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THE SMALE CONJECTURE FOR HYPERBOLIC 3-MANIFOLDS: ISOM $(M^3) \simeq$ Diff (M^3)

DAVID GABAI

Abstract

The main result of this paper is, "If M is a closed hyperbolic 3-manifold, then the inclusion of the isometry group $\mathrm{Isom}(M)$ into the diffeomorphism group $\mathrm{Diff}(M)$ is a homotopy equivalence."

1. Introduction

The main result of this paper is:

Theorem 1.1. *If M is a closed hyperbolic 3-manifold, then the inclusion of the isometry group $\mathrm{Isom}(M)$ into the diffeomorphism group $\mathrm{Diff}(M)$ is a homotopy equivalence.*

Theorem 1.1 had been proven for Haken manifolds in 1976 by Hatcher [9] and Ivanov [11, 12]. We showed in [6], [7] that $\pi_0(\mathrm{Diff}(M))$ is canonically bijective with $\pi_0(\mathrm{Isom}(M))$, which is well known to be finite. Thus, if $\mathrm{Diff}_0(M)$ denotes the path component of $\mathrm{Diff}(M)$ containing id_M, then this result is equivalent to:

Theorem 1.2. *If M is a closed hyperbolic 3-manifold, then $\mathrm{Diff}_0(M)$, is contractible.*

The proof of Theorem 1.1 follows along the same lines as the proof of Theorem 0.1 ii) [6]. Namely use the contractibility of the space of Riemannian metrics on M in conjunction with the insulator theory of [6] to reduce to the Haken case.

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The main technical innovations of this paper are the *Canonical Solid Torus Theorem,* Theorem 3.4, together with the *Non-Encroachment Lemma*, Lemma 4.3. Roughly speaking the Canonical Solid Torus Theorem asserts that if δ is a special type of oriented simple closed geodesic in a closed hyperbolic 3-manifold M, then associated to any Riemannian metric r on M there exists a canonically immersed solid torus V_r whose interior $\overset{\circ}{V}_r$ is embedded. Furthermore, if γ is a core of $\overset{\circ}{V}_r$, then it is canonically oriented and with that orientation is isotopic to δ. Finally, if r is a hyperbolic metric and δ_r is the oriented geodesic in M freely homotopic to δ, then δ_r is a core of $\overset{\circ}{V}_r$ and its orientation agrees with the canonical orientation given to cores of $\overset{\circ}{V}_r$. While these immersed tori may not vary continuously with r, the Non-Encroachment Lemma provides sufficient control to approximately pass from one V_r to another $V_{r'}$, when r' is close to r. (More or less, the worst that can happen is that a sequence of solid tori gets a 3-ball pinched off in the limit.)

Remarks 1.3. The canonical solid torus depends on an a priori chosen non-coalescable insulator family for the geodesic δ.

A discussion of the difference between the insulator construction of [6] and the *full insulator construction* developed in this paper is given in §3.

The Mostow Rigidity theorem [16] together with the results in [7] allow us to equate $\mathrm{Diff}_0(M)$ with $\mathrm{Hyp}(M)$, the space of hyperbolic metrics on M. Thus we obtain:

Theorem 7.3. *The space* $\mathrm{Hyp}(M)$ *of hyperbolic metrics on a complete hyperbolic* 3*-manifold* M *of finite volume is contractible.*

This paper is organized as follows. In §2 we present a detailed outline of the proof of Theorem 1.1. In particular we show how Theorem 1.1 is deduced from the Coarse Torus Isotopy Theorem 4.6 and the Local Contractibility Theorem 6.3. In §3 we give the proof of Theorem 3.4. In §4 we give the proof of the Coarse Torus Isotopy Theorem. Section 5 gives a new formulation of Hatcher's theorem (the Smale Conjecture). The proof of the Local Contractibility Theorem is given in §6 and applications are presented in §7.

Definition 1.4. $\mathrm{Diff}(M)$ will denote the space of diffeomorphisms of M with the C^∞ topology. $\mathrm{Diff}_0(M)$ will denote the path component of $\mathrm{Diff}(M)$ containing id_M, i.e., $\mathrm{Diff}_0(M)$ is the set of diffeomorphisms isotopic to id_M. If X, Y are smooth manifolds then $\mathrm{Emb}(X, Y)$ is the space of smooth embeddings of X into Y. If X is a subspace of Y, then $\mathrm{Emb}_0(X, Y)$ is the subspace consisting of embeddings isotopic to the standard inclusion.

If $E \subset Y$, then $N(k, E) = \{y \in Y | d(y, E) \leq k\}$. Similarly if $x \in Y$, then $B(k, x) = \{y \in Y | d(y, x) \leq k\}$. The symbol ρ will always represent the standard hyperbolic metric on $\mathbb{H}^3$. The particular model of hyperbolic space will be clear from context. We will assume that ρ is induced from a metric also called ρ on the closed hyperbolic 3-manifold M. We will use notations such as $N_\rho(k, E)$ or $d_r(x, y)$ when the metric ρ or r is not clear from context. $|E|$ will denote the number of components of E and $\overset{\circ}{E}$ will denote the interior of E. If $X \subset Y$ then $\mathrm{Bd}(X) = \overline{X} - \overset{\circ}{X}$. If Δ is a cellulation, then Δ^k denotes its k-skeleton.

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2. The proof of Theorem 1.1

By [6] and [7] the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ induces a bijection of $\mathrm{Isom}(M)$ with $\pi_0(\mathrm{Diff}(M))$. Therefore to prove Theorem 1.1 it remains to show that $\mathrm{Diff}_0(M)$ is contractible. By [18] $\mathrm{Diff}(M)$ is an ANR and by [14, p. 35] an ANR has the homotopy type of a CW complex. Thus by J. H. C. Whitehead's theorem it suffices to show that all the homotopy groups of $\mathrm{Diff}_0(M)$ are trivial.

In this section we explain how Theorem 1.1 follows directly from the following results which are established in §4 and §7 respectively.

Coarse Torus Isotopy Theorem 4.6. *Let M be a closed orientable hyperbolic 3-manifold with geodesic δ satisfying the insulator condition. If $f : S^n \to \mathrm{Diff}_0(M)$, then there exists a cellulation Δ of B^{n+1} and a function which associates to each cell $\sigma \in \Delta$ an embedded* *solid torus* V_σ *such that:*

i) *If* κ *is a proper face of* σ*, then* $V_\kappa \subset \mathring{V}_\sigma$ *and* V_κ *is isotopic to the standard embedding in* V_σ*.*

ii) *If* $x \in \sigma \cap S^n$*, then* $f_x(\delta)$ *is a core of* $\mathring{V}_\sigma$*.*

Idea of the proof. Associated to $x \in S^n$, there is the push forward hyperbolic metric $(f(x))_*\rho$. This gives rise to a map $h : S^n \to \mathrm{RM}(M)$ the space of Riemannian metrics on M. The contractibility of $\mathrm{RM}(M)$ enables us to extend h to $B^{n+1} \to \mathrm{RM}(M)$. By Theorem 3.4 associated to each $x \in B^{n+1}$, there is a canonical solid torus V_x immersed in M, with embedded interior. Furthermore, the curve δ is isotopic to a core of V_x. While the various V_x's do not vary continuously in x, the Non-Encroachment Lemma 4.3 enables us to maintain sufficient control to obtain a cellulation Δ^* and find embedded solid tori by shrinking various V_x's which together satisfy the conclusions of Theorem 4.6 except that, in Conclusion i) the statement $V_\kappa \subset \mathring{V}_\sigma$ is replaced by the condition $V_\sigma \subset \mathring{V}_\kappa$. Roughly speaking our desired cellulation Δ is the cellulation dual to Δ^*, with V_σ being the solid torus V_{σ^*}, where σ^* is the cell dual to σ.

Local Contractibility Theorem 6.3. *Let* δ *be an oriented simple geodesic in the closed hyperbolic* 3*-manifold* M *and* V *a solid torus embedded in* M*. If* $H : S^n \to \mathrm{Diff}_0(M)$ *is such that* $H_t(\delta) \subset \mathring{V}$ *for each* $t \in S^n$*, then* H *extends to a map* $G : B^{n+1} \to \mathrm{Diff}_0(M)$ *such that* $G_s(\delta) \subset \mathring{V}$ *for each* $s \in B^{n+1}$*.*

Idea of the proof. Suppose that that $n > 0$, V is a closed regular neighborhood $N(\delta)$ of δ and the restriction of H_t to $N(\delta)$ is the identity for all $t \in B^{n+1}$. In that case $M - \mathring{N}(\delta)$ is Haken, and Theorem 6.3 follows by the Hatcher, Ivanov theorem [9], [11], [12] for Haken manifolds. (Actually they proved this in the PL category, but as noted in [9], the proof can be promoted to Diff using [10].) In general one can control the maps $H_t|N(\delta)$, since $\mathrm{Emb}_0(D^2 \times S^1, \mathbb{R}^2 \times S^1) \simeq S^1 \times S^1$ (an equivalent formulation of the Smale conjecture), and thereby reduce to the previous case.

The proof that $\mathrm{Emb}_0(D^2 \times S^1, \mathbb{R}^2 \times S^1) \simeq S^1 \times S^1$ is carried out in §5 and the proof of Theorem 6.3, including the case $n = 0$, is given in §6.

Proof of Theorem 1.1. Given $f : S^n \to \mathrm{Diff}(M^3)$ construct a cellulation Δ of B^{n+1} and solid tori V_σ as in Theorem 4.6. That theorem implies that for each $\sigma \in \Delta$, δ is isotopic to a core of $\overset{\circ}{V}_\sigma$. Therefore to each $x \in \Delta^0 - S^n$ there exists an $f_x \in \mathrm{Diff}_0(M)$ such that $f_x(\delta)$ is a core of V_x. Assume by induction that f has been extended to $S^n \cup \Delta^i$ such that if $x \in \sigma$ a k-cell, $0 \leq k \leq i$, then $f_x(\delta)$ is a core of $\overset{\circ}{V}_\sigma$. By Theorems 4.6 and 6.3 we can extend f to $S^n \cup \Delta^{i+1}$ so that if $x \in \sigma$ a k-cell, $0 \leq k \leq i+1$, then $f_x(\delta)$ is a core of $\overset{\circ}{V}_\sigma$. The proof is complete when $i = n$. □

Remarks 2.1. i) Here is another view of the structure of the proof of Theorem 1.1 offered by the referee. Consider the fibration $p : \mathrm{Diff}_0(M) \to \mathrm{Emb}_0(V, M)$ where V is as above. By Hatcher and Ivanov [9], [11], [12], each component of $\mathrm{Diff}(M, V)$, the subspace of $\mathrm{Diff}(M)$ which fixes V pointwise, is contractible. Thus, using Lemma 6.1 it suffices to show that the induced map p_* in the long exact homotopy sequence is trivial. Theorem 4.6 can be viewed as an approximate statement that p_* is trivial, with Theorem 6.3 providing the necessary refinement.

ii) The proof of Theorem 6.3 does not rely on the insulator technology.

3. The Canonical Solid Torus Theorem

The reader should be familiar with the notions of *noncoalescable insulator family* and *trilinking* (see 0.4-0.5 [6] as well as the minimal surface theory developed in §3 [6].

In what follows the set $\{\lambda_{ij}\}$ will denote a $(\pi_1(M), \{\partial\delta_i\})$ noncoalescable insulator family where δ is a simple closed oriented geodesic and $\delta_0, \delta_1, \delta_2, \ldots$ denote the lifts of δ to $\mathbb{H}^3$. Let $g \in \pi_1(M)$ denote the primitive element which fixes δ_0, such that if δ_0 is oriented from the repelling fixed point of $\partial\delta_0$ to the attracting one, then δ_0 inherits the orientation induced from δ.

Definition 3.1. By an *immersed solid torus* $V \subset M$ we mean that V is the image of an immersion $f : D^2 \times S^1 \to M$. By the *interior* of V or $\overset{\circ}{V}$ we mean $f(\overset{\circ}{D}{}^2 \times S^1)$. By ∂V we mean $f(\partial D^2 \times S^1)$.

Review of the insulator construction [6] 3.2. Let r be any Riemannian metric on M and let r also denote the induced metric on

$\widetilde{M} = \mathbb{H}^3$. Let $\{\sigma_{ij}\}$ be a family of $\pi_1(M)$ equivariant r-least area D^2-limit laminations which span the $\{\lambda_{ij}\}$. To each σ_{0i} let H_{0i} denote the component of $\mathbb{H}^3 - \sigma_{0i}$ whose closure contains $\partial\delta_0$. The set $H_0 = \cap_i H_{0i}$ contains a unique component that is invariant by the maximal cyclic group $\langle g \rangle$ which fixes δ_0 and projects to an open solid torus $\overset{\circ}{V}$ which is the interior of an immersed solid torus V. The orientation on δ induces an orientation on δ_0 and hence on $\partial\delta_0$ and hence on each core of $\overset{\circ}{V}$. Any positively oriented core of $\overset{\circ}{V}$ is isotopic in M to δ.

To first approximation, one should think of σ_{0i} as a properly embedded r-least area plane spanning the smooth circle λ_{0i} and H_{0i} as the open half space of $\mathbb{H}^3 - \sigma_{0i}$ which contains $\partial\delta_0$. Finally H_0 is the intersection of these half spaces and a component projects down to the open solid torus $\overset{\circ}{V}$.

Given M, δ and the insulator family $\{\lambda_{ij}\}$, the construction of V depends only on the Riemannian metric r and the choice of spanning laminations. The main result of this chapter eliminates dependence on the spanning laminations.

Definition 3.3. Let r be the Riemannian metric on $\mathbb{H}^3$ induced from the Riemannian metric r on M. For each λ_{ij}, let $\{\sigma_{ij}^\alpha\}_{\alpha\in J}$ denote the collection of r-least area D^2-limit laminations which span λ_{ij}. Let H_{ij}^α denote the component of $\mathbb{H}^3 - \sigma_{ij}^\alpha$ which contains the ends of δ_i. Let $H_{ij} = \cap_\alpha H_{ij}^\alpha$, $H_i = \cap_j H_{ij}, \Sigma_{ij} = \cup_\alpha \sigma_{ij}^\alpha$ and $\Sigma_i = \cup_j \Sigma_{ij}$.

We will show that H_0 contains a unique $\overset{\circ}{D}{}^2 \times \mathbb{R}$ component which projects to the interior $\overset{\circ}{V}_r$ of an immersed solid torus V_r in M. This V_r is said to arise from the *full insulator construction*. The adjective *full* stresses the fact that all possible spanning laminations of each λ_{0j} are used in the construction. This should be contrasted with the insulator construction of [6], reviewed above, where a single spanning lamination is used.

Canonical Solid Torus Theorem 3.4. *Let δ be an oriented simple closed geodesic in the closed orientable hyperbolic 3-manifold M possessing a non-coalescable insulator family $\{\lambda_{ij}\}$. Given any Riemannian metric r on M the full insulator construction gives rise to a canonically immersed solid torus V_r having the following properties:*

i) *$\overset{\circ}{V}_r$ is embedded.*

ii) *If γ is a core of $\overset{\circ}{V}_r$, then it is canonically oriented and with that orientation is isotopic to δ.*

iii) *If r is a hyperbolic metric and δ_r is the oriented geodesic in M freely homotopic to δ, then δ_r is a core of $\overset{\circ}{V}_r$. Its orientation agrees with the canonical orientation given to cores of $\overset{\circ}{V}_r$.*

Lemma 3.5. *For all $i, j \in \mathbb{N}, \Sigma_{0i} \cap \Sigma_{0j} \neq \emptyset$ implies $\lambda_{0i} \cap \lambda_{0j} \neq \emptyset$.*

Proof. This follows from Lemma 3.5 viii) [6]. q.e.d.

Lemma 3.6.

i) *After reordering the λ_{0j}'s there exists a finite set $\{\lambda_{01}, \lambda_{02}, \dots, \lambda_{0m}\}$ which are representatives of the outermost $\langle g\rangle$-orbits of $\{\lambda_{0j}\}_{j\in\mathbb{N}}$. I.e., given any λ_{0k}, either $\lambda_{0k} = g^q(\lambda_{0i})$ for some $q \in \mathbb{Z}$ and $1 \leq i \leq m$ or there exists an $n \in \mathbb{Z}$ and $j \in \{1, \dots, m\}$ such that $\lambda_{0k} \subset Y(g^n(\lambda_{0j}))$, where $Y(g^n(\lambda_{0j}))$ is the component of $S^2_\infty - g^n(\lambda_{0j})$ which does not contain $\partial\delta_0$.*

ii) *$H_0 = \cap_{i=1}^m \cap_{n\in\mathbb{Z}} g^n(H_{0j})$. In words, to construct H_0 we need only consider the collection of spanning laminations corresponding to $\{\lambda_{01}, \lambda_{02}, \dots, \lambda_{0m}\}$ and their $\langle g\rangle$-translates.*

Proof. This follows as on p. 63 [6]. q.e.d.

Lemma 3.7.

i) *If $h \in \langle g\rangle$, then $h(H_0) = H_0$ and $h(\Sigma_0) = \Sigma_0$.*

ii) *$H_i \cap H_j \neq \emptyset$ if and only if $i = j$.*

iii) *There exists an $a > 0$ (which depends on r) so the $H_0 \subset N_\rho(a, \delta_0)$.*

Proof. Again these facts follow as in Step 2, p. 63 [6]. q.e.d.

Definition 3.8. If $t > 0$ let $\Sigma^t = \overset{\circ}{N}(t, \delta_0) \cap \Sigma_0$. We say that L is a leaf of Σ_0 (resp. Σ^t), if L is a leaf of some σ_{0j}^α (resp. L is a component of $R \cap \overset{\circ}{N}(t, \delta_0) \cap \Sigma_0$, where R is a leaf of some σ_{0j}^α).

This paper makes extensive use of the notion of convergence of *sequences* of embedded surfaces or laminations in Riemannian 3-manifolds. The reader is advised to read Definition 3.2 [6] for the definition of the word *converges*.

Correction 3.9. In this paper and [6] all convergence takes place in the C^k*-topology, all* $k < \infty$ rather than the C^∞*-topology.*

Lemma 3.10. *Let* $w \in \mathbb{H}^3$ *and* $L_1, L_2, \ldots$ *a sequence of leaves of the laminations* $\sigma_{0i_j}^{\alpha_j}$ *such that* $\mathrm{Lim}\ d_\rho(L_i, w) \to 0$*. After passing to subsequence, each* L_j *is a leaf of* $\sigma_{0i}^{\alpha_j}$ *for some fixed* i *and the* L_j*'s converge to an* r*-least area* D^2*-limit lamination* $\sigma_{0i}^{\alpha_\infty}$ *which contains* w*.*

Lemma 3.10 can be thought of as saying that Σ_{0i} and Σ_0 are closed in the C^k-topology, all $k < \infty$.

Proof. By the local finiteness of $\{\lambda_{0j}\}$, the last sentence of Proposition 3.9 [6] and the fact that each L_j meets a fixed neighborhood of w, we can pass to a subsequence so that each L_i is a leaf of a lamination spanning the same insulator λ_{0i}. By passing to another subsequence, and invoking Proposition 3.10 [6] we conclude that these leaves converge to a r-least area D^2-limit lamination $\sigma_{0i}^{\alpha_\infty}$. q.e.d.

Corollary 3.11. H_0 *and each* H_{0i} *are open sets.*

Lemma 3.12.

i) *Each leaf* L *of* Σ^t *is properly embedded in* $\overset{\circ}{N}(t, \delta_0)$*.*

ii) *If* $t > a, L$ *separates* $\overset{\circ}{N}(t, \delta_0)$ *and one component of* $L - \overset{\circ}{N}(t, \delta_0)$ *contains both ends of* δ_0*.*

iii) *The leaves of* Σ^t *have uniformly bounded area.*

iv) (*Leaves of* Σ^t *are closed in the* C^k*-topology, all* $k < \infty$.) *If* $L_1, L_2, \ldots$ *are leaves of* Σ^t *respectively containing points* $x_1, x_2, \ldots$ *and* $\mathrm{Lim}\, x_i \to x \in \overset{\circ}{N}(t, \delta_0)$*, then there exists a leaf* L_x *of* Σ^t *containing* x *and a subsequence of* $\{L_i\}$ *such that the following holds. If* K *is any compact subsurface in* L_x *containing* x*, then there exist embeddings* $f_i : K \to L_i$ *such that* $f_i \to id_K$ *in the* C^k*-topology all* $k < \infty$ *and for all* i*,* $x_i \in f_i(K)$*.*

Proof. i) The last sentence of Proposition 3.9 [6] together with the fact that L is contained in a leaf of a D^2-limit lamination implies that there exists an embedded disc $D \subset \mathbb{H}^3$ such that $\partial D \cap \overset{\circ}{N}(t, \delta_0) = \emptyset$ and L is a component of $D \cap \overset{\circ}{N}(t, \delta_0)$.

ii) Apply Lemma 3.7 iii).

iii) By the local finiteness of λ_{0i} and the last sentence of Proposition 3.9 [6] only finitely many $\langle g\rangle$-orbits of Σ_{0i} can intersect $\overset{\circ}{N}(t,\delta_0)$. Applying Proposition 3.9 [6] to these orbits we conclude that there exists a $C > 0$ such that if L is a leaf of Σ^t, then $L \subset B_\rho(C,z)$ for some $z \in \delta_0$. The disc D in the proof of i) can be chosen to have boundary in $\partial B_\rho(2C,z)$ or some 2-sphere which is an arbitrarily close perturbation of it. Thus for some uniform constant $C_1 > 0$, $\text{area}_r(L) < C_1\,\text{area}_\rho(L) \leq C_1\,\text{area}_\rho(D) \leq C_1\,\text{area}_\rho(\partial B_\rho(2C,z))$.

iv) This follows from Lemma 3.10 and Definition 3.2 [6]. q.e.d.

Definition 3.13. A leaf L_x of Σ^t satisfying the conclusion of Lemma 3.12 iv) is said to be obtained as a *limit of leaves* $L_1, L_2, \dots$ of Σ^t.

Lemma 3.14. *Let U denote $\overset{\circ}{B}_\rho(\eta,x)$ or $\overset{\circ}{B}_r(\eta,x)$. For every $t > 0$ there exists $\epsilon > 0$ so that if $\eta \leq \epsilon, d_\rho(x,\delta_0) \leq .9t$ and L leaf of Σ^t, then $L \cap U$ is empty or a properly embedded disc D whose closure in $\overline{U}$ is a properly embedded disc transverse to $\partial\overline{U}$.*

Proof. Since the leaves of Σ_0 are stable minimal surfaces with respect to a metric induced from a compact manifold, it follows by Schoen [19] that the leaves have uniformly bounded normal curvature. Therefore there exists $\epsilon_0 > 0$ such that if $\epsilon_1 \leq \epsilon_0$ and either $U = \overset{\circ}{B}_r(\epsilon_1,x)$ or $\overset{\circ}{B}_\rho(\epsilon_1,x)$, then each component D of $\Sigma_0|U$ is a properly embedded disc whose closure in $\overline{U}$ is a properly embedded disc transverse to $\partial\overline{U}$. Furthermore if $U' = \overset{\circ}{B}_\rho(\epsilon',x)$ or $\overset{\circ}{B}_r(\epsilon',x)$ and $U' \subset U$, then $D \cap U'$ is connected or empty.

Suppose that $M_1, M_2, \dots$ is a sequence of leaves of Σ^t which hit progressively smaller ρ-balls about $y_1, y_2, \dots$ in multiple components, where $\text{Lim}\; y_i \to x \in N_\rho(.9t,\delta_0)$. Let $L_1, L_2, \dots$ denote the leaves of Σ^{2t} which respectively contain the $M_1, M_2, \dots$. Let L_x denote a limit of $L_1, L_2, \dots$ which contains x. By Lemma 3.12 and the previous paragraph there exists $\epsilon_2 < \epsilon_1$ so that $L_x \cap B_\rho(\epsilon_2,x)$ is connected and transverse to $\partial B_\rho(\epsilon_2,x)$. Let $K \subset L_x$ be a compact subsurface containing x, such that $d_\rho(\partial K,\delta_0) > 1.5t$. Let $f_i(K) \subset L_i$ be as in Lemma 3.12 iv). Since $f_i(K) \to K$, it follows that for i sufficiently large, $d_\rho(\partial f_i(K),\delta_0) > t$ and hence $M_i \subset f_i(K)$. Therefore by Schoen's theorem if $d_\rho(y_j,x) \leq \epsilon_2 - \epsilon'$, then $\overset{\circ}{B}_\rho(\epsilon',y_j) \cap M_i$ is connected or empty for all i sufficiently large.

q.e.d.

Notation 3.15. In what follows, J will denote a component of $H_0, f = 2a$ (see Lemma 3.7) and m will be as in Lemma 3.6. Taking $t = a$, fix ϵ to satisfy the conclusion of Lemma 3.14.

Our next goal, carried out in sections 3.16–3.36, is to show that $\bar{J}$ is a simply connected manifold with boundary.

Definition 3.16. Given a component J of H_0, normally orient each leaf L of Σ^f to point into the component of $\mathring{N}_\rho(f, \delta) - L$ which contains J. This orientation is called the *J-normal orientation*. The + *side* of L is the side facing J.

Warning 3.17. J-normal orientations may not in general induce a consistent transverse orientation on leaves of Σ_0.

Lemma 3.18.

i) *If K_L denotes the component of $\mathring{N}(f, \delta_0) - L$ which contains J, then J is a component of $\cap_{L \text{ leaf of } \Sigma^f} K_L$. If J' is another component of $\cap_{L \text{ leaf of } \Sigma^f} K_L$, then J' is a component of H_0.*

ii) (*Continuity of J-normal orientation*). *If the leaf $L \in \Sigma^f$ is a limit of leaves $L_1, L_2, \dots$ of Σ^f and $\bar{J} \cap L \neq \emptyset$ then the J-normal orientation on L induces the J-normal orientation on L_i for i sufficiently large.*

Proof. i) The first assertion is immediate. The hypothesis of the second implies that for each σ^α_{0j}, J and J' lie in the same component of $\mathring{N}(f, \delta_0) - \sigma^\alpha_{0j}$ and hence in the same component of $\mathbb{H}^3 - \sigma^\alpha_{0j}$.

ii) Let $U = \mathring{B}_\rho(\epsilon, x)$ where $x \in \bar{J} \cap L$. If $D_i = L_i \cap U$, then $D_1, D_2, \dots$ is a sequence of discs converging in the C^k-topology, all $k < \infty$, to D. For i sufficiently large D_i and D nearly coincide, hence if their J-normal orientations are opposite, then $J \cap U$ would be confined to the region "between" D_i and D. Thus, if for i sufficiently large all such J-normal orientations were opposite, then $J \cap U = \emptyset$. q.e.d.

Definition 3.19. If two leaves L_1, L_2 of Σ^f are tangent at x with opposite J-normal orientations then we say that L_1, L_2 are *antitangent at x* and x is an *antitangential point*. If U is a sufficiently small η-neighborhood of x, then the components of $U \cap L_1$ and $U \cap L_2$ containing x are discs which meet along a saddle or multi-saddle tangency. Call the closure of a component of $U - L_1 \cup L_2$ lying on the + side of both L_1 and L_2 a *wedge of L_1, L_2 at x*.

Two minimal surfaces S and T which are tangent at y either coincide or meet at y along a multi-saddle. We say that the multi-saddle is of *multiplicity* $m(y)$, if locally $S \cap T$ consists of $2m(y) + 2$ arcs which meet at y. If x is an antitangential point, then define the $m(x)$, the *multiplicity* of x, to be the supremum of multiplicity of pairs of leaves that are antitangent at x.

We say that $x \in Bd(J)$ is a *spike point* if:

i) There exist a triple of leaves A, B, C of Σ^f having a common tangent vector at x, and no pair of A, B, C are tangent at x.

ii) There exists no $v \in T_x(\mathbb{H}^3)$ which is transverse to A, B, C and whose direction agrees with the three J-normal orientations at x.

If x is a spike point and U is a neighborhood of x which intersects each A, B, C in discs, then a *spike region* is the closure of a component of $U - A \cup B \cup C$ which lies on the $+$ side of A, B, C and limits on x.

Example 3.20. A spike point and spike region can be found in Figure 4.1 [6]. Note that if the normal orientation of one of the leaves was reversed, then these leaves would not define a spike point. To understand spike regions, consider very small spheres centered at x, and the possibly empty tiny triangles in these spheres which lie on the $+$ sides of the leaves. A spike region is the closure of a continuum of such triangles which limit on x.

Lemma 3.21.

i) *If L_1 and L_2 are antitangent at x, then J lies in at most one wedge of L_1 and L_2 at x.*

ii) *If leaves A, B, C define a spike point x, then at most one spike region of A, B, C at x intersects J.*

Proof. i) Let σ_{0i}^{α} and σ_{0j}^{β} be respectively laminations whose leaves contain L_1 and L_2. If J' is the component of $\mathbb{H}^3 - \sigma_{0j}^{\alpha} \cup \sigma_{0i}^{\beta}$ which contains J, then Lemma 4.1 [6] implies that J' can contain at most one wedge of L_1 and L_2 at x.

ii) Repeat the above argument using the laminations $\sigma_{0a}^{\alpha_a}, \sigma_{0b}^{\alpha_b}, \sigma_{0c}^{\alpha_c}$ which respectively contain leaves A, B and C. q.e.d.

Lemma 3.22.

i) *There exists an* $M < \infty$ *such that every antitangential point of* $\mathrm{Bd}(J)$ *has multiplicity* $\leq M$.

ii) *If* $x_1, x_2, \ldots \to x$ *is a sequence of antitangential points of* $\mathrm{Bd}(J)$, *then* $\mathrm{Lim}\, sup_{i\to\infty} m(x_i) < m(x)$.

iii) *The points of* $\mathrm{Bd}(J)$ *which are spike points but not antitangential points is a discrete set. Any limit point is an antitangential point.*

Proof. Suppose that $x_1, x_2, \ldots$ are a sequence of antitangential points of $\mathrm{Bd}(J)$ which limit on x. Suppose that $A_1, B_1; A_2, B_2; \ldots$ are leaves of Σ^f with antitangencies respectively at $x_1, x_2, \ldots$. By passing to subsequence, restricting to an $\eta \leq \epsilon$ open ball U about x, and letting D_i (resp. E_i) denote $A_i \cap U$ (resp. B_i), then by Lemmas 3.10 and 3.12 we can assume that these D_i's and E_i's are discs containing x_i which converge in the C^k-topology, all $k < \infty$, to antitangent discs D, E at x. Note that D and E cannot coincide in a neighborhood of x, else they would coincide in U and therefore $J \cap U = \emptyset$.

If we give U Euclidean coordinates and let Q (resp. Q_i) denote the tangent plane to D (resp. D_i) at x (resp. x_i), then D and E (resp. D_i and E_i) are the graphs of functions on Q (resp. Q_i) and the difference function w (resp. w_i) satisfies the following properties. After changing coordinates on Q by a linear transformation T, $w(z) = p(z)+q(z)$ where p is a linear homogenous harmonic polynomial of degree $d, 2 \leq d < \infty$ and $|q(z)| + |z||\nabla q(z)| + \cdots + |z|^d|\nabla^d q(z)| \leq C|z|^{d+1}$. Here $z = (z_1, z_2)$ denotes a point in $\mathbb{R}^2$ and $|\nabla^n q(z)|$ denotes the Euclidean norm of the vector of n^2 degree-n partial derivatives evaluated at z. This is the well known local description of tangent minimal surfaces, which is powered by the Bers - Vekua theorem on general continuation [1], [20]. A proof of the above result is given in Colding-Minicozzi II [5] and we acknowledge here the use of their notation and error term which appears sharper than that found in the literature.

i) Since $D_i \to D$ in the C^k-topology, $k < \infty$, it follows that

$$\mathrm{Lim}\ \mathrm{sup}_{i\to\infty} m(x_i) \leq m(x).$$

Using the compactness of $\mathrm{Bd}(J)/\langle g\rangle$ we obtain conclusion i).

ii) Now suppose that for all $i, m(x_i) = m(x) = d - 1$. It follows from [5] that after passing to subsequence, and changing coordinates on Q_i by a linear transformation T_i, with $T_i \to T$, that $w_i = p_i + q_i$ where p_i is a linear homogenous harmonic polynomial of degree $d, p_i \to p$, and $|q_i(z)| + |z||\nabla q_i(z)| + \cdots + |z|^d|\nabla^d q_i(z)| \leq C|z|^{d+1}$. Here C is the same uniform constant and the original coordinates on Q are related to those of Q_i via an orthogonal change of coordinates. In words, around a uniformly sized neighborhood of x, these (multi)-saddles are geometrically extremely close.

Since each of A_i and B_i meets U in a single disc it follows that $J \cap U$ must lie on the $+$ side of each of D, E and each D_i and E_i. Therefore for each $i, \bar{J} \cap U$ lies in $W_i \cap U$ (resp. $W \cap U$) where W_i (resp. W) is a unique wedge of $D_i \cup E_i$ emanating from x_i. This implies that for all i and j, $x_i \in W_j \cap U$ and $x_j \in W_i \cap U$ and finally $\{x_1, x_2, \dots\} \in W \cap U$.

Using the above equations it follows that there exists $\eta > 0, N < \infty$ so that if $U = B_\rho(\eta, x)$ and $i \geq N$, then W (resp. W_i) is contained in $K \cap U$ (resp. $K_i \cap U$) where K is a cone based at x (resp. x_i) of uniform angle $c < \pi$. Thus $\{x, x_1, x_2, \dots\} \subset K \cap_{i \geq N} K_i$ which is evidently impossible.

iii) If $x_1, x_2, \dots$ is a sequence of distinct spike points limiting on x, then after passing to subsequence there exist triples of leaves R_i, S_i, T_i of Σ^f defining the spikes which converge to a triple R, S, T which have a common tangent vector at x. If no pair of R, S and T are tangent at x, then using Schoen's normal curvature lemma it follows that there exists $\eta > 0, N < \infty$ so that if $i \geq N$ and $U = \mathring{B}_\rho(\eta, x)$ then $\bar{J} \cap U$ is contained in $K \cap U$ (resp. $K_i \cap U$) where K is a cone based at x (resp. x_i) of uniform angle $c < \pi$. Thus one obtains a contradiction as above.

If say R and S are tangent at x but not antitangent, then T must be antitangent to both. Otherwise for i sufficiently large the triple R_i, S_i, T_i would not satisfy the normal orientation requirement of Definition 3.19.

q.e.d.

Definition 3.23 (Linear Model near a boundary point x)**.** Suppose that $x \in Bd(J)$. Let T^1_x denote the sphere of unit vectors in $T(\mathbb{H}^3)$ through x. To each leaf L of Σ^f through x, let A_L denote the closed hemisphere in T^1_x bounded by the tangent plane through L and lying on the positive side of that plane. Let $C_L = \partial A_L$.

It follows from Lemma 3.10 that the collection of such C_L's is closed (in the space of great circles). Thus we obtain:

Lemma 3.24. *$A_x \overset{\text{def}}{=} \cap A_L$ is either*

i) *a closed disc,*

ii) *a closed interval,*

iii) *one or two points, or*

iv) *empty.*

Lemma 3.25. *If* $\operatorname{Lim} x_i \to x$ *where* $x_i \in J$ *and* $x \in \operatorname{Bd}(J)$ *and the* x_i*'s approach* x *asymptotically along a tangential direction* $v \in T_x^1$*, then* $v \in A_L$*, for all leaves of* Σ^f *through* x*.*

Proof. If not let $E \subset T_x^1 - A_L$ be a small round disc such that $v \in \mathring{E}$. If y is sufficiently close to x and in a small cone based at x defined by the directions in E, then y lies on the non $+$ side of L. This implies that $y \notin J$. q.e.d.

Definition 3.26. A point $x \in Bd(J)$ is of *Type* I if either there exists a vector u based at x which is transverse to each leaf of Σ^f passing through x or x is a spike point which is not an antitangent point. Otherwise we say $x \in \operatorname{Bd}(J)$ is of *Type* 0. Let $\mathcal{O}$ denote the collection of Type 0 points of $\operatorname{Bd}(J)$.

Lemma 3.27. *If x is of Type* 0*, then x is an antitangent point.*

Proof. We need to show that if $\cup_{L \text{ leaf through } x} C_L = T_x^1$, then x is either a spike point or an antitangential point. Under this hypothesis A_x is not a disc and by Lemma 3.25, $A_x \neq \emptyset$.

If A_x is an interval, then let $w \in \mathring{A}_x$ and $[-1,1] \subset T_x^1$ a small geodesic interval orthogonal to A_x and passing through w at 0. We will show that x is an antitangential point by showing that there exist leaves R and S such that $A_x \subset C_R$ (resp. $A_x \subset C_S$) and the $+$ side of R (resp. S) points towards $-1 \in [-1,1]$ (resp. $+1$). If not we derive a contradiction as follows. If $t > 0$, and L_t is a leaf with $t \notin A_{L_t}$, then C_{L_t} hits $[-1,1]$ at some unique point in $(0,t)$ with normal pointing towards 0. Since C_{L_t} is disjoint from $\mathring{A}_x$ it follows that as $t \to 0, C_{L_t}$ becomes nearly tangent to A_x. Using Lemmas 3.12, 3.18, we conclude that a limit corresponds to a leaf R such that C_R contains A_x with normal pointing towards -1. In a similar manner one constructs a leaf S through x so that C_S contains A_x with normal pointing to $+1$.

If $w \in A_x$ and A_x is one or two points, then let $T \subset T^1_x$ be the circle of vectors orthogonal to w. Using a limiting argument as above, for each $t \in T$, there exists a leaf L_t through x such that $t \notin \overset{\circ}{A}_{L_t}$ but $w \in C_{L_t}$. Just consider a limit of L_{t_s}'s where $t_s \to w$ and t_s lies on the geodesic arc from t to w and $t_s \notin A_{L_{t_s}}$. This implies that either $\cup_{t \in T} \overset{\circ}{A}_{L_t} = S^2_\infty - \{w, -w\}$ or $S^2_\infty - \gamma$ where γ lies in a great semi-circle from w to $-w$. In the latter case a limiting argument implies that x is an antitangent point. In the former case a compactness argument implies that for a finite set $\{t_1, t_2, \dots, t_n\}$, $\cup_{i=1}^n \overset{\circ}{A}_{L_{t_i}} = S^2_\infty - \{w, -w\}$. If no pair A_{t_i}, A_{t_j} are antitangent at x, then a combinatorial argument implies $n = 3$ suffices and hence x is a spike point. q.e.d.

Corollary 3.28. *Type* 0 *boundary points of* J *of multiplicity* 1 *are a closed discrete subset of* $\mathbb{H}^3$.

Definition 3.29. If $X \subset Y$, then we say that X is 0-LC at $x \in \mathrm{Bd}(X)$ if for each open set U of Y containing x, there exists an open set V of Y about x such that if $y, z \in X \cap V$, then there exists a path from y to z lying in $X \cap U$.

Lemma 3.30. *If* $x \in Bd(J)$ *is Type* I, *then* $\bar{J}$ *is a manifold near* x.

Proof. The point of this paragraph and the next is to show that a Type I spike point can be treated just like any Type I point which satisfies the first sentence of Definition 3.26. Suppose the leaves A, B, and C of Σ^f define a spike at x with S the germ of the spike region emanating from x and v a unit vector tangent to each of the three leaves. Let $\mathcal{P}$ denote the union of A, B and C together with the collection of leaves of Σ^f which pass through x and which are limits of leaves $L_1, L_2, \dots$ of Σ^f such that for all i, there exists $x_i \in L_i$ with $x_i \in \bar{J} - x$ and Lim $x_i = x$. As $\eta \to 0, B_\rho(\eta, x) \cap S$ is contained in narrower and narrower hyperbolic cones based at x. Therefore v must be tangent to every leaf in $\mathcal{P}$. By Lemmas 3.12 iv) and 3.18 ii) it follows that either x is an antitangential point or there exists a nonzero vector u based at x and transverse to each leaf in $\mathcal{P}$.

If x is Type I because there exists a vector u transverse to each leaf of Σ^f through x, then define Σ to be Σ^f. Otherwise x is not antitangential and there exists a u as in the preceding paragraph. Let $U = \overset{\circ}{N}_\rho(\eta, x)$ and define Σ to be the collection of leaves of Σ^f which hit $\bar{J} \cap U - x$ together with all limits of such leaves which pass through x. Choose

$\eta \leq \epsilon$ sufficiently small so that if L is a leaf of Σ passing through x, then L is transverse to the vector u constructed in the previous paragraph. Such an η exists, else one can find a sequence $L_1, L_2, \dots$ of leaves containing x which hit $\mathrm{Bd}(J)$ in points closer and closer to x and have u as a tangent vector. A limit of such leaves lies in $\mathcal{P}$ and is tangent to u, a contradiction.

Since x is of Type I, and leaves have bounded normal curvature, there exists a closed neighborhood of the form $V = D^2 \times [-1, 1]$, where x is identified with the origin, each $z \times [-1, 1]$ is transverse to Σ and if L is a leaf of $\Sigma|V$, and $L \cap \bar{J} \neq \emptyset$, then L is a closed disc properly embedded in $D^2 \times (-1, 1)$. Call a leaf L of Σ, $+$ if its normal vector points up and $-$ otherwise. If no $+$ leaf meets x, then by restricting the size of V we can assume that the set of $+$ leaves is empty. Similarly for the $-$ leaves. If there exist $+$ leaves (resp. $-$ leaves) define the function $A : D^2 \to [-1, 1]$ (resp. $B : D^2 \to [-1, 1]$) by $A(z) = \max\{t|\text{ some } + \text{ leaf hits } (z, t)\}$ (resp. $B(z) = \min\{t|\text{ some } - \text{ leaf hits } z \times t\}$). That Σ is closed in the C^k-topology implies such maxima and minima exist.

Claim 1. A and B are continuous.

Proof. We prove continuity of A. Let $y_1, y_2, \dots \to y$ be a convergent sequence in D^2. By Lemma 3.18 ii) and 3.12 iv) it follows that $\text{Lim sup } A(y_i) \leq A(y)$. Since there exists a $+$ leaf L through $(y, A(y))$ and transverse to $y \times [-1, 1]$ it follows that $A(y) \leq \text{ Lim inf } \{A(y_i)\}$.

q.e.d.

If there exists no $-$ leaves, then from the continuity of A one readily deduces that $\bar{J}$ is a manifold near x.

From now on we will assume that there exist both $+$ and $-$ leaves. Applying the standard results for intersections of least area surfaces, the Meeks-Yau exchange roundoff trick and the fact that $J \cap V \neq \emptyset$ to our setting we have:

Claim 2. A $+$ leaf and a $-$ leaf cannot coincide. If L_1 and L_2 are distinct leaves of Σ, then L_1 is transverse to L_2 except at finitely many (multi)-saddle tangencies. $L_1 \cap L_2$ contains no embedded simple closed curve. q.e.d.

Let $\pi : \bar{J} \cap V \to D^2 \overset{\text{def}}{=} D^2 \times 0$ be the projection onto the D^2-factor. Let J_1 be a component of $V \cap J$ and $E_1 \overset{\text{def}}{=} \pi(\bar{J}_1) \subset D^2$.

Claim 3. $\mathring{E}_1 = \pi(J_1 \cap \mathring{V})$ and $\mathring{E}_1$ is a disc.

Proof. The failure of the either claim would give rise to plus and minus leaves L_1, L_2 of Σ violating Claim 2. q.e.d.

Claim 4. $\overset{\circ}{E}_1$ is 0-LC at every point x of $\mathrm{Bd}(\overset{\circ}{E}_1)$.

Proof. It follows directly from the next paragraph that given $\kappa > 0$ each component of $B_{D^2}(\kappa/2, x) \cap \overset{\circ}{E}_1$ is contained in one of finitely many components of $B_{D^2}(\kappa, x) \cap \overset{\circ}{E}_1$. The subsequent paragraph will show that for some $\eta < \kappa/2$ at most one of the components of $B_{D^2}(\kappa, x) \cap \overset{\circ}{E}_1$ can meet $B_{D^2}(\eta, x)$. These facts imply Claim 4.

We will assume that $x \in \overset{\circ}{D}{}^2$ and $d_{D^2}(x, \partial D^2) > \kappa$, for the general case is similar but notationaly messier. We first show that if $\epsilon_1 < \epsilon_2 \leq \kappa$, and A is the annulus spanning the circles S_1, S_2 of radius ϵ_1, ϵ_2 about x, then only finitely many distinct components of $\overset{\circ}{E}_1 \cap A$ can hit both S_1 and S_2. Otherwise let $\alpha_1, \alpha_2, \ldots$ be properly embedded arcs lying in distinct components of $\overset{\circ}{E}_1 \cap A$ whose endpoints meet both S_1 and S_2. By passing to subsequence, reordering and choosing a correct basepoint and orientation on S_1 we can assume that the α_i's are ordered according the linear ordering of the points $\{\alpha_i \cap S_1\}$. Let S_3 be the circle of radius ϵ_3 about x, where $\epsilon_1 < \epsilon_3 < \epsilon_2$ and let $R_i \subset A$ denote the region between α_i and α_{i+1}. Finally for all i, let $s_i \in S_3 \cap R_i$ be such that $s_i \cap \pi(J) = \emptyset$ and s a limit point of $s_1, s_2, \ldots.$ Let L_i^+ (resp. L_i^-) denote a + leaf (resp. − leaf) through $(s_i, A(s_i))$ (resp. $(s_i, B(s_i))$. Let $A_i = \{z \in R_i | L_i^+$ is above L_i^- at $z\}$. Since least area surfaces intersect transversely or in multi-saddles or coincide it follows that $A_i \neq \emptyset$. The A_i's being disjoint implies that Lim area$(A_i) \to 0$. Let $i_1 < i_2 < \ldots$ be a sequence such that $L_{i_1}^+, L_{i_2}^+, \ldots$ limits to $L^+, L_{i_1}^-, L_{i_2}^-, \ldots$ limits to L^- and L^+, L^- respectively pass through $(s, A(s)) = (s, B(s))$. If $Z = \{z \in A | L^+$ is above L_- at $z\}$ and $C_1 > 0$ is the area of the smallest (of the finitely many) components of $\overset{\circ}{Z}$ which limit on s, then for i sufficiently large, area$(A_i) > \frac{1}{2} C_1 > 0$.

The previous paragraph implies that if $\eta \leq \kappa$ and F is a component of $B(\eta, x) \cap \overset{\circ}{E}_1$ which limits on x, then for $y \in F$ there exists an embedded path from y to x which, except for x, lies in F. If two such components F_1, F_2 limit on x, then there exists a simple closed curve α in E_1 which intersects $\mathrm{Bd}(\overset{\circ}{E}_1)$ only at x. If D_α was the disc bounded by α, then $\overset{\circ}{D}_\alpha \cap \mathrm{Bd}(\overset{\circ}{E}_1) \neq \emptyset$. Let $z \in \overset{\circ}{D}_\alpha$ be such that $A(z) = B(z)$. If L_1

and L_2 are respectively plus and minus leaves passing through $(z, A(z))$, then $L_1 \cap L_2$ gives rise to a contradiction to Claim 2. q.e.d.

Claim 5. For every $\kappa > 0, \overset{\circ}{E}_1$ is the union of finitely many connected subsets of of diameter less than κ.

Proof Use Claim 4 and the compactness of $\mathrm{Bd}(\overset{\circ}{E})$ to find an $\eta > 0$ so that $\overset{\circ}{N}_{D^2}(\eta, \mathrm{Bd}(\overset{\circ}{E})) \cap \overset{\circ}{E}$ is contained in the union of finitely many connected subsets of $\overset{\circ}{E}$ of diameter $< \kappa$. Any maximal collection $\{y_i\}$ of points in $\overset{\circ}{E} - \overset{\circ}{N}_{D^2}(\eta, \mathrm{Bd}(\overset{\circ}{E}))$, with the property $i \neq j$ implies $d_{D^2}(y_i, y_j) \geq \eta$ gives rise to a finite set F of open η-discs centered at the points of F which contain $\overset{\circ}{E} - \overset{\circ}{N}_{D^2}(\eta, \mathrm{Bd}(\overset{\circ}{E}))$. q.e.d.

Claim 6. E_1 is a disc.

Proof. Using techniques similar to those of [3, pp. 26–32] and Claim 4 it follows that E is the image of an immersed disc whose interior maps to $\overset{\circ}{E}$. Actually it is an embedding otherwise one obtains a contradiction to Claim 2. What follows is a formal argument.

In the classical language [22] a metric space satisfying the conclusion of Claim 5 is said to satisfy *Property S.* By Theorem 4.2 [22], $\mathrm{Bd}(\overset{\circ}{E}_1)$ is a locally connected compact continuum such that each point of $\mathrm{Bd}(\overset{\circ}{E}_1)$ is *accessible from all sides from* $\overset{\circ}{E}_1$.

If $\mathrm{Bd}(\overset{\circ}{E}_1)$ had a cut point y, then being accessible from all sides implies that there exists a simple closed curve α in E_1 which meets $\mathrm{Bd}(\overset{\circ}{E}_1)$ only at y and the disc bounded by α contains points of $\mathrm{Bd}(\overset{\circ}{E}_1)$ in it's interior. This leads to a contradiction to Claim 2.

Thus $\mathrm{Bd}(\overset{\circ}{E}_1)$ is a Peano continuum without cut points, p. 76 [23], and so by Corollary 3.32a [23] through each point of $\mathrm{Bd}(\overset{\circ}{E}_1)$ there exists a simple closed curve lying in $\mathrm{Bd}(\overset{\circ}{E}_1)$.

$\mathrm{Bd}(\overset{\circ}{E}_1)$ must be a simple closed curve, else $\mathrm{Bd}(\overset{\circ}{E}_1)$ contains an embedded graph θ of Euler characteristic -1. Being connected, $\overset{\circ}{E}$ lies in only one component of $D^2 - \theta$, contradicting the fact that each point of $\mathrm{Bd}(\overset{\circ}{E}_1)$ is a limit point of $\overset{\circ}{E}_1$. By the Schoenflies theorem $\mathrm{Bd}(\overset{\circ}{E}_1)$ bounds a disc which is evidently E_1. q.e.d.

Claim 7. $\pi(x)$ has a neighborhood that intersects exactly one component of $\pi(J \cap V)$.

Proof. First suppose that there exist two components F_1, F_2 of $\pi(J \cap V)$ with closures E_1, E_2 which contain x. By Claim 6 these E_i's are discs and the connectivity of J implies that both must intersect ∂D^2. Let $y_1, y_2 \in \partial D^2$ be such that $A(y_i) = B(y_i)$ and y_1, y_2 separate $F_1 \cap \partial D^2$ from $F_2 \cap \partial D^2$. Let $L_i^+, L_i^-, i = 1, 2$ be leaves of Σ respectively of plus and minus type which pass through $(y_i, A(y_i))$. Finally consider the (at most 4) laminations $\sigma_{0k}^{\alpha_i}$ which contain these leaves. If J' is the component of $\mathbb{H}^3 - \cup\sigma_{0k}^{\alpha_i}$ which contains J, then J' is either non-simply connected or J^* the closure of J' with respect to the induced path metric is not injectively immersed. Either situation contradicts Lemma 4.1 [6].

If every neighborhood of $\pi(x)$ hits infinitely many components of $\pi(J \cap V)$, then we obtain a contradiction in a manner similar to that of the second paragraph of the proof of Claim 4. q.e.d.

Claims 6 and 7 imply that after reducing the size of the D^2 and reparametrization $\pi(J \cap V) = \{(x, y) \in D^2 | y > 0\}$. Therefore $\bar{J} \cap V = \{(x, y, t) | y \geq 0 \text{ and } A(x, y) \leq t \leq B(x, y)\}$ which is a tamely embedded 3-cell, thereby completing the proof. q.e.d.

Lemma 3.31. *J is simply connected.*

Proof. Let $\mathcal{F}$ be the foliation of $\mathbb{H}^3$ by totally geodesic hyperbolic planes orthogonal to δ_0. Let $\alpha \subset J$ be a simple closed curve homotopically nontrivial in J. Choose α to be transverse to $\mathcal{F}$ except at finitely many points and assume that α has been chosen to minimize this number. Our α may lie completely in a leaf of $\mathcal{F}$.

Case 1. α bounds an embedded disc D in $\mathbb{H}^3$.

Proof. Let $N \subset J$ be a closed tubular neighborhood of α and N_1 a smaller tubular neighborhood with $N_1 \subset \mathring{N}$. Modify the metric r to r_1 so that $r | \mathbb{H}^3 - N = r_1 | \mathbb{H}^3 - N$ and $\mathbb{H}^3 - \mathring{N}_1$ has strictly convex boundary with respect to metric r_1. Furthermore if $E \subset \mathbb{H}^3 - \mathring{N}_1$ is a disc with $\partial E \cap N = \emptyset$, then $\text{area}_{r_1}(E) \geq \text{area}_r(E)$. By [15] there exists an essential properly embedded r_1-least area disc $D \subset \mathbb{H}^3 - \mathring{N}_1$ with $\partial D \subset \partial N_1$. Furthermore D is least area among all essential immersed discs with boundary ∂D on ∂N_1. If $D \cap L \neq \emptyset$ where L is a leaf of some σ_{0j}^{α}, then since L is a leaf of a D^2-limit lamination, there exists an embedded disc $F \subset L$, with $\partial F \cap D = \emptyset$ and $F \cap D \neq \emptyset$. Since F is an r-least area disc and D is locally r-least area near $D \cap F$, D is transverse to F except at finitely many tangencies of the standard saddle or multi-saddle type. A simple closed curve in $D \cap F$ bounds subdiscs $D' \subset D$,

$F' \subset F$ with $\partial D' = \partial F'$. If $\text{area}_{r_1} D' \geq \text{area}_{r_1} F'$, then the Meeks-Yau exchange roundoff technique gives rise to a disc contradicting the r_1-area minimality of D. If $\text{area}_{r_1} D' < \text{area}_{r_1} F'$, then the exchange roundoff technique gives rise to a disc contradicting the r-area minimality of F, since by construction $\text{area}_r D' \leq \text{area}_{r_1} D'$.

q.e.d.

Case 2. α is homotopically nontrivial in J.

Proof. We first show that α is homologically trivial in J. Let r and r_1 be Riemannian metrics as in Case 1. By [15] there exists an oriented genus minimizing, properly embedded surface $S \subset \mathbb{H}^3 - \mathring{N}_1$ with connected boundary which is an r_1-least area surface among all immersed surfaces properly homotopic to S rel ∂S in $\mathbb{H}^3 - \mathring{N}_1$. ($S$ can be thought of as a minimal genus Seifert surface for α.) Again if $S \cap \sigma^\alpha_{0j} \neq \emptyset$, there exists a disc F lying in a leaf of σ^α_{0j} which nontrivially intersects S and $\partial F \cap S = \emptyset$. Let $F' \subset F$ be a disc such that $\partial F' \subset S$. Since S is π_1-injective in $\mathbb{H}^3 - \mathring{N}_1$ it follows that $\partial F'$ bounds a disc $D' \subset S$. We now obtain a contradiction as in Case 1.

Therefore if T is a (connected) leaf of $\mathcal{F}|J$ and T is transverse to α, then $|T \cap \alpha| \geq 2$. If for all leaves T of $\mathcal{F}|J$, $|T \cap \alpha| \leq 2$, then α is unknotted in $\mathbb{H}^3$. Finally, if $|T \cap \alpha| > 2$, and $x, y \in T \cap \alpha$ are such that the oriented α points "up" at both x and y, then we obtain a homotopically nontrivial curve in J with fewer points of tangency with $\mathcal{F}$ via the following procedure. Concatenate the appropriate component of $\alpha - \{x, y\}$ with an arc in T with endpoints x, y and isotope slightly.

q.e.d.

Definition 3.32. Given $x \in Bd(J)$, $\eta < \epsilon$ and L a leaf of Σ^f we say that L is *x_η − relevant* if $L \cap \bar{J} \cap B_\rho(\eta, x) \neq \emptyset$.

Lemma 3.33. *For each $x \in Bd(J)$ there exists $\epsilon_x \leq \epsilon$ such that if $\eta \leq \epsilon_x$ and L is x_η-relevant, then $L \cap B_\rho(\eta, x)$ is a disc transverse to $\partial B_\rho(\eta, x)$ and each component of $\partial B_\rho(\eta, x) - L$ has area at least $.40\,\text{area}(\partial B_\rho(\eta, x)))$.*

Proof. It follows from Schoen's bounded normal curvature lemma [19] that there exists a constant C such that if η is sufficiently small, $\beta < C\eta$ and L any leaf of $\Sigma^f|B_\rho(\eta, x)$ which intersects $B_\rho(\beta, x)$, then L satisfies the conclusion of the lemma. Since the metric r is induced from a metric on a closed hyperbolic 3-manifold, C can be chosen independent of x.

Suppose that $L_1, L_2, \ldots$ were a sequence of x_{η_i}-relevant leaves which failed the conclusion of the lemma, where $\eta_i \to 0$. Let y_i be the ρ-closest point of L_i to x. By passing to subsequence we can assume that y_i approaches x asymptotically along a ray. Since $d_\rho(L_i, x) > C\eta$ it follows again by Schoen, that for i sufficiently large and j sufficiently larger $L_j \cap L_i \cap B_\rho(\eta_i, x) = \emptyset$ and $L_j \cap B_\rho(\eta_i, x)$ separates x from $L_i \cap B_\rho(\eta_i, x)$. Since x lies on the $+$ side of L_j it follows that $\bar{J} \cap L_i \cap B_\rho(\eta_i, x) = \emptyset$, a contradiction. q.e.d.

Lemma 3.34. *If $x \in \bar{J}$ is a Type 0 point, then $\bar{J}$ is a manifold near x.*

Proof. The proof will follow by induction on $m(x)$.

Step 1. $m(x) = 1$.

Proof of Step 1. By Lemma 3.22 there exists ϵ_1 such that $0 < \epsilon_1 < \epsilon_x$ and $B_\rho(\epsilon_1, x)$ contains a unique Type 0 point. Let B_t denote $B_\rho(t, x)$.

Claim 1. If $0 < t \leq \epsilon_1$, then there exists $t_1 > 0$ so that for each component X of $B_t \cap J$ either $x \in \overline{X}$ or $d_\rho(X, x) > t_1$.

Proof. If not then $B_t \cap J$ has infinitely many components which hit $B_{t/2}$. Let $y_1, y_2, \ldots$ a sequence of points lying in distinct components of $B_t \cap J$, with $y_i \in \partial B_{t/2}$. If y is a limit point of $\{y_i\}$, then $y \in \bar{J}$ and is of Type I. But this contradicts the fact that $\bar{J}$ is a manifold near y. q.e.d.

Claim 2. There exists a unique component J_i of $B_{\epsilon_1} \cap J$ which limits on x.

Proof. It follows from Claim 1 (with $t = \epsilon_1$) that at least one component limits on x.

Suppose that two components J_1 and J_2 limit on x. For $i = 1, 2$ let $\mathring{E}_i$ be the component of $\partial B_{\epsilon_1} \cap J_i$ such that there exists a path $\sigma \subset (J - \mathring{B}_{\epsilon_1})$ from E_1 to E_2. Each $\mathring{E}_i$ is an open disc else:

i) There exists a leaf L tangent to $\partial B_\eta, \eta \leq \epsilon_1$ at a point $y \in \bar{J}$. This contradicts Lemma 3.33.

ii) There exist leaves L_1 and L_2 such that $L_1 \cap L_2 \neq \emptyset$ but $L_1 \cap \partial B_{\epsilon_1}$ and $L_2 \cap \partial B_{\epsilon_1}$ lay in disjoint components of $\partial B_{\epsilon_1} - E_i$. This implies that $L_1 \cap L_2$ contains a simple closed curve, which is a contradiction.

A proof similar to that of Lemma 3.30 shows that each $\overline{E}_i$ is a closed disc.

To complete the proof it suffices to show that there exists a finite set of leaves of $\Sigma^f|B_{\epsilon_1}$ whose boundaries separate $\overset{\circ}{E}_1$ from $\overset{\circ}{E}_2$. Indeed, if $\sigma_{0i_1}^{\alpha_1}, \dots, \sigma_{0i_n}^{\alpha_n}$ were the laminations containing these leaves and $\hat{J}$ the component of $\mathbb{H}^3 - \cup_{k=1}^n \sigma_{0i_k}^{\alpha_k}$ which contained J, then $\hat{J}^*$ is either not simply connected or not injectively immersed in $\mathbb{H}^3$. (Here $\hat{J}^*$ is the closure of $\hat{J}$ with respect the induced path metric.) This violates Lemma 4.1 [6].

Since x is an antitangency point there exists a wedge W containing $B_{\epsilon_2} \cap J$. By reducing the size of ϵ_1, if necessary, we can assume that $\text{area}_\rho(D_W) < .01\,\text{area}\,\partial B_{\epsilon_1}$ where $D_W = W \cap \partial B_{\epsilon_1}$. Suppose that W is defined by the antitangent leaves A, B. We will show that $n \leq 4$ where two of the leaves are A and B. The other one or two leaves will intersect E_1.

Parametrize ∂E_1 by $[0,1]$ mod 1. (From the observer standing on B_{ϵ_1}, choose the parametrization to correspond to a clockwise path about ∂E_1.) If $t \in \partial E_1$, then L_t will denote a leaf of $\Sigma^f \cap B_{\epsilon_1}$ through t. L_t is not in general unique. Since $E_1 \cup E_2 \subset D_W$ it follows from the previous paragraph and Lemma 3.33, that for any $t, \partial L_t \cap \partial D_W \neq \emptyset$.

If there exists a leaf L_0 such that $\partial L_0 \cup \partial D_W$ separates $\overset{\circ}{E}_1$ from $\overset{\circ}{E}_2$ then L_0, A, B are our desired leaves. Otherwise fix a L_0. Next consider a L_t. If $\partial L_0 \cup \partial L_t \cup \partial D_W$ separate $\overset{\circ}{E}_1$ from $\overset{\circ}{E}_2$ we are done, otherwise there exists a path α from $\overset{\circ}{E}_2$ to $\overset{\circ}{E}_1$ missing $\partial L_0 \cup \partial L_t \cup \partial D_W$. We say L_t is to the *left of* 0 if α can be chosen to miss $(0,t)$. Otherwise we say that L_t is to the *right of 0*. If there exists leaves L and L' both of which pass through t with one to the left of 0 and the other to the right of 0, then $\partial L \cup \partial L' \cup \partial D_W$ separate $\overset{\circ}{E}_1$ and $\overset{\circ}{E}_2$. Again we are done if there is a L_t to the left (resp. right) and a $L_{t'}$ to the right (resp. left) with $0 < t' \leq t$ (resp. $0 < t \leq t'$). If $L_{t_1}, L_{t_2}, \dots$ limits to L_s and each L_{t_i} is to the left (resp. right) of 0, then either L_s is to the left (resp. right) of 0 or $\partial L_s \cup \partial L_0 \cup \partial D_W$ separate. If $L_t, t \neq 0$ is to the left and L' is a limit of leaves $L_t, t \in (0,1), t \to 1$, then $\partial L_0 \cup \partial D_W \cup \partial L'$ separate.

q.e.d.

By Claims 1 and 2 there exists an embedded path $\alpha : [0,1] \to \mathbb{H}^3$ such that $\alpha([0,1)) \subset J$ and $\alpha(1) = x$. We can assume α is transverse to $\cup_{i=1}^\infty \partial B_{s_i}$, where $s_i = \epsilon_1/i$. Also if $\overset{\circ}{F}$ is a component of $\partial B_{s_i} \cap J$, then $|\overset{\circ}{F} \cap \alpha| \leq 1$. Let $\overset{\circ}{F}_1, \overset{\circ}{F}_2, \dots$ denote the collection of such discs ordered by how they are hit by α. Let $t_i \in \{s_i\}$ denote that value so that

$F_i \subset \partial B_{t_i}$. If $j > i$ and $j, i \in \mathbb{N}$, then let $R_{(i,j)}$ denote the component of J between $\mathring{F}_i$ and $\mathring{F}_j$. Let R_i denote $R_{(i,i+1)}$.

The region R_i is simply connected since each of $\mathring{F}_i, \mathring{F}_{i+1}$ and J are simply connected. The continuity of α and Claim 1 imply that for each $t \leq \epsilon_1$ there exists $N_t > 0$ such that $R_{(i,j)} \subset B_t$ for $j > i \geq N_t$. We now show that for $i > N_{\epsilon_1}, \overline{R}_i$ is a 3-ball. First, each point of $\mathrm{Bd}(R_i)$ not on $\mathring{F}_i$, is a Type I point of $\mathrm{Bd}(J)$. Second, by Lemma 3.33 ∂B_{t_i} is transverse to each x_{t_i}-relevant leaf. Thus the argument of Lemma 3.30 shows that R_i is a manifold near each point of $\mathrm{Bd}(F_i) \cup \mathrm{Bd}(F_{i+1})$. Thus $\overline{R}_i$ is an irreducible compact simply connected manifold and hence is a 3-ball. A similar argument shows that for $j > i, \overline{R}_{(i,j)}$ is a 3-ball. By the second sentence of this paragraph the sequence of balls $\overline{R}_j$, limit only on x. Thus, if X is the component of $J - F_i$ which limits on x, then $\overline{X}$ is homeomorphic to $\cup_{j \geq i} \overline{R}_j \cup x$ where the latter space is topologized with the 1-point compactification with x being the point at infinity. Thus $\overline{X}$ is topologically a 3-ball. This completes the proof of Step 1. q.e.d.

Step 2. $1 < m(x) \leq M$.

Proof of Step 2. Assume by induction that the theorem is true for $\{x | m > m(x) \geq 1\}$. We will prove it for $m(x) = m$. The proof of Step 2 follows exactly like the proof of Step 1 with the following minor modifications. By Lemma 3.22, for almost all s, $\partial B_s \cap \mathcal{O} = \emptyset$. Choose ϵ_1 so that $\partial B_{\epsilon_1} \cap \mathcal{O} = \emptyset$. In the proof of Claim 1 use only $\{t | (\partial B_t \cup \partial B_{t/2}) \cap \mathcal{O}\} = \emptyset$. In the paragraph after the proof of Claim 2, choose $\{s_i\}$ so that $\epsilon_1 > s_1 > s_2 > \ldots \to 0$ and for each $i, \partial B_{s_i} \cap \mathcal{O} = \emptyset$. q.e.d.

Remark 3.35. It follows immediately from the proof of Lemma 3.30 that $\partial \bar{J}$ is tamely embedded in $\mathbb{H}^3$ at each Type I point of ∂J. A more extensive argument shows that $\partial \bar{J}$ is tamely embedded at all the Type 0 points. Therefore by Bing [3] and Moise [Me] $\partial \bar{J}$ is tamely embedded in $\mathbb{H}^3$.

Proposition 3.36. *If J is a component of H_0, then $\bar{J}$ is a simply connected manifold with boundary whose interior is J.*

Proof. Apply Lemmas 3.30, 3.31 and 3.34. q.e.d.

A collection of smooth simple closed curves $\hat{\lambda}_i$ in S^2_∞ is *locally finite* if in the unit ball model, for every $\eta > 0$ there are only finitely many $\hat{\lambda}_i$'s of diameter $> \eta$.

Lemma 3.37. *Let $x, y \in S^2_\infty$ and $\{\hat{\lambda}_i\}$ be a locally finite collection of smoothly embedded simple closed curves in $S^2_\infty - \{x, y\}$, such that no $\hat{\lambda}_i$ separates x from y. Let r be a Riemannian metric induced from a closed hyperbolic 3-manifold. To each $\hat{\lambda}_i$, let $\hat{\Sigma}_i$ be the union of r-least area D^2-limit laminations which span $\hat{\lambda}_i$. Let $\hat{\Sigma} = \cup \hat{\Sigma}_i$. Let $\hat{H} = \mathbb{H}^3 - \hat{\Sigma}$.*

If $\hat{J}$ is a component of $\hat{H}$, then $\hat{J}$ is open and simply connected.

Proof. The proof of Lemma 3.10 shows that $\hat{H}$ is an open set. Simple connectivity follows from the proof of Lemma 3.31. q.e.d.

Lemma 3.38. *Let $x, y \in S^2_\infty$, r a Riemannian metric on $\mathbb{H}^3$ induced from the closed 3-manifold M and $\hat{\lambda}_1, \ldots, \hat{\lambda}_m$ smooth simple closed curves in $S^2_\infty - \{x, y\}$ such that no $\hat{\lambda}_i$ separates x from y. For each i, let $\hat{\Sigma}_i$ be the union of r-least area D^2-limit laminations which span $\hat{\lambda}_i$ and let H_i be the complementary region of $\mathbb{B}^3 - \hat{\Sigma}_i$ which contains x, y. Then one of the following must hold:*

i) *x, y lie in the same component H of $\cap_{i=1}^m H_i$.*

ii) *There exist $\hat{\lambda}_i, \hat{\lambda}_j, \hat{\lambda}_k$ such that $\hat{\lambda}_i \cup \hat{\lambda}_j \cup \hat{\lambda}_k$ separate x from y in S^2_∞.*

Proof. This is Lemma 4.3 [6] with $\hat{\Sigma}_i$ in place of σ_i. If $m \leq 3$ and ii) does not hold, then x, y lie in the same component of $S^2_\infty - \cup \hat{\lambda}_i$, so i) follows by Proposition 3.9 [6]. Assuming inductively that the lemma is true for $m < p$, we will establish it for cardinality p. Therefore either ii) holds or

$(*)$ for every $j \leq p$, x and y lie in the same component of $\cap_{i \neq j} H_i$.

We show that if $(*)$ holds, then either i) holds or for each j and k, $\hat{\lambda}_j \cap \hat{\lambda}_k \neq \emptyset$. Let $\tau_j \subset \cap_{i \neq j} H_i$ (resp. $\tau_k \subset \cap_{i \neq k} H_i$) be a properly embedded path from x to y. By Lemma 3.37, each component of $\mathrm{int}(\cap_{i \notin \{j,k\}} H_i)$ is simply connected. Thus, there exists $h : I \times I \to \cap_{i \notin \{j,k\}} H_i$ a homotopy from τ_j to τ_k. Either $\hat{\Sigma}_k \cap \hat{\Sigma}_j \neq \emptyset$ and hence $\tau_k \cap \tau_j \neq \emptyset$ by the proof of Lemma 3.5, or $h^{-1}(\hat{\Sigma}_k)$ and $h^{-1}(\hat{\Sigma}_j)$ are disjoint closed sets which are disjoint from $\partial I \times I$. Also $\hat{\Sigma}_k$ (resp. $\hat{\Sigma}_j$) is disjoint from $I \times 0$ (resp. $I \times 1$). Thus, $\tau_k \cap \tau_j = \emptyset$ implies that there exists an embedded path τ from $0 \times I$ to $1 \times I$ disjoint from $h^{-1}(\hat{\Sigma}_j \cup \hat{\Sigma}_k)$. Finally, $h \circ \tau$, is a path from x to y in $\cap_{i=1}^p H_i$ and so conclusion i) holds. (To construct τ, first engulf $h^{-1}(\hat{\Sigma}_j)$ and $h^{-1}(\hat{\Sigma}_k)$ in two disjoint families, each of which is a finite union of closed smooth regions disjoint from $\partial I \times I$ and $I \times 0$ (for $h^{-1}(\hat{\Sigma}_k)$) or $I \times 1$ (for $h^{-1}(\hat{\Sigma}_j)$).)

Now argue as in the last paragraph of the proof of Lemma 4.3 [6].

q.e.d.

Lemma 3.39. *There exists a unique unbounded component of H_0. There exists a uniform bound on the diameter of the bounded regions.*

Proof. By Proposition 3.9 [6] each σ^α_{0j} lies in an e-neighborhood of the ρ-convex hull of λ_{0j} and by Lemma 3.6 there are only finitely many outermost $\langle g\rangle$-orbits of λ_{0j}'s. Thus there exists $N_0 > 0$ such the image under ρ-orthogonal projection of any σ^α_{0j} into δ_0 has ρ-diameter $< N_0/2$ and hence the image under orthogonal projection of any Σ_{0j} has ρ-diameter $< N_0$. By Lemma 3.7 iii), $H_0 \subset N_\rho(a, \delta_0)$. Let $N_1 = 2(\text{length}_\rho(\delta) + N_0 + a)$. Parametrize $\mathbb{B}^3 - \partial\delta_0$ by $D^2 \times \mathbb{R}$ so that in these coordinates $g(x,t) = (x, t+L)$, where $L = \text{length}(\delta)$ and $0 \times \mathbb{R}$ parametrizes δ_0 by arclength. Let $p : \mathbb{H}^3 \to \delta_0$ denote ρ-orthogonal projection.

In the next two paragraphs we show that there exists at most one component of H_0 with diameter $\geq N_1$ and that any component of diameter $\geq N_1$ is unbounded. If J_1 and J_2 are distinct components of H_0 such that $[-N_0, N_0] \subset p(J_1) \cap p(J_2)$ then let β_1, β_2 be paths respectively in J_1, J_2 whose orthogonal projections are paths from $-N_0$ to N_0. Let $\Sigma^* = \cup\{\Sigma_{0j} | 0 \in p(\Sigma_{0j})\}$. By choice of $N_0, p(\Sigma^*) \subset (-N_0, N_0)$ and so J_1 and J_2 lie in the same component C of $\mathbb{H}^3 - \Sigma^*$. Indeed there exists a closed loop $\beta \subset \mathbb{H}^3 - \Sigma^*$ obtained by concatenating β_1, β_2 with arcs lying in $D^2 \times -N_0$ and $D^2 \times N_0$. By construction $D^2 \times 0 \cap \Sigma^* = D^2 \times 0 \cap \Sigma_0$. Thus some component of $(D^2 \times 0) \cap C$ is hit algebraically nonzero times by β. This contradicts Lemma 3.37.

If J_3 is a component of H_0 of diameter $\geq N_1$ such that $p(J_3) \neq \delta_0$, then there exist distinct $\langle g\rangle$-translates J_1, J_2 of J_3 such that $[-N_0, N_0] \subset p(J_1) \cap p(J_2)$, which contradicts the previous paragraph. Thus if J_1, J_2 are distinct components of diameter $\geq N_1$, then they both must project to all of δ_0 and one thereby obtains a contradiction as in the previous paragraph.

To complete the proof we need to show that there exists a component of H_0 of diameter $\geq N_1$. Let $Z^* = \cup\{\Sigma_{0j} | p(\Sigma_{0j}) \cap [-N_1, N_1] \neq \emptyset$ and $\lambda_{0j} = h(\lambda_{0i})$ for some $i \in \{1, 2, \ldots, m\}$ and $h \in \langle g\rangle\}$. (Recall Lemma 3.6.) So $Z^* = \Sigma_{0j_1}, \ldots, \Sigma_{0j_k}$. It follows from Lemma 3.6 that $H_0 \cap D^2 \times [-N_1, N_1] = \cap_{i=1}^k H_{0j_i} \cap D^2 \times [-N_1, N_1]$. By Lemma 3.38, there exists a component J_1 of $\mathbb{B}^3 - Z^*$ which contains a properly embedded path β connecting $\partial\delta_0$. For each i, β must lie in H_{0j_i}. Thus $\beta \cap D^2 \times [-N_1, N_1]$ lies in some component of H_0 and $\text{diam}_\rho(\beta \cap D^2 \times [-N_1, N_1]) \geq 2N_1$.

q.e.d.

Remark 3.40. This proof together with the usual convergence of laminations results implies that if r lies in a compact region K of the space of Riemannian metrics on M, then there exists a uniform bound $N(K)$ for the ρ-diameter of bounded regions of H_0.

Lemma 3.41. *If $P' : \mathbb{H}^3 \to M_\delta = \overset{\circ}{D}{}^2 \times S^1$ is the quotient map under the action of $\langle g \rangle$, then $P'(H_0)$ is the union of open balls and one open solid torus denoted $\overset{\circ}{T}_r$. The closure $\overset{\circ}{T}_r$ is a solid torus denoted T_r. Therefore $H_0 \subset \mathbb{H}^3$ is a union of uniformly bounded open balls and one component $\overset{\circ}{U}_r$ whose $\mathbb{H}^3$-closure is a $D^2 \times \mathbb{R}$ denoted U_r whose ends limit on $\partial\delta_0$.*

Proof. Each component Z of $P'(H_0)$ has $\pi_1(Z) \in \{1, \mathbb{Z}\}$ since it is covered by a component of H_0, which by Lemma 3.31 is simply connected, with covering translations contained in $\langle g \rangle$. If Z is covered by a bounded region J, then by Proposition 3.36 and Lemma 3.39 $\bar{J}$ is a uniformly bounded ball and hence Z is a uniformly bounded embedded open ball. If Z is covered by the unique unbounded region $\overset{\circ}{U}_r$, then the uniqueness of this manifold together with its $\langle g \rangle$-invariance and the fact that its closure U_r is a manifold implies that $\overline{Z}$ is a compact manifold with boundary. Z is irreducible since it is covered by an irreducible manifold. Therefore $\overline{Z}$ is a solid torus and hence U_r is a $D^2 \times \mathbb{R}$ whose ends limit on $\partial\delta_0$. q.e.d.

Remark 3.42. By further generalizing the argument of Step 3, p. 63 [6] it is not difficult to show that if Z is an open ball, then $\overline{Z}$ is a closed ball.

Lemma 3.43. *Every core of T_r is a core of M_δ.*

Proof. Let $\{\sigma_{ij}^{\alpha_k}\}$ be a $\pi_1(M)$-invariant collection of r-least area D^2-limit laminations such that for $1 \le i \le m$, each λ_{0i} is spanned by $0 < n_i < \infty$ laminations. Since only finitely many $\langle g \rangle$-orbits of laminations are involved, the technology of [6] applies and we obtain an immersed solid torus V_α in M. Also if $\overset{\circ}{T}_\alpha$ is the injective lift of $\overset{\circ}{V}_\alpha$ to M_δ, then its closure T_α is an embedded solid torus. Similarly if $\overset{\circ}{U}_\alpha$ denotes the lift of $\overset{\circ}{V}_\alpha$ to $\mathbb{H}^3$ which contains $\overset{\circ}{U}_r$, then U_r the closure of $\overset{\circ}{U}_r$, is an embedded $D^2 \times \mathbb{R}$ limiting on $\partial\delta_0$. Because δ lifts to a core of M_δ, C_α lifts to a core $\widetilde{C}_\alpha \subset \overset{\circ}{T}_\alpha$ of M_δ.

Using the local structure of ∂U_r it is not difficult to show that given

$\eta > 0$ there exists a collection of laminations $\{\sigma_{ij}^{\alpha_k}\}$ as above so that $U_r \subset U_\alpha \subset N_\rho(\eta, U_r)$ and hence $T_r \subset T_\alpha \subset N_\rho(\eta, T_r)$.

Let C_r be a core of $\overset{\circ}{V}_r$ with $\widetilde{C}_r$ its closed lift to M_δ. To show that C_r is a core of M_δ it suffices to show that $\pi_1(M_\delta - \widetilde{C}_r) = \mathbb{Z} \oplus \mathbb{Z}$. This is equivalent to showing that if T is a solid torus standardly embedded in M_δ and $T_r \subset T$, then every loop $\kappa \subset M_\delta - \widetilde{C}_r$ is homotopic in $M_\delta - \widetilde{C}_r$ to a loop in $M_\delta - T$.

Let κ be a loop in $M_\delta - \widetilde{C}_r$. Since $\widetilde{C}_r$ is a core of T_r, κ is homotopic in $M_\delta - \widetilde{C}_r$ to a loop κ_1 in $M_\delta - T_r$. Let $\eta < d_\rho(\kappa_1, T_r)$ and choose T_α so that $T_\alpha \subset N_\rho(\eta, T_r)$. As in the proof of Lemma 4.1 [6], each point of ∂U_α is of Type I and hence ∂T_α is tamely embedded in M_δ. Since $\widetilde{C}_\alpha$ is isotopically standardly embedded in M_δ this implies that T_α is isotopically standard in M_δ. Therefore κ_1 (resp. κ) can be homotoped off of any compact set in M_δ via a homotopy disjoint from T_α (resp. C_r).

q.e.d.

Proof of the Solid Torus Theorem: i) Our desired canonical immersion of a $D^2 \times S^1$ (up to reparametrization) into M is given by $q|T_r$, where $q : M_\delta \to M$ is the covering projection associated to the subgroup $\langle g \rangle \subset \pi_1(M)$. Note that $q|\overset{\circ}{T}_r$ is an embedding since $\overset{\circ}{T}_r \subset \overset{\circ}{T}_\alpha$ and by [6] $q|\overset{\circ}{T}_\alpha$ is an embedding. Here $\overset{\circ}{T}_\alpha$ is any $\overset{\circ}{D}{}^2 \times S^1$ obtained by applying the insulator construction of [6]. Let V_r denote the immersed solid torus $q|T_r$ and $\overset{\circ}{V}_r = q(\overset{\circ}{T}_r)$. (Recall Definition 3.1.)

ii) By Lemma 3.43 if C_r is a core of $\overset{\circ}{V}_r$, then C_r lifts to a core $\widetilde{C}_r$ of M_δ. Let $\overset{\circ}{V}_\alpha \subset M$ be obtained from the insulator construction of [6] and C_α a core of $\overset{\circ}{V}_\alpha$. Since C_α is isotopic to δ and $C_r \subset C_\alpha$ it suffices to show that C_r is a core of $\overset{\circ}{V}_\alpha$. This in turn is equivalent to showing that $\widetilde{C}_r$ is a core of $\overset{\circ}{T}_\alpha$. But as noted in the proof of Lemma 3.43, T_α is isotopically a standard solid torus in M_δ. Therefore $\widetilde{C}_r$ is isotopic to $\widetilde{C}_\alpha$ in $\overset{\circ}{T}_\alpha$ if and only if is isotopic in M_δ. By Lemma 3.43 they are both isotopic to a core of M_δ.

As indicated in 3.2, the orientation on δ induces an orientation on the core C_r of $\overset{\circ}{V}_r$. This orientation has the property that if γ_0 is the lift of C_r with ends limiting on $\partial\delta_0$, then γ_0 is oriented from the negative to the positive endpoint. If γ_0 and δ_0 are viewed as properly embedded arcs in $\mathbb{B}^3$, then any isotopy of C_r to δ lifts to a proper isotopy of γ_0 to

δ_0. This implies that C_r and δ are isotopic as oriented curves.

iii) By the convexity property of Definition 0.4 [6] there exists, for each j, a totally geodesic plane P_j such that ∂P_j separates $\partial\delta_0$ from λ_{0j}. Therefore P_j separates δ_0 from any ρ-least area D^2-limit lamination σ_{0j} spanned by λ_{0j}. Thus $\delta_0 \subset \overset{\circ}{U}_\rho$ and $\delta \subset \overset{\circ}{V}_\rho$. Reversing the proof of ii) we see that δ is a core of $\overset{\circ}{V}_\rho$. That argument also shows that the orientation on δ, viewed as a core of V_ρ coincides with the given orientation on δ. q.e.d.

4. Proof of the Coarse Torus Isotopy Theorem

Notation 4.1. Let $\mathrm{RM}(M)$ denote the space of Riemannian metrics on M and $f : B^n \to \mathrm{RM}(M)$ continuous. Let δ be an oriented simple closed geodesic in the closed orientable hyperbolic 3-manifold M possessing a non-coalescable insulator family $\{\lambda_{ij}\}$. Let V_x denote the canonical immersed solid torus in M associated to the Riemannian metric $f(x)$. If $x \in B^n$, then let $d_x(p,q)$ denote distance in M measured by the Riemannian metric $f(x)$ and let $d_{B^n}(x,y)$ denote distance measured in the standard metric on B^n. Let $W_x^\epsilon = \{y \in V_x | d_x(y, \partial V_x) \geq \epsilon\}$.

Lemma 4.2. *Let $f : B^n \to \mathrm{RM}(M)$ be continuous, where M is a closed hyperbolic 3-manifold. There exists $e > 0$ such that if $x \in B^n, \lambda$ a smooth simple closed curve in S^2_∞ and σ an x-least area D^2-limit lamination spanning λ, then $\sigma \subset N_\rho(e, C(\lambda))$ where $C(\lambda)$ is the hyperbolic convex hull of λ.*

Proof. This statement is exactly the last sentence of Proposition 3.9 [6], except the e is uniform over all of B^n. The proof of Proposition 3.9 [6] shows that an e will work for a fixed metric x, provided that e satisfy the conclusion of Lemma 3.7 [6]. Lemma 3.7 states that if $g : B^1 \to \mathrm{RM}(M)$, then there exists a uniform e which works for all the metrics $g(x)$. The proof of Lemma 3.7 works equally well for B^n as it does for B^1, the essential point being the compactness of the parameter space. q.e.d.

While the V_x's may not vary continuously in x (see Remark 4.2 [6]) we do have the following key result.

Non-Encroachment Lemma 4.3. *Let M be a closed oriented hyperbolic 3-manifold, δ a simple closed geodesic possessing the $(\pi_1(M), \{\partial\delta_i\})$ non-coalescable insulator family $\{\lambda_{ij}\}$ and $f : B^n \to \mathrm{RM}(M)$ a continuous map.*

If $x \in B^n$ and $\epsilon > 0$, then there exists $\eta > 0$ such that $W_x^\epsilon \subset W_y^{\epsilon/2}$ if $d_{B^n}(x, y) < \eta$.

Proof. For $t \in B^n$, let U_t denote the lift of V_t to $\mathbb{H}^3$ whose ends limit on $\partial\delta_0$. We need to show that if $z \in U_x$ and $d_\rho(z, \partial U_x) \geq \epsilon$, then for t sufficiently close to x, $z \in U_t$ and $d_\rho(z, \partial U_t) \geq \epsilon/2$.

If this is false, then by the compactness of $U_t/\langle g\rangle$, there exists $x_1, x_2, \ldots$ converging to x and a $z \in U_x$ such that $d_\rho(z, \partial U_x) \geq \epsilon$ and for all i either $z \notin U_{x_i}$ or $d_\rho(z, \partial U_{x_i}) \leq \epsilon/2$. Therefore we can assume one of the following holds:

i) For all i, $z \in U_{x_i}$.

ii) For all i, $z \notin U_{x_i}$.

If i) occurs, then by Lemmas 3.6 and 4.2 after passing to subsequence we can find a fixed λ_{0k} such that for all $i \in \mathbb{N}$ there exists a $x_i - D^2$-limit lamination σ_{0k}^i spanning λ_{0k} and $y_i \in \sigma_{0k}^i$ such that $d_\rho(y_i, z) \leq \epsilon/2$. Again passing to subsequence we can assume that $y_i \to y$, where $d_\rho(y, z) \leq \epsilon/2$ and $\sigma_{0k}^i \to \sigma_{0k}$ a x-least area D^2-limit lamination spanning λ_{0k}. (This uses Proposition 3.10 [6] which is stated for metrics parametrized by B^1 rather than B^n, but the proof works for the more general setting.) By definition of convergence, $y \in \sigma_{0k}$, which implies the contradiction $d_\rho(x, \partial U_x) \leq \epsilon/2$.

We now show that ii) cannot occur. Since $z \in \mathring{U}_x, q(z) \in \mathring{V}_x$ and hence there exists a core γ of $\mathring{V}_x$ passing through $q(z)$. Here $q : \mathbb{H}^3 \to M$ is the universal covering space projection. The lift $\widetilde{\gamma} \subset U_x$ passing through z limits on $\partial\delta_0$. Arguing as in the previous paragraph it follows that if $y \in B^n$ is sufficiently close to x, then $\widetilde{\gamma} \cap \sigma_{0k}^\alpha = \emptyset$ for all y-least area D^2-limit laminations σ_{0k}^α. Since $\widetilde{\gamma}$ lies in the component of $\mathbb{B}^3 - \sigma_{0k}^\alpha$ containing $\partial\delta_0$, for every σ_{0k}^α, it follows that $\widetilde{\gamma} \subset U_y$ and hence $z \in U_y$.
q.e.d.

Remark 4.4. The η of Lemma 4.3 is probably not uniform in x. Indeed, the regions U_{x_i} may limit to a disconnected region, one component of which is U_x. In the limit a piece of the U_{x_i}'s may get pinched off at an antitangential point or a spike point.

Lemma 4.5. *Let M be a closed hyperbolic 3-manifold with geodesic δ satisfying the insulator condition. If $f : S^n \to \mathrm{Diff}_0(M)$, then there exists a cellulation Δ^* of B^{n+1} such that for each cell σ of Δ^*, there exists a solid torus T_σ such that:*

i) *If κ is a proper face of σ, then T_σ is a core of $\overset{\circ}{T}_\kappa$.*

ii) *If $x \in \sigma \cap S^n$, then $f_x(\delta)$ is a core of $\overset{\circ}{T}_\sigma$.*

Proof. Define $h : S^n \to \mathrm{RM}(M)$ by $h(x) = (f_x)_*(\rho)$, the push forward of ρ. By the contractibility of the space of Riemannian metrics on a closed 3-manifold, there exists $g : B^{n+1} \to \mathrm{RM}(M)$ extending h. Fix a non-coalescable insulator family for δ. Thus we can define the canonical solid tori V_x and the sets $W_x^\epsilon \subset \overset{\circ}{V}_x$ for each $x \in B^{n+1}$.

For each $x \in B^{n+1}$ let $T_x^0 \subset \overset{\circ}{V}_x$ be a $D^2 \times S^1$ unknotted in $\overset{\circ}{V}_x$. If $x \in S^n$, then choose T_x^0 so that $f_x(\delta) \subset \overset{\circ}{T^0}_x$ (Note that $f_x(\delta)$ is the geodesic homotopic to δ in M with the hyperbolic metric $(f_x)_*(\rho)$.) Thus for each $x \in B^{n+1}$, there exists $\epsilon_x^0 > 0$ so that $T_x^0 \in W_x^{\epsilon_x^0}$ and by the non-encroachment lemma for $y \in B^{n+1}$ sufficiently close to x, say y in the open set A_x^0, then $T_x^0 \subset W_y^{\epsilon_x/2}$. For $x \in S^n$ choose A_x^0 to have the additional property that if $y \in A_x^0 \cap S^n$, then $f_y(\delta) \subset \overset{\circ}{T^0}_x$.

Fix a finite cover of B^{n+1} by elements of $\{A_x^0\}$, say using $\{x_1, \ldots, x_p\} \overset{\text{def}}{=} \mathcal{X}^0$ so that S^n is covered by a subset $\{A^0_{x_{i_k}}\}$, with each $x_{i_k} \in S^n \cap \mathcal{X}^0$. Let Δ_0 be any piecewise smooth cell division such that if $\sigma \in \Delta_0$, then $\sigma \subset A^0_{x_i}$ for some $x_i \in \mathcal{X}^0$. Furthermore if $\sigma \cap S^n \neq \emptyset$, then $\sigma \subset A^0_{x_i}$ for some $x_i \in S^n \cap \mathcal{X}^0$. Finally, define $T_\sigma^0 = T_{x_i}^0$ for some x_i as above. If possible, choose $x_i \in S^n$.

For each $x \in \Delta_0^n$ construct a solid torus T_x^1 unknotted in $\overset{\circ}{V}_x$ satisfying the following property. If $\sigma_1, \ldots, \sigma_r$ are the cells of Δ_0 which contain x, then $T^0_{\sigma_1} \cup \ldots \cup T^0_{\sigma_r} \subset \overset{\circ}{T^1}_x \subset T_x^1 \subset \overset{\circ}{V}_x$. For $x \in \Delta_0^n$ let ϵ_x^1 be such that $T_x^1 \subset W_x^{\epsilon^1}$. By the non-encroachment lemma, there exists a neighborhood A_x^1 of x such that if $y \in A_x^1$, then $T_x^1 \subset W_y^{(\epsilon_x^1)/2}$.

Fix a finite cover of Δ_0^n by elements of $\{A_x^1\}$, say using $\{x_1^1, \ldots, x_q^1\} \overset{\text{def}}{=} \mathcal{X}^1$. Let Δ_1 be a piecewise smooth subdivision of Δ_0 that only nontrivially subdivides Δ^n, such that if $\sigma \in \Delta_1$ has dimension $\leq n$, then $\sigma \subset A^1_{x_i^1}$ for some $x_i^1 \in \mathcal{X}^1$. Furthermore if in addition $\sigma \cap S^n \neq \emptyset$, then $\sigma \subset A^1_{x_i^1}$ for some $x_i^1 \in S^n \cap \mathcal{X}^1$. Finally, define $T_\sigma^1 = T_{x_i^1}$ for some x_i^1 as above. If possible, choose $x_i^1 \in S^n$. If $\dim(\sigma) = n+1$, then $T_\sigma^1 \stackrel{\text{def}}{=} T_\sigma^0$.

In a similar way construct $\Delta_2, \dots, \Delta_{n+1}$. Here Δ_i is obtained from Δ_{i-1} by subdividing only the $(n-i+1)$-skeleton, and has the feature that if σ is a proper face of τ and $\dim(\tau) \geq n-i+2$, then $T_\tau^i \subset \overset{\circ}{T^i}_\sigma$. Finally take $\Delta^* = \Delta_{n+1}$ and $T_\sigma = T_\sigma^{n+1}$. q.e.d.

Theorem 4.6 (Coarse Torus Isotopy Theorem)**.** *Let M be a closed orientable hyperbolic 3-manifold with geodesic δ satisfying the insulator condition. If $f : S^n \to \mathrm{Diff}_0(M)$, then there exists a cellulation Δ of B^{n+1} and a function which associates to each cell $\sigma \in \Delta$ a solid torus V_σ such that:*

i) *If κ is a proper face of σ, then $V_\kappa \subset \overset{\circ}{V}_\sigma$ and V_κ is isotopic to the standard embedding in V_σ.*

ii) *If $x \in \sigma \cap S^n$, then $f_x(\delta)$ is a core of $\overset{\circ}{V}_\sigma$.*

Proof. Let Δ^* be the cell structure arising from Lemma 4.5 and let Δ' be the relative cell structure of B^{n+1} dual to Δ^*. Given a cell $\sigma^* \in \Delta^*$, let σ denote the dual cell. This induces a natural 1-1 correspondence between cells of Δ^* and Δ'. The restriction of Δ' to S^n gives rise to a regular cell division Σ of S^n. The union of cells of Δ' and Σ gives a regular cell division of B^{n+1} which we call Δ.

For $\sigma \in \Delta'$, define $V'_\sigma = T_{\sigma^*}$. If $\sigma \in \Sigma$ is a k-cell, then $\sigma = \tau \cap S^n$ for a unique $(k+1)$-cell $\tau \in \Delta'$. Define $V'_\sigma = V'_\tau$. The collection $\{V'_\sigma\}_{\sigma \in \Delta'}$ satisfies all our conclusions except that if $\sigma = \tau \cap S^n$, then $V'_\sigma = V'_\tau$ (rather than $V'_\sigma \subset \overset{\circ}{V'}_\tau$.) By appropriately shrinking the $\{V'_\sigma | \sigma \in \Sigma\}$ and maintaining the $\{V'_\tau | \tau \in \Delta'\}$ we obtain the desired collection of $D^2 \times S^1$'s which we denote $\{V_\sigma\}_{\sigma \in \Delta}$. q.e.d.

5. Another Formulation Of Hatcher's Theorem

Let $\mathrm{Emb}(D^2 \times S^1, \mathbb{R}^2 \times S^1)$ denote the space of smooth embeddings with the C^∞ topology which take the core of each $D^2 \times S^1$ to a curve isotopic to $0 \times S^1$, the core of $\mathbb{R}^2 \times S^1$. Let $\mathrm{Emb}_0(D^2 \times S^1, \mathbb{R}^2 \times S^1)$ denote the smooth embeddings isotopic to the standard one.

In [10], Hatcher listed 17 equivalent formulations of the Smale Conjecture. Here is another along the same lines.

Theorem 5.1. $\mathrm{Emb}_0(D^2 \times S^1, \mathbb{R}^2 \times S^1) \simeq S^1 \times S^1$.

Proof. View $\mathrm{Emb}_0(D^2 \times S^1, \mathbb{R}^2 \times S^1)$ as $\mathrm{Emb}_0(\frac{1}{2}D^2 \times S^1, \overset{\circ}{D}{}^2 \times S^1)$, where $\frac{1}{2}D^2 \times S^1$ is the concentric solid torus of radius $\frac{1}{2}$. The map $\mathrm{Diff}_0(D^2 \times S^1) \to \mathrm{Emb}_0(\frac{1}{2}D^2 \times S^1, \overset{\circ}{D}{}^2 \times S^1)$ defined by restricting to $\frac{1}{2}D^2 \times S^1$ is surjective and is a fibration by [17], [4]. The fiber is all diffeomorphisms which restrict to the identity on $\frac{1}{2}D^2 \times S^1$. By [9] or [11, 12] the fiber is contractible and hence $\mathrm{Emb}_0(D^2 \times S^1, \mathbb{R}^2 \times S^1)$ is homotopy equivalent to $\mathrm{Diff}_0(D^2 \times S^1)$. (Actually they proved this in the PL category, but as noted in [9], the proof can be promoted to Diff using [10].) Again restriction is a fibration $\mathrm{Diff}_0(D^2 \times S^1) \to \mathrm{Diff}_0(S^1 \times S^1)$ with fiber $\mathrm{Diff}(D^2 \times S^1, \partial D^2 \times S^1)$, the space which fixes $\partial D^2 \times S^1$ pointwise. By [10] $\mathrm{Diff}(D^2 \times S^1, \partial D^2 \times S^1)$ is contractible and the contractibility of that space is equivalent to the Smale conjecture. Thus $\mathrm{Diff}_0(D^2 \times S^1)$ is homotopy equivalent to $\mathrm{Diff}_0(T^2)$. Fix a basepoint $t_0 \subset T^2$. There is a fibration $\mathrm{Diff}_0(T^2) \to \mathrm{Emb}(t_0, T^2) = T^2$ by restriction. The fiber $\mathrm{Diff}_0(T^2, t_0)$ consists of the basepoint preserving maps of T^2 isotopic to id_{T^2}. Since the latter space is contractible [8] the proof of Theorem 5.1 is complete. q.e.d.

Remark 5.2. By tracing back this homotopy equivalence we see that the $S^1 \times S^1$ corresponds to maps of $\mathrm{Diff}_0(D^2 \times S^1)$ which are compositions of *shifts* and *rolls*. Shifts are maps of the form $(z, t) \to (z, e^{i\theta}t)$ and rolls are of the form $(z, t) \to (e^{i\theta}z, t)$, where $z \in D^2$ and $\theta, t \in S^1$.

A proof similar to that of Theorem 5.1 yields:

Theorem 5.3. $\mathrm{Emb}(D^2 \times S^1, \mathbb{R}^2 \times S^1) \simeq O(2) \times O(2) \times \Omega(SO(2))$.

Remark 5.4. The $O(2)$'s in Theorem 5.3 allow for reflections in the core and cocore directions of $D^2 \times S^1$. Finally given an element ϕ of ΩS^1, the loop space of S^1, one obtains a diffeomorphism $h : D^2 \times S^1 \to D^2 \times S^1$ by $h(z, t) = (e^{i\phi(t)}z, t)$.

6. The Local Contractibility Theorem

Lemma 6.1. *Let δ be a simple closed geodesic in the closed hyperbolic 3-manifold M. If $f : M \to M$ is a diffeomorphism homotopic to id_M such that $f|\delta = \mathrm{id}_\delta$, then f is isotopic to $g : M \to M$ such that:*

(1) $g|N(\delta) = \mathrm{id}_{N(\delta)}$.

(2) $g|(M - \overset{\circ}{N}(\delta))$ *is isotopic to* id *rel* $\partial N(\delta)$.

Furthermore, given any neighborhood J of δ there exists a g as above and a $N(\delta) \subset J$ such that the isotopy from f to g can be supported in J.

Proof. Given a neighborhood J of δ, let $N_1(\delta)$ be a small closed regular neighborhood such that $f(N_1(\delta)) \subset J$. After an isotopy supported in J we can assume that f fixes $N_1(\delta)$ setwise. A priori $f|N_1(\delta)$ is isotopic to a finite number of Dehn twists about a meridian, however $M - \overset{\circ}{N}_1(\delta)$ is atoroidal, anannular and Haken and so by [13] has a finite mapping class group. This implies that $f|N_1(\delta)$ is isotopic to $\mathrm{id}_{N_1(\delta)}$ and hence we can assume that after another isotopy supported in J, that $f|N_1(\delta) = \mathrm{id}$.

Let $q : \mathbb{H}^3 \to M$ be the universal covering projection and $W^1 \overset{\text{def}}{=} q^{-1}(N_1(\delta))$ with $\{W_i^1\}$ the components of W^1. Since f is homotopic to id_M there exists a lift $\widetilde{f}$ such that $\widetilde{f}|S^2_\infty = \mathrm{id}$. Since $f|N_1(\delta) = \mathrm{id}$, for all i $\widetilde{f}|W_i^1 = W_i^1$ and is a translation by some $n \in \mathbb{Z}$ fundamental domain units, n being independent of i. Choose $N(\delta) \subset \overset{\circ}{N}_1(\delta)$ and let f_1 be the map $T^{-n}f$ where T is the following Dehn twist about a torus. $T|N(\delta) \cup (M - \overset{\circ}{N}_1(\delta))) = \mathrm{id}$, where the restriction to each concentric torus about $N(\delta)$ in $N_1(\delta)$ is a shift, the θ (of Remark 5.2) varying from 0 to 2π as one goes from $\partial N(\delta)$ to $\partial N_1(\delta)$. Note that T is isotopic to id_M via an isotopy supported in $N_1(\delta)$ and if $\widetilde{f_1}$ is the lift of f_1 isotopic to $\widetilde{f}$, then $\widetilde{f_1}|W = \mathrm{id}_W$. Here $W = q^{-1}(N(\delta))$. (This paragraph replaces the inaccurate second sentence of second paragraph of p. 48 [6].)

To complete the proof, proceed as in the rest of the second paragraph of p. 48 [6]. q.e.d.

Lemma 6.2. *If δ is a simple closed oriented geodesic in the hyperbolic* 3*-manifold M and δ can be isotoped into the solid torus $V \subset M$, then the isotoped δ is a core of V. If δ_1 and δ_2 are two isotoped images of δ both lying in V, then δ_1 and δ_2 are isotopic* (*as oriented curves*) *in V.*

Proof. Let γ be a curve in V isotopic to δ. Since δ represents a primitive element of $\pi_1(M), \gamma$ represents a generator of $\pi_1(V)$. Thus δ, γ and V lift to the covering space M_δ with fundamental group $\langle\delta\rangle$. In

M_δ we have $\gamma \subset V \subset M_\delta = \overset{\circ}{D}{}^2 \times S^1$ and δ, hence γ, is a core of M_δ. This implies that γ is a core of V and V is unknotted in M_δ, thereby proving the first assertion.

The failure of the second assertion implies that δ_1 and δ_2 represent oppositely oriented cores. That in turn would imply that the geodesic δ is isotopic to itself oppositely oriented, an impossibility in a hyperbolic 3-manifold. q.e.d.

Theorem 6.3 (Local Contractibility Theorem)**.** *Let δ be an oriented simple geodesic in the closed hyperbolic* 3*-manifold M and V a solid torus embedded in M. If $H : S^n \to \mathrm{Diff}_0(M)$ such that $H_t(\delta) \subset \overset{\circ}{V}$ for each $t \in S^n$, then H extends to a map $G : B^{n+1} \to \mathrm{Diff}_0(M)$ such that $G_s(\delta) \subset \overset{\circ}{V}$ for each $s \in B^{n+1}$.*

Proof. Fix $t_0 \in S^n$. After replacing H by $H_{t_0}^{-1}H$ and V by $H_{t_0}^{-1}(V)$, we can assume that $H_{t_0} = \mathrm{id}_M$ and $\overset{\circ}{V}$ is a neighborhood of δ.

We start with the case $n = 0$ where $t_0 = 1 \in S^0$. Use Lemma 6.2 to extend H to $[-1, 0] \cup \{1\}$ so that $H_0|\delta = H_1|\delta = \mathrm{id}_\delta$. Next use Lemma 6.1 to extend H to $[-1, \frac{1}{2}] \cup \{1\}$ so that $H_{\frac{1}{2}}|N(\delta) = H_1|N(\delta) = \mathrm{id}_{N(\delta)}$, and $H_{\frac{1}{2}}|(M - \overset{\circ}{N}(\delta))$ is homotopic to id rel $\partial N(\delta)$. Finally apply [21] to extend H to $[-1, 1]$ so that the isotopy $H|[\frac{1}{2}, 1]$ is an isotopy fixed on $N(\delta)$. Since, by Lemmas 6.1 and 6.2, these constructions can be carried out so that $H_t(\delta) \subset \overset{\circ}{V}$ for all $t \in [-1, 1]$, the case of $n = 0$ is complete.

Claim. If $n > 0$, then there exists a solid torus regular neighborhood X of δ and $K : B^{n+1} \to \mathrm{Emb}_0(X, \overset{\circ}{V})$ such that $H_t|X = K_t$ for each $t \in S^n$.

Proof of Claim. Choose X so that $H_t(X) \subset \overset{\circ}{V}$ for all $t \in S^n$. Note that V and X can be parametrized so that $H_{t_0}|X \in \mathrm{Emb}_0(X, \overset{\circ}{V})$ is the standard inclusion. If $n > 1$, then the Claim follows directly from Theorem 5.1. Now assume that $n = 1$ and $H|X : S^1 \to \mathrm{Emb}_0(X, \overset{\circ}{V})$ represents a nontrivial element of $\pi_1(\mathrm{Emb}_0(X, \overset{\circ}{V}))$. Theorem 5.1 and Remark 5.2 imply that there exists a map $L : S^1 \times I \to \mathrm{Emb}_0(X, \overset{\circ}{V})$ such that for $t \in S^1$, $L_{(t,0)} = H_t|X$; for $s \in I$, $L_{(t_0,s)} = \mathrm{id}_X$; for $t \in S^1$, $L_{(t,1)}$ is a composition of twists and rolls and $(L|S^1 \times 1)|\partial X$ represents a nontrivial element of $\pi_1(\mathrm{Diff}_0(\partial X))$. By the Palais-Cerf covering isotopy theorem [17], [4], L extends to a map $K : S^1 \times I \to \mathrm{Diff}_0(M)$ such that $K|S^1 \times 0 = H$ and $K_{(t_0,s)} = \mathrm{id}_M$ if $s \in I$. Therefore, if $N = M - \overset{\circ}{X}$, then $B_t \overset{\mathrm{def}}{=} K_{(t,1)}|N$, $t \in S^1$, represents a loop in $\mathrm{Diff}_0(N)$, based at the identity, which restricts to a nontrivial loop in $\mathrm{Diff}_0(\partial N)$. This contradicts the fact that $\overset{\circ}{N}$ is a hyperbolic 3-manifold. (The loop $B_t, t \in S^1$ lifts to a path of maps $\widetilde{B}_t, t \in [0,1]$ of $\mathbb{H}^3$ starting at the identity. The end of this path must be a nontrivial covering transformation. This contradicts the fact that the diameters of the homotopy tracks $\{\widetilde{B}_t(x)|t \in [0,1]\}$ are uniformly bounded.) q.e.d.

By the covering isotopy theorem there exists a map $J : B^{n+1} \to \mathrm{Diff}_0(M)$ which extends K and satisfies $J_{t_0} = \mathrm{id}_M$. The map $E : S^n \to \mathrm{Diff}_0(M)$ defined by $E_t = J_t^{-1} H_t$ satisfies $E_{t_0} = \mathrm{id}_M$ and $E_t|X = \mathrm{id}_X$ for all $t \in S^n$. Since $M - \overset{\circ}{X}$ is Haken it follows by [9] or [11, 12] that E extends to $E^* : B^{n+1} \to \mathrm{Diff}_0(M)$ such that for $z \in B^{n+1}$ E^*_z fixes X pointwise. Define $G : B^{n+1} \to \mathrm{Diff}_0(M)$ by $G_z = J_z E^*_z$. If $t \in S^n, G_t = J_t E^*_t = J_t E_t = J_t J_t^{-1} H_t = H_t$ and if $z \in B^{n+1}$, then $G_z(X) = J_z E^*_z(X) = J_z(X) = K_z(X) \subset \overset{\circ}{V}$. q.e.d.

7. Applications

Definition 7.1. Let $\mathrm{Hyp}(M)$ denote the subspace of the space of Riemannian metrics on M consisting of metrics of constant curvature -1.

Lemma 7.2. *If M is a complete hyperbolic 3-manifold, then* $\mathrm{Hyp}(M)$ *is homeomorphic to* $\mathrm{Diff}_0(M)$.

Proof. We will show that $\phi : \mathrm{Diff}_0(M) \to \mathrm{Hyp}(M)$ by $f \to f_*(\rho)$ is bijective. Since id_M is the only isometry of M (with respect to a fixed hyperbolic metric) which is homotopic to id_M it follows that ϕ is injective. Conversely if ρ' is a hyperbolic metric on M, then by Mostow there exists an isometry $h : M_\rho \to M_{\rho'}$ such that h is homotopic to id_M. By [7] $h \in \mathrm{Diff}_0(M)$. q.e.d.

Theorem 7.3. *The space* $\mathrm{Hyp}(M)$ *of hyperbolic metrics on a complete hyperbolic* 3*-manifold of finite volume M is contractible.*

Proof. If M is closed, then the contractibility of $\mathrm{Diff}_0(M)$ follows from Theorem 1.1. If M is noncompact, then the contractibility of $\mathrm{Diff}_0(M)$ follows from [9] or [11, 12]. Now apply Lemma 7.2. q.e.d.

Definition 7.4. If δ is a smooth oriented simple closed curve in the manifold M, then let $\mathrm{Emb}_\delta(S^1, M)$ denote the space of smooth embeddings of an oriented S^1 into M whose image is isotopic, as oriented curves, to δ.

The following application was suggested by Allen Hatcher.

Theorem 7.5. *Let δ be an oriented simple closed curve in the closed hyperbolic 3-manifold M. If $M - \delta$ is atoroidal, then*

$$\mathrm{Emb}_\delta(S^1, M) \simeq S^1.$$

Proof. The fibration $\mathrm{Diff}_0(M) \to \mathrm{Emb}_\delta(S^1, M)$ defined by restricting to δ has fiber $\mathrm{Diff}_0(M, \delta)$, the subspace of $\mathrm{Diff}_0(M)$ that fixes δ pointwise. The path components of this space are naturally parametrized by $\mathbb{Z}$, the various components corresponding to the $2\pi n$ *shifts* of δ. If $Y \subset M$, then let $\mathrm{Diff}_{00}(M, Y)$ denote the subspace of $\mathrm{Diff}_0(M)$ consisting of all maps isotopic to the identity, via an isotopy fixing Y pointwise. The theorem follows from the long exact homotopy sequence and the fact [9], [11, 12] that $\mathrm{Diff}_{00}(M, \delta)$ is contractible. (Actually it follows directly from [9],[11, 12] that $\mathrm{Diff}_{00}(M, N(\delta))$ is contractible, but that result can be promoted to the desired one via standard differential topology techniques.) q.e.d.

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Caltech, Pasadena, CA 91125
Princeton University, Princeton NJ 08544