_2 = \|(C_{g_i} + C_{g_{i+1}})|_{Z_i^+}\|_2.$$

We write now $g(t) = g_i(t_i + t)$ and $g'(t) = g_{i+1}(s_{i+1} - t)$; by the Corollary 5.9, the above norm is $\|(C_g - C_{g'})|_{Z_i^+}\|_2$. In this situation, we appeal to Proposition 5.12, observing that by the definition of η

$$Z_i^+ \subseteq \bigsqcup_{m \geq 5\varrho+11} Y_m.$$

We may, therefore, conclude

$$\|\omega(\xi_0, \xi_1, \xi_2)|_{Z_i^+}\|_2 \leq \sum_{m \geq 5\varrho+11} C_4 L^{m/2} \sqrt{\mu^2(Z_i^+ \cap Y_m)}.$$

Lemma 5.17 gives $\sqrt{\mu^2(Z_i^+ \cap Y_m)} \leq 2$, so we can estimate the above by a geometric series

$$\|\omega(\xi_0, \xi_1, \xi_2)|_{Z_i^+}\|_2 \leq 2 C_4 \sum_{m \geq 1} L^{m/2} =: K_2.$$

We find in the same way, the same bound for $\|\omega(\xi_0, \xi_1, \xi_2)|_{Z_i^-}\|_2$. Putting everything together in (13), we obtain the statement of the theorem for $K = 3 K_1 + 6 K_2$. q.e.d.

The proof of the next fact is reminiscent of Lemma 5.16; nevertheless, we provide it for completeness.

Lemma 5.18. *Let (g_0, g_1, g_2) be an ideal triangle in X . Then, the function $C_{g_0} + C_{g_1} + C_{g_2}$ is not identically zero.*

Proof. Let D, s_i, t_i be as in Lemma 5.8 and write $a = (\varrho - t_1 + s_1)/2$. Recall $|t_i - s_i| \leq D$ and $\varrho \geq 4D + 6$, so that

$$(15) \quad a \geq (3D + 6)/2 \geq 0.$$

We set $x = g_1(s_1 - a)$ and $y = g_1(t_1 + a)$. In particular, we have $d(x, y) = \varrho$ and thus, (6) gives $C_{g_1}(x, y) = \varrho/4$. To show that $C_{g_0} + C_{g_1} + C_{g_2}$ does not vanish at (x, y) , it is enough to show that both C_{g_0} and C_{g_2} vanish at this pair. Suppose for a contradiction that at least one of them does not vanish. Due to the symmetry of the situation, we may, upon replacing (g_0, g_1, g_2) by $(\tilde{g}_2, \tilde{g}_1, \tilde{g}_0)$ and swapping x with y , assume $C_{g_2}(x, y) \neq 0$. Set $x' = P_{g_2}x$. Since $|t_2 - s_2| \leq D$, one of the following two cases must occur:

Either there is $t \geq 0$ such that $d(x', z) \leq D/2$ for $z = g_2(t_2 + t)$, or there is $t' \geq 0$ such that $d(x', z') \leq D/2$ for $z' = g_2(s_2 - t')$.

In the first case, both $d(g_0(s_0 - t), z)$ and $d(g_0(t_0 + a), x)$ are bounded by one because of Lemma 5.8 so that

$$\begin{aligned} |t_0 - s_0 + a + t| &= d(g_0(t_0 + a), g_0(s_0 - t)) \\ &\leq 2 + d(x, x') + d(x', z) \leq 2 + 1/2 + D/2 \end{aligned}$$

because $N_{g_2}(x) \neq 0$ forces $d(x, x') < 1/2$. But the left-hand side above is bounded below by $a + t - D \geq a - D$. Summing up, we obtain $2a \leq 3D + 5$, which contradicts (15).

In the second case, Lemma 5.8 gives $d(g_1(t_1 + t'), z') \leq 1$ so that

$$\begin{aligned} |t_1 - s_1 + t' + a| &= d(g_1(t_1 + t'), x) \\ &\leq d(g_1(t_1 + t'), z') + d(z', x') + d(x', x) \\ &\leq 1 + D/2 + 1/2. \end{aligned}$$

The left-hand side is again at least $a - D$ so that we obtain the impossible inequality $2a \leq 3D + 3$. q.e.d.

End of proof of Theorem 1.5. Recall from Proposition 5.2 that the unitary H -representation $L^2(X, \mu)$ is contained in $\bigoplus_{n=1}^{\infty} L^2(H)$. It follows that the representation $\mathcal{H} \cong L^2(X, \mu) \otimes L^2(X, \mu)$ is also contained in $\bigoplus_{n=1}^{\infty} L^2(H)$, and we still denote by ω the resulting H -equivariant bounded cocycle

$$\omega : \partial X \times \partial X \times \partial X \longrightarrow \bigoplus_{n=1}^{\infty} L^2(H)$$

obtained via Theorem 5.14.

Let us show that $\omega|_{\partial^3 X}$ does not vanish. For every (ξ_0, ξ_1, ξ_2) in $\partial^3 X$, one can choose by visibility an ideal triangle (g_0, g_1, g_2) with $\xi_i = g_i(+\infty)$. Now, the element $C_{g_0} + C_{g_1} + C_{g_2}$ of $C_b(X^2)$ is non-zero by Lemma 5.18. The corresponding element in \mathcal{H} is non-zero since μ has full support.

Since it is enough to consider compactly supported continuous functions on X^2 to induce the weak topology on \mathcal{H} , the inclusion map $C_b(X^2) \cap \mathcal{H} \rightarrow \mathcal{H}$ is weakly continuous when the left-hand side is endowed with the topology of uniform convergence on compact sets. Therefore, the weak continuity follows from the corresponding continuity of $\omega : (\partial X)^3 \rightarrow C_b(X^2)$. q.e.d.

Remark 5.19. If X is the n -dimensional real hyperbolic space \mathbf{H}^n , then H acts transitively on $\partial^3 X$. Therefore, by equivariance, the norm $\|\omega\|$ is constant on $\partial^3 X$; this constant cannot be zero since ω does not vanish on $\partial^3 X$. On the other hand, ω vanishes on the set of non-distinct triples, which is in the closure of $\partial^3 X$. This shows that the cocycle ω is not norm continuous in general, compare with Remark 2.14.

6. Cohomology vanishing in higher rank

We start with the following statement for the ambient group of k points.

Theorem 6.1. *Let $G = \mathbf{G}(k)$, where k is a local field and \mathbf{G} is a connected almost k -simple group with $\text{rank}_k \mathbf{G} \geq 2$. For any separable coefficient G -module F , the inclusion map $i : F^G \rightarrow F$ induces an isomorphism*

$$i_* : H_{\text{cb}}^2(G, F^G) \cong H_{\text{cb}}^2(G, F).$$

Proof. We may assume that \mathbf{G} is k isotropic since otherwise G is compact and H_{cb}^2 always vanishes. Let $P < G$ be a minimal parabolic subgroup. Then, it follows from Mautner's lemma that G/P is a strong boundary for G , see [30, No. 11.2.1]. Therefore, according to Corollary 2.6, we only have to show that any G -equivariant measurable essentially bounded alternating cocycle $\omega : (G/P)^3 \rightarrow F$ ranges in F^G .

Lemma 6.2. *For any parabolic proper subgroup $Q < G$ and any ω as above, there is a Q -equivariant, measurable, essentially bounded alternating map $\alpha : (G/P)^2 \rightarrow F$ such that $\omega - d\alpha$ ranges in F^G .*

In other words, the restriction map $\text{res}_{G \rightarrow Q}$ factors through $H_{\text{cb}}^2(Q, F^G)$.

Proof of the Lemma. Let R be the soluble radical of Q . Since R is amenable, the inclusion $F^R \rightarrow F$ induces an isomorphism for $H_{\text{cb}}^\bullet(Q, -)$ by Corollary 2.2. Hence, the restriction must factor through $H_{\text{cb}}^2(Q, F^R)$. By Mautner's Lemma (in the generality stated in [30, No. 11.2.4]), we have $F^R = F^G$, hence the statement. q.e.d.

Let now Q, Q' be two different maximal parabolic subgroups containing P . (The fact that one can take $Q \neq Q'$ is our only use of the rank assumption.) Applying Lemma 6.2 to Q and Q' , there are alternating measurable bounded maps $\alpha, \alpha' : (G/P)^2 \rightarrow F$ which are respectively

Q - and Q' -equivariant and such that both $\omega - d\alpha$ and $\omega - d\alpha'$ range in F^G . Now, $\alpha - \alpha'$ defines a class in $H_{cb}^1(Q \cap Q', F)$; the latter, however, is trivial since F is a separable coefficient module, see [30, No. 11.4.1]. Thus, there is a $Q \cap Q'$ -equivariant measurable map $\beta : G/P \rightarrow F$ such that $\alpha - \alpha' = d\beta$. Since $Q \cap Q'$ contains P , Mautner's property forces β to be essentially constant. Thus, $\alpha = \alpha'$ and this map is therefore $Q \cup Q'$ -equivariant. Since these two parabolics are maximal and different, they generate G and so α is G -equivariant. This shows that the cocycle ω is G -cohomologous to the cocycle $\omega - d\alpha$ ranging in F^G , as required (observe that by Mautner's property and alternation α has to vanish so that actually ω itself ranges in F^G , in accordance with Corollary 2.6). q.e.d.

We can now pass to lattices using (the injectivity of) induction.

Theorem 6.3. *Let Γ be a lattice in $\mathbf{G}(k)$, where k is a local field and \mathbf{G} a connected almost k -simple group with $\text{rank}_k \mathbf{G} \geq 2$. For any separable dual Banach Γ -module E , the inclusion map $i : E^\Gamma \rightarrow E$ induces an isomorphism*

$$i_* : H_b^2(\Gamma, E^\Gamma) \cong H_b^2(\Gamma, E).$$

Proof. Write $G = \mathbf{G}(k)$. Since the Furstenberg boundary G/P considered in the above proof is also a strong boundary for Γ (see [30, No. 11.1.10]), the Corollary 2.6 applied to Γ implies that we have only to show that any Γ -equivariant measurable essentially bounded alternating cocycle $\omega : (G/P)^3 \rightarrow E$ ranges in E^Γ . Consider the separable coefficient G -module $F = L^2 \mathbf{I}_\Gamma^G E$ and the induced G -equivariant cocycle $\mathbf{i}\omega : (G/P)^3 \rightarrow F$. As we saw in Theorem 6.1, $\mathbf{i}\omega$ must range in F^G which is here $L^2(G/G, E)^\Gamma$. This means precisely that ω ranges in E^Γ . q.e.d.

Remark 6.4. One could have argued at a more formal level: for any lattice Γ in a locally compact group G and any coefficient Γ -module E , we have $(L^2 \mathbf{I}_\Gamma^G E)^G \cong E^\Gamma$ and a commutative diagram in all degrees

$$\begin{array}{ccccc}
 & & H_b^\bullet(\Gamma, E^\Gamma) & \xrightarrow{i_*} & H_b^\bullet(\Gamma, E) \\
 & \nearrow \text{res} & \downarrow \mathbf{i} & & \downarrow \mathbf{i} \\
 H_{cb}^\bullet(G, (L^2 \mathbf{I}_\Gamma^G E)^G) & \longrightarrow & H_{cb}^\bullet(G, L^2 \mathbf{I}_\Gamma^G(E^\Gamma)) & \longrightarrow & H_{cb}^\bullet(G, L^2 \mathbf{I}_\Gamma^G E)
 \end{array}$$

In the situation at hand, we know that in degree two, the induction \mathbf{i} is injective and that the bottom row is an isomorphism, so that i_* is onto, which is what was to be shown.

Proof of Theorem 1.4. In view of Theorems 6.1 and 6.3, it remains only to justify the following claim:

$$\dim H_b^2(\Gamma) = \dim H_{cb}^2(G) = \begin{cases} 1 & \text{if } k = \mathbf{R} \text{ and } \pi_1(G) \text{ is infinite,} \\ 0 & \text{in all other cases} \end{cases}$$

for trivial coefficients \mathbf{R} . This is shown in [8], [9], but we sketch the argument since it is simplified by Theorem 6.1. The kernel of the natural map $H_{cb}^2(G) \rightarrow H_c^2(G)$ is easily seen to identify with the quotient of the space of continuous quasi-morphisms $G \rightarrow \mathbf{R}$ by the subspaces of bounded continuous functions. Using contracting properties of inner automorphisms, one checks that all continuous quasi-morphisms are bounded (Lemma 6.1 in [8]; the simple connectedness assumption is superfluous there). Thus, $H_{cb}^2(G)$ injects into $H_c^2(G)$. The latter vanishes if k is non-Archimedean. If k is Archimedean, then a result of Guichardet–Wigner [20] states that $H_c^2(G)$ vanishes unless $\pi_1(G)$ is infinite, in which case it is one dimensional. The assumption on $\pi_1(G)$, moreover, excludes $k = \mathbf{C}$. To conclude the claim for G , it remains to observe that the cocycle given in [20] is bounded.

For the lattice Γ , consider the commutative diagram

$$\begin{array}{ccc} & & H_b^2(\Gamma) \\ & \nearrow \text{res} & \downarrow \mathbf{i} \\ H_{cb}^2(G) & \longrightarrow & H_{cb}^2(G, L^2(G/\Gamma)) \end{array}$$

Since the restriction and induction maps are injective ([30, No. 8.6.2] and Corollary 2.9) and the bottom arrow is an isomorphism by Theorem 6.1, we conclude that all three maps are isomorphisms. q.e.d.

Corollary 6.5. *Let X be the symmetric space or Bruhat–Tits building associated to a group G as in Theorem 1.4 and ∂X the geometric boundary of X . Assume that X is not a symmetric space of Hermitian type and let F be any separable coefficient G -module. Then, any measurable G -equivariant alternating bounded cocycle*

$$\omega : \partial X \times \partial X \times \partial X \longrightarrow F$$

vanishes almost everywhere on $(\partial X)^3$.

Proof. The generic G -orbits in ∂X are of the form G/P , where $P < G$ is a minimal parabolic subgroup. On the other hand, G/P is a strong boundary for G [9]; therefore, the conclusion follows from:

$$ZL_{\text{alt}}^\infty((G/P)^3, F)^G \cong H_{\text{cb}}^2(G, F) = 0,$$

which is the combination of Corollary 2.6 and of Theorem 1.4. q.e.d.

We recall from [30, No. 13.1] that a rough action (by isometries) of a group Γ on a Banach space E is a map $\varrho : \Gamma \rightarrow \text{Isom}(E)$ such that the expression

$$\sup_{x,y \in \Gamma} \sup_{v \in E} \|\varrho(x)\varrho(y)v - \varrho(xy)v\|_E$$

is finite. Trivial examples of rough actions are given by bounded perturbation of actual actions. Theorem 6.3 has the following consequence.

Corollary 6.6. *Let Γ be a lattice in $\mathbf{G}(k)$, where k is a local field and \mathbf{G} is a connected almost k -simple group with $\text{rank}_k \mathbf{G} \geq 2$. Then, any rough action of Γ on a separable reflexive Banach space is a bounded perturbation of an actual Γ -action.*

Proof. It is shown in [30, No. 13.1] that a rough action $\varrho : \Gamma \rightarrow \text{Isom}(E)$ determines an isometric linear representation π on E and a class κ in the kernel of the natural map

$$H_{\text{b}}^2(\Gamma, E) \longrightarrow H^2(\Gamma, E)$$

(for the representation π) such that κ vanishes if and only if ϱ is a bounded perturbation of a Γ -action. Since E is reflexive, the linear representation π is adjoint and thus (π, E) is a separable coefficient module. Therefore, by Theorem 6.3, it is enough to show that the map $H_{\text{b}}^2(\Gamma, E^\Gamma) \rightarrow H^2(\Gamma, E^\Gamma)$ is injective, where E^Γ is the space of vectors that are Γ -invariant for π . This follows from the injectivity of $H_{\text{b}}^2(\Gamma) \rightarrow H^2(\Gamma)$ mentioned in the proof of Theorem 1.4. q.e.d.

7. Non-vanishing results

The constructions of Sections 4 and 5 can also be used to associate cohomological invariants to actions on negatively curved spaces in a more general way than what we used above for superrigidity; that is

the aim of this section. Besides their intrinsic interest, these invariants are essential for our results [31] on measure equivalence. More stability/invariance properties of these objects are considered in [31].

More precisely, we introduce the class

$$\mathcal{C}_{\text{reg}} = \{\Gamma : H_b^2(\Gamma, \ell^2(\Gamma)) \neq 0\}.$$

Before proceeding, we observe the following, recalling that a group is called ICC if all non-trivial conjugacy classes are infinite (it is easy to see that this is the case for any torsion-free group in \mathcal{C}_{reg} , see [31]).

Proposition 7.1. *Let Γ be a (discrete) group in the class \mathcal{C}_{reg} . Then, there exist uncountably many non-isomorphic unitary Γ -representations π with $H_b^2(\Gamma, \pi) \neq 0$. If, in addition, Γ is an ICC group (e.g., if it is torsion free), then $H_b^2(\Gamma, \pi) \neq 0$ for every unitary subrepresentation π of $\ell^2(\Gamma)$, and if $\ell^2(\Gamma) = \int^{\oplus} \pi_x$ is any direct integral decomposition of the regular Γ -representation, then $H_b^2(\Gamma, \pi_x) \neq 0$ for almost every x .*

Proof. It follows readily from Corollary 2.7 that for any unitary Γ -module \mathcal{H} , there is a maximal closed submodule \mathcal{H}_0 with the property that $H_b^2(\Gamma, \mathcal{H}_0) = 0$. Denoting by \mathcal{H}_b its orthogonal complement, it follows that every Γ -invariant closed submodule of \mathcal{H}_b has non-vanishing H_b^2 , and that, moreover, if $\pi_b = \int^{\oplus} \pi_x$ is a direct integral decomposition of the Γ -representation π_b on \mathcal{H}_b , then for almost every x one has $H_b^2(\Gamma, \pi_x) \neq 0$ (see the remark at the end of Corollary 2.7). Thus, the first statement of the Proposition holds for any group Γ which admits some unitary representation π with $H_b^2(\Gamma, \pi) \neq 0$, such that π has no irreducible sub-representations. It is a well known classical fact that for any discrete group, its regular representation has the latter property, thus implying the first statement.

To see the second, assume now that Γ is furthermore ICC, and let \mathcal{H}_0 be as above where $\mathcal{H} = \ell^2(\Gamma)$. Because the Γ -right translation action on $\ell^2(\Gamma)$ is an equivariant isometry for its left action, it follows from the maximality of \mathcal{H}_0 that it is also right Γ -invariant. Since the left-right $\Gamma \times \Gamma$ -representation on $\ell^2(\Gamma)$ is irreducible for any ICC group, and $\mathcal{H}_0 \neq \ell^2(\Gamma)$ by our assumption, we may now conclude using the discussion of the previous paragraph. q.e.d.

7.1. Groups acting on CAT(-1) spaces. Our geometric cocycle allows us to give the following algebraic characterization of non-elementary actions.

Theorem 7.2. *Let (X, d) be a proper $CAT(-1)$ space and $H = \text{Isom}(X)$. There is a class $\kappa \in H_{\text{cb}}^2(H, \bigoplus_{n=1}^{\infty} L^2(H))$ such that for any group Γ and any homomorphism $\varrho : \Gamma \rightarrow H$ the following are equivalent:*

- (i) ϱ is elementary;
- (ii) $\varrho^* \kappa = 0$ in $H_{\text{b}}^2(\Gamma, \bigoplus_{n=1}^{\infty} L^2(H))$.

We mention on this occasion that in the special case of the real hyperbolic spaces \mathbf{H}^n a much stronger dichotomy holds. Recall that $\text{Isom}(\mathbf{H}^n)$ is the group $H = O(n, 1)_+$ consisting of the elements of $O(n, 1)$, which preserve the upper sheet in the paraboloid model.

Proposition 7.3. *Let Γ be any group and $\varrho : \Gamma \rightarrow H = O(n, 1)_+$ be any homomorphism. The following assertions are equivalent:*

- (i) ϱ is elementary;
- (ii) the map $\varrho^* : H_{\text{cb}}^n(H, E) \rightarrow H_{\text{b}}^n(\Gamma, E)$ is zero for every separable coefficient H -module E and all $n \geq 1$;
- (iii) there exists some separable coefficient H -module E such that the map $\varrho^* : H_{\text{cb}}^2(H, E) \rightarrow H_{\text{b}}^2(\Gamma, E)$ is non-injective.

It follows from the structure of $O(n, 1)_+$ that (i) is equivalent to

(i') *The closure of $\varrho(\Gamma)$ in H is amenable,*

which is an obvious sufficient condition for (ii).

Remark 7.4. The situation changes if one considers complex hyperbolic spaces $\mathbf{H}_{\mathbb{C}}^n$. For simplicity, let us first discuss $H_0 = \mathbf{PSU}(n, 1) < \text{Isom}(\mathbf{H}_{\mathbb{C}}^n)$. The Hermitian structure on $\mathbf{H}_{\mathbb{C}}^n$ yields a generator of $H_{\text{c}}^2(H_0)$ via the van Est isomorphism; this class comes actually from $H_{\text{cb}}^2(H_0)$, which has also dimension one; see e.g., [8], [9]. However, if $\varrho : \Gamma \rightarrow H_0$ is a homomorphism such that Γ preserves a totally real subspace of $\mathbf{H}_{\mathbb{C}}^n$, then the pull-back of this class to $H_{\text{cb}}^2(\Gamma)$ vanishes since it factors through some $H_{\text{cb}}^2(SO(k, 1)_+)$ which is zero [8]. Thus, the dichotomy of Proposition 7.3 does not hold.

More interestingly, Burger–Iozzi [6] show that the pull-back of this class through a non-elementary homomorphism $\varrho : \Gamma \rightarrow H_0$ vanishes if and only if Γ preserves a totally real subspace of $\mathbf{H}_{\mathbb{C}}^n$.

The conclusion of Proposition 7.3 also fails for the group $H = \text{Isom}(\mathbf{H}_{\mathbb{C}}^n)$: it is straightforward to adapt the previous example to H by using an induction argument (which is particularly simple since H_0 has finite index in H).

Let us get to the proofs.

Lemma 7.5. *Let Γ be a group and $\varrho : \Gamma \rightarrow H = \text{Isom}(X)$ an isometric action on a proper $\text{CAT}(-1)$ space X . Then, ϱ is elementary if and only if $\varrho|_\Lambda$ is elementary for every finitely generated subgroup $\Lambda < \Gamma$.*

Proof. The “only if” part is tautological. For the converse, assume that $\varrho|_\Lambda$ is elementary for every $\Lambda \in \mathcal{F}$, where \mathcal{F} stands for the set of finitely generated subgroups of Γ . For every subgroup $\Lambda < \Gamma$, define $F_\Lambda \subseteq \overline{X}$ by

$$F_\Lambda = \{x \in \overline{X} : |\Lambda x| \leq 2\},$$

so that $\varrho|_\Lambda$ is elementary if and only if $F_\Lambda \neq \emptyset$. We have to show $F_\Gamma \neq \emptyset$. We have $F_\Gamma \supseteq \bigcap_{\Lambda \in \mathcal{F}} F_\Lambda$, and one checks that the sets F_Λ are closed. Therefore, by compactness of \overline{X} , it is enough to show that the family $(F_\Lambda)_{\Lambda \in \mathcal{F}}$ has the finite intersection property.

Let $\Lambda_1, \dots, \Lambda_n \in \mathcal{F}$; the group Λ generated by them is still in \mathcal{F} so $F_\Lambda \neq \emptyset$ by assumption. We are now done because $F_\Lambda \subseteq \bigcap_{i=1}^n F_{\Lambda_i}$. q.e.d.

Proof of Theorem 7.2. Write $\mathcal{H} = \bigoplus_{n=1}^\infty L^2(H)$ and let $\kappa \in H_{\text{cb}}^2(H, \mathcal{H})$ be the class defined by the cocycle ω of Theorem 1.5 as in Remark 5.1. Let Γ be a group and $\varrho : \Gamma \rightarrow H$ a homomorphism. The main part of the proof is

(ii) \Rightarrow (i). Suppose for a contradiction that ϱ is non-elementary. By Lemma 7.5, there is some finitely generated subgroup $\Lambda < \Gamma$ such that $\varrho|_\Lambda$ is non-elementary. Since the map $(\varrho|_\Lambda)^*$ induced at the level of H_{b}^2 factors as

$$H_{\text{cb}}^2(H, \mathcal{H}) \xrightarrow{\varrho^*} H_{\text{b}}^2(\Gamma, \mathcal{H}) \xrightarrow{\text{res}} H_{\text{b}}^2(\Lambda, \mathcal{H}),$$

we have $(\varrho|_\Lambda)^*\kappa = 0$ in $H_{\text{b}}^2(\Lambda, \mathcal{H})$.

By Theorem 2.5, there is a strong boundary (B, β) for Λ . Since the Λ -action on B is amenable, there is a Furstenberg map to (∂X) , that is, a Λ -equivariant measurable map $\varphi : B \rightarrow \mathcal{P}(\partial X)$, see e.g., [42], Proposition 4.3.9. Since the diagonal Λ -action on $B \times B$ is ergodic, we may apply the Corollary 3.2 in [10] (which we generalized above in Lemmas 3.4 and 3.5) and deduce that φ ranges β -essentially in the Dirac masses. Thus, keeping the notation φ , we have a Λ -equivariant measurable map $\varphi : B \rightarrow \partial X$ and by Proposition 2.12 the class $(\varrho|_\Lambda)^*\kappa$ is represented by the cocycle

$$\omega \circ (\varphi^3) : B^3 \longrightarrow \mathcal{H}.$$

By Corollary 2.6, the vanishing of $(\varrho|_\Lambda)^*\kappa$ forces $\omega \circ (\varphi^3)$ to vanish β^3 -a.e. Therefore, since the continuous map $\omega|_{\partial^3 X}$ vanishes nowhere, the

support of $\varphi_*\beta$ contains at most two points. Since this support is non-empty and Λ -equivariant, we have a contradiction with the fact that Λ acts non-elementarily.

(i) \Rightarrow (ii). Assume now that ϱ is elementary. If Γ fixes a point in X , then the closure $K = \overline{\varrho(\Gamma)} < H$ is compact. In particular, $H_{cb}^n(K, -)$ vanishes for all $n > 0$ and all Banach coefficients. Since ϱ^* factors through the restriction map $H_{cb}^2(H, -) \rightarrow H_{cb}^2(K, -)$, we conclude that $\varrho^*\kappa$ vanishes.

If Γ does not fix a point in X , it fixes a point z or a set of two points $\{z, z'\}$ in ∂X . Apply Proposition 2.12 there by replacing S and H by Γ , replacing G by H , and let $\varphi(\gamma) = \gamma z$. Then, the class $\varrho^*\kappa$ is represented by the cocycle $z^*\omega : \Gamma^3 \rightarrow \mathcal{H}$ defined by $z^*\omega(\gamma_0, \gamma_1, \gamma_2) = \omega(\gamma_0 z, \gamma_1 z, \gamma_2 z)$; this cocycle vanishes since ω is alternating and the Γ -orbit of z does not contain three distinct points. q.e.d.

The following consequence is relevant for our work [31] on Orbit Equivalence.

Corollary 7.6. *Let Γ be a discrete group acting non-elementarily and properly by isometries on some proper CAT(-1) space. Then, $H_b^2(\Gamma, \ell^2(\Gamma)) \neq 0$.*

Proof. By assumption, there is a proper CAT(-1) space X and a homomorphism $\varrho : \Gamma \rightarrow H = \text{Isom}(X)$ such that the associated action is proper and non-elementary. Assume for a contradiction that $H_b^2(\Gamma, \ell^2(\Gamma))$ vanishes. Set $\Gamma_0 = \varrho(\Gamma)$. Since the action is proper, $\ker(\varrho)$ is finite and thus, the Γ -representation $\ell^2(\Gamma_0)$ is contained in $\ell^2(\Gamma)$. Since in the unitary setting sub-representations are always complemented, this forces the space $H_b^2(\Gamma, \ell^2(\Gamma_0))$ to vanish as well (see e.g., [30, No. 8.2.9]). Moreover, the properness of the action makes Γ_0 discrete in H , so that the Γ -representation $L^2(H)$ is a multiple of $\ell^2(\Gamma_0)$. Applying Corollary 2.7, we deduce $H_b^2(\Gamma, L^2(H)) = 0$. Applying it again, we have the vanishing of $H_b^2(\Gamma, \bigoplus_{n=1}^\infty L^2(H))$ which contradicts Theorem 7.2. q.e.d.

Proof of Proposition 7.3. Retain the notation of the proposition; we may assume $n \geq 2$ since otherwise the statement is tautological. The implication (i) \Rightarrow (ii) is clear in view of (i') since the map ϱ^* always factors through the continuous bounded cohomology of the closure of $\varrho(\Gamma)$ in H . On the other hand, (ii) \Rightarrow (iii) is trivial once there is E with

$H_{cb}^2(H, E) \neq 0$; this follows from Theorem 7.2 since $n \geq 2$. Thus, we are left with.

(iii) \Rightarrow (i): Suppose for a contradiction that ϱ is non-elementary. By assumption there is E and a non-zero class $c \in H_{cb}^2(H, E)$ such that $\varrho^*(c)$ vanishes in $H_b^2(\Gamma, E)$. The geometric boundary $\partial\mathbf{H}^n$ is an amenable H -space, so by Theorem 2.1, there is an alternating measurable H -equivariant cocycle $\omega : (\partial\mathbf{H}^n)^3 \rightarrow E$ representing c . Since the set of distinct triples $\partial^3\mathbf{H}^n$ is a single H -orbit, the norm $\|\omega\|_E$ is essentially constant on $\partial^3\mathbf{H}^n$; since, moreover, we may set ω to zero on the complement of $\partial^3\mathbf{H}^n$, it follows from the transitivity of H on $\partial^3\mathbf{H}^n$ that ω coincides a.e. with an alternating Borel (strict) cocycle ω' whose norm is some constant $K \neq 0$ on $\partial^3\mathbf{H}^n$. As in the proof of Theorem 7.2, we have a finitely generated subgroup $\Lambda < \Gamma$ such that $\varrho|_\Lambda$ is non-elementary and a Λ -equivariant measurable map $\varphi : B \rightarrow \partial\mathbf{H}^n$, where B is a strong boundary for Λ (Theorem 2.5). It is shown by Burger–Iozzi (appendix to [9]) that $\varphi^*\omega'$ indeed represents the class $(\varrho|_\Lambda)^*(c)$. This class vanishes since $(\varrho|_\Lambda)^*$ factors through ϱ^* , but this contradicts the fact that $\varphi^*\omega'$ is not essentially zero in view of Corollary 2.6. q.e.d.

7.2. Trees and amalgams. We begin with a statement parallel to Theorem 7.2:

Theorem 7.7. *Let $T = (V, E)$ be a countable tree and $\text{Aut}(T)$ its automorphism group. There is a class $\kappa \in H_b^2(\text{Aut}(T), \ell^2(E \curvearrowright E))$ such that for any group Γ and any homomorphism $\varrho : \Gamma \rightarrow \text{Aut}(T)$, the following are equivalent:*

- (i) ϱ is elementary,
- (ii) $\varrho^*\kappa = 0$ in $H_b^2(\Gamma, \ell^2(E \curvearrowright E))$.

Recall that $E \curvearrowright E$ denotes the set of pairs of successive non-backtracking edges. Thus, the above characterization clearly also holds if we replace $\varrho^*\kappa$ by its image in $H_b^2(\Gamma, \ell^2(E \times E))$.

Corollary 7.8. *Let Γ be a group with a non-elementary action by automorphisms on some countable tree $T = (V, E)$. Then, $H_b^2(\Gamma, \ell^2(E \curvearrowright E)) \neq 0$.*

In particular, if the Γ -action on E is proper, then $H_b^2(\Gamma, \ell^2(\Gamma)) \neq 0$.

Appealing to Bass–Serre theory, we get the following corollaries for amalgamated products, wherein we call $A *_C B$ B -elementary if the associated Bass–Serre tree is finite or linear. In other words, $A *_C B$ is

non-elementary as soon as, say, $A \neq C$ and $[B:C] > 2$; this is equivalent to non-elementarity of the action on the associated tree.

Corollary 7.9. *Let $\Gamma = A * B$ be a free product of countable groups with $|A| > 2$ or $|B| > 2$. Then, $H_b^2(\Gamma, \ell^2(\Gamma)) \neq 0$.*

*The statement holds more generally if Γ is a non-elementary amalgamated product $\Gamma = A *_C B$ over a finite group C .*

In fact, the assumption that the action on edges is proper is much more than is needed to deduce the non-vanishing of $H_b^2(\Gamma, \ell^2(\Gamma))$; what is really relevant are the Γ -orbits in $E \vee E$. It follows that in the case of amalgamated products, there is another more general simple condition that is sufficient for our purposes.

Recall that (following Baumslag) a subgroup H of a group G is called malnormal if $H \neq G$ and $H \cap gHg^{-1}$ is trivial for all $g \in G$ with $g \notin H$. More generally, call H almost malnormal if the latter condition is replaced by requiring $H \cap gHg^{-1}$ to be finite only.

Corollary 7.10. *Let $\Gamma = A *_C B$ be a non-elementary amalgamated product of countable groups. If C is malnormal either in A or in B , then $H_b^2(\Gamma, \ell^2(\Gamma)) \neq 0$.*

The statement holds more generally if C is almost malnormal in one of the factors.

Remark 7.11. Actually, for the cocycle constructed for Proposition 4.6, we may replace pairs of successive non-backtracking edges by simplicial paths of any length n (as we did when presenting this cocycle in the note [32]). Thus, it is straightforward to obtain a chain of more and more general assumptions on the amalgamated product, where the first case $n = 2$ corresponds to almost malnormality. We leave this to the reader.

We turn now to the proofs.

Proof of Theorem 7.7. The scheme of the proof goes exactly as for Theorem 7.2, replacing the cocycle of Theorem 1.5 by the cocycle ω of Section 4.2 and using the techniques of Section 4 instead of the usual arguments for proper GNC spaces. Here are the relevant details.

We take $\kappa = j[\omega]$ as in Remark 4.8. Note that an action $\varrho : \Gamma \rightarrow \text{Aut}(T)$ is elementary if and only if there is a Γ -invariant subset of cardinality one or two in \overline{T} (e.g., the endpoints of a fixed unoriented edge). Thus, the compactness of \overline{T}^σ implies the statement analogous

to Lemma 7.5. Now, the proof of (ii) \Rightarrow (i) goes as in Theorem 7.2 upon replacing all the Furstenberg map arguments with the arguments of Section 4.3.

For (i) \Rightarrow (ii), we do not have at our disposal the compactness of stabilizers, but since ω is defined on the whole of \overline{T} , the argument using Proposition 2.12 given in the proof of Theorem 7.2 can be used as soon as there is any orbit of cardinality ≤ 2 , so that we are done. q.e.d.

Remark 7.12. Observe that the proof above also applies to show the following: If H is any locally compact σ -compact group with a continuous action on a simplicial tree $T = (V, E)$, then the space $H_{\text{cb}}^2(H, \ell^2(E \vee E))$ is non-zero. In particular, if the H -action on the edges is proper, then $H_{\text{cb}}^2(H, L^2(H))$ is non-zero. Here, continuity of the action amounts to say that the vertex stabilizers are open subgroups of H .

It remains to give the

Proof of Corollary 7.10. Assume e.g., that C is almost malnormal in B and let $T = (V, E)$ be the Bass–Serre tree associated to the amalgamated product. Let $e \in E$ be the edge stabilized by C and such that A is the stabilizer of $s(e)$ and B of $t(e)$. Pick $b \in B$, $b \notin C$ and write $e' = b\bar{e}$. Then, $(e, e') \in E \vee E$ and the stabilizer of (e, e') is $C \cap bCb^{-1}$, hence is finite. Therefore, the Γ -orbit of (e, e') realizes $\ell^2(\Gamma)$ as a Γ subrepresentation of $\ell^2(E \vee E)$; let $P : \ell^2(E \vee E) \rightarrow \ell^2(\Gamma)$ be the associated orthogonal Γ -equivariant projection. This P is just the restriction to the orbit of (e, e') and therefore, the explicit description of the cocycle ω in Lemma 4.7 shows that $P_*\omega$ never vanishes on $\partial^3 T$; therefore, arguing as for Theorem 7.7, it gives a non-trivial class in $H_{\text{b}}^2(\Gamma, \ell^2(\Gamma))$. q.e.d.

7.3. Hyperbolic groups and spaces. Using Mineyev’s homological bicombing [28], it is not difficult to deduce the following result for hyperbolic groups (*à la* Gromov).

Theorem 7.13. *Let Γ be a non-elementary hyperbolic group with vanishing first ℓ^2 -Betti number. Then, $H_{\text{b}}^2(\Gamma, \ell^2(\Gamma)) \neq 0$.*

Recall that a hyperbolic group is elementary if it is virtually cyclic; it is well known that a non-elementary hyperbolic group is non-amenable [16]. In fact, the condition on the ℓ^2 -Betti number can be removed: The following statement, conjectured in the first version of this paper, was meanwhile proved in joint work [29] with Mineyev.

Theorem. *The space $H_b^2(\Gamma, \ell^2(\Gamma))$ is non-trivial for every non-elementary hyperbolic group Γ .*

Proof of Theorem 7.13. Let $X = (V = \Gamma, E)$ be the Cayley graph of Γ for some finite generating system $S = S^{-1} \not\ni e$ and let d be the corresponding distance function for Γ . Let $C_{00}(E)$ be the space of finitely supported functions on E with its natural Γ -action and write $\|\cdot\|_p$ for the p norms on $C_{00}(E)$. Mineyev’s bicombing (Theorem 10 in [28]) provides us with a Γ -equivariant map $q : \Gamma \times \Gamma \rightarrow C_{00}(E)$ such that the following hold:

- (i) $\forall x, y \in \Gamma : \|q(x, y)\|_1 \geq d(x, y)$,
- (ii) $\exists T \in \mathbf{R} \forall x, y, z \in \Gamma : \|q(y, z) - q(x, z) + q(x, y)\|_1 \leq T$,
- (iii) $\exists C \in \mathbf{R} \forall x, y \in \Gamma$: the support of $q(x, y)$ is in the C -neighbourhood of any geodesic from x to y .

(Here (i) comes from the fact that q is a combing.)

We view now q as a map $\Gamma \times \Gamma \rightarrow \ell^2(E)$ and define a map $\omega : \Gamma^3 \rightarrow \ell^2(E)$ as the coboundary $\omega = dq$.

Lemma 7.14. *There is a constant K such that for all $1 < p < \infty$ and all $x, y \in \Gamma$ one has $\|q(x, y)\|_p \geq K^{\frac{1-p}{p}} d(x, y)^{1/p}$.*

Proof of the lemma. The condition (iii) on q implies that there is a sequence $x_0 = x, x_1, \dots, x_{d(x,y)} = y$ in Γ such that

$$\text{Supp}(q(x, y)) \subseteq \bigcup_{j=1}^{d(x,y)-1} B(x_j, C + 1),$$

where by abuse of notation B denotes balls in E . There is a constant K depending only on C and $|S|$ bounding the number of elements of any such ball; therefore, the number of edges in the support of $q(x, y)$ is at most $d(x, y)K$. Thus, Hölder’s inequality implies

$$\|q(x, y)\|_1 \leq \|q(x, y)\|_p \cdot (d(x, y)K)^{\frac{p-1}{p}}.$$

This, together with property (i), yields the claimed inequality. q.e.d.

Since $\|\cdot\|_2 \leq \|\cdot\|_1$ the property (ii) implies that ω is bounded and thus defines a class $[\omega]$ in $H_b^2(\Gamma, \ell^2(E))$. In view of $\ell^2(E) \cong \bigoplus_{s \in S} \ell^2(\Gamma)$, it remains only to show that $[\omega]$ is non-trivial. Suppose thus for a contradiction that there is a bounded equivariant map $\alpha : \Gamma^2 \rightarrow \ell^2(E)$ with $\omega = d\alpha$. Then, $q - \alpha$ determines a class in $H^1(\Gamma, \ell^2(E))$. Since

Γ is non-elementary, it is non-amenable; therefore, by Hulanicki's criterion [21], $H^1(\Gamma, \ell^2(\Gamma))$ is Hausdorff and thus our assumption on the first ℓ^2 -Betti number implies $H^1(\Gamma, \ell^2(\Gamma)) = 0$ because Γ is finitely generated. In other words, the affine action associated to $q - \alpha$ has a fixed point, so that $q - \alpha$ is bounded. We conclude that q itself is bounded, in contradiction to Lemma 7.14. q.e.d.

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