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Arithmetic Properties of Solutions of Certain Functional Equations with Transformations Represented by Matrices Including a Negative Entry

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Abstract. Mahler's method gives algebraic independence results for the values of functions of several variables satisfying certain functional equations under the transformations of the variables represented as a kind of the multiplicative action of matrices with integral entries. In the Mahler's method, the entries of those matrices must be nonnegative; however, in the special case stated in this paper, one can admit those matrices to have a negative entry. We show the algebraic independence of the values of certain functions satisfying functional equations under the transformation represented by such matrices, expressing those values as linear combinations of the values of ordinary Mahler functions.

1. Introduction and the result

Let $\Omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with nonnegative integer entries a, b, c, and d. We define a transformation $\Omega : \mathbf{C}^2 \to \mathbf{C}^2$ by $\Omega(z_1, z_2) = (z_1^a z_2^b, z_1^c z_2^d)$. For any 2×2 matrix A with nonnegative integer entries, defining $A(z_1, z_2)$ by the same way as $\Omega(z_1, z_2)$, we see that $A(\Omega(z_1, z_2)) = (A\Omega)(z_1, z_2)$, and hence $\Omega^k(z_1, z_2)$ is well-defined for any nonnegative integer k.

In what follows, for any field F we denote by $F(x_1, \dots, x_n)$ and $F[[x_1, \dots, x_n]]$ the field of rational functions and the ring of formal power series in the variables $x_1, \dots, x_n$ with coefficients in F, respectively. Let K be an algebraic number field. Let $f_1(z_1, z_2), \dots, f_m(z_1, z_2) \in K[[z_1, z_2]]$ converge in some neighborhood of the origin of $\mathbf{C}^2$ and satisfy

$$f_i(z_1, z_2) = a_i f_i(\Omega(z_1, z_2)) + b_i(z_1, z_2) \quad (1 \le i \le m), \tag{1}$$

where $a_1, \dots, a_m$ are nonzero algebraic numbers and $b_i(z_1, z_2)$ $(1 \le i \le m)$ are rational functions in the variables z_1 and z_2 with algebraic coefficients such that $b_i(0, 0)$ $(1 \le i \le m)$ are defined.

Theorem 1 below was originally proved by Mahler [4] (see also [5]) and improved by Loxton and van der Poorten [3]. These three authors dealt with functions $f_i(z_1, \dots, z_n)$

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of n variables $z_1, \dots, z_n$ satisfying functional equations such as (1); however, it is rather complicated to state here the precise conditions assumed in their theorems and hence we mention here only the case of $n = 2$. Nishioka [6] gave more easily understandable conditions than those of the three authors for the algebraic independence of the values of the functions $f_i(z_1, \dots, z_n)$.

THEOREM 1 (A special case of Loxton and van der Poorten [3]). *Suppose that $a + d > 0$ and the characteristic polynomial $X^2 - (a+d)X + ad - bc$ of Ω is irreducible over $\mathbf{Q}$. Let α be an algebraic number with $0 < |\alpha| < 1$. If $f_i(z_1, z_2) \in K[[z_1, z_2]]$ $(1 \le i \le m)$ satisfying* (1) *are algebraically independent over $\mathbf{C}(z_1, z_2)$ and convergent at $(1, \alpha)$ and if $b_i(\Omega^k(1, \alpha))$ $(1 \le i \le m)$ are defined for all $k \ge 0$, then the values $f_i(1, \alpha)$ $(1 \le i \le m)$ are algebraically independent.*

Here the condition that the entries of Ω are nonnegative is crucial; however, in this paper we establish the algebraic independence of the values of certain functions satisfying functional equations such as (1) with Ω having a negative entry.

THEOREM 2. *Let $\{R_k\}_{k \in \mathbf{Z}}$ be a sequence of integers with integral subscripts satisfying*

$$R_{k+2} = aR_{k+1} + R_k \quad (k \in \mathbf{Z}),$$

where a is a positive integer. Suppose there exists an integer k_0 such that $R_{2k_0} = 0$ and $R_{2k_0+1} > 0$. Let $\gamma_1, \dots, \gamma_m$ be nonzero distinct algebraic numbers. Define the series

$$F_{\mu\xi}(z_1, z_2) = \sum_{k \in \mathbf{Z}} \frac{\xi^k z_1^{R_{2k+1}} z_2^{R_{2k-1}}}{1 - \gamma_\mu z_1^{R_{2k+1}} z_2^{R_{2k-1}}} \quad (1 \le \mu \le m)$$

and

$$G_{\mu\eta}(z_1, z_2) = \sum_{k \in \mathbf{Z}} \frac{\eta^k z_1^{R_{2k+2}} z_2^{R_{2k}}}{1 - \gamma_\mu z_1^{R_{2k+2}} z_2^{R_{2k}}} \quad (1 \le \mu \le m),$$

where ξ and η are algebraic numbers with $|\xi| > 1$ and $|\eta| > 1$. Then $F_{\mu\xi}(z_1, z_2)$ and $G_{\mu\eta}(z_1, z_2)$ $(1 \le \mu \le m)$ are power series which converge in the domain

$$D = \{(z_1, z_2) \in \mathbf{C}^2 \mid \lambda \log|z_1| + \log|z_2| < 0,\ \log|z_1| + \lambda \log|z_2| < 0\}, \tag{2}$$

where $\lambda = (a^2 + 2 + a\sqrt{a^2+4})/2$, and satisfy the following two properties:

(i) *Let*

$$\Omega = \begin{pmatrix} a^2+2 & 1 \\ -1 & 0 \end{pmatrix}$$

and define the transformation $\Omega : \mathbf{C}^2 \to \mathbf{C}^2$ by $\Omega(z_1, z_2) = (z_1^{a^2+2} z_2, z_1^{-1})$. Then $F_{\mu\xi}(z_1, z_2)$ and $G_{\mu\eta}(z_1, z_2)$ $(1 \le \mu \le m)$ satisfy the functional equations

$$F_{\mu\xi}(z_1, z_2) = \xi F_{\mu\xi}(\Omega(z_1, z_2)) \quad \text{and} \quad G_{\mu\eta}(z_1, z_2) = \eta G_{\mu\eta}(\Omega(z_1, z_2)), \tag{3}$$

respectively.

(ii) *Let α be an algebraic number with $0 < |\alpha| < 1$ such that $\alpha^{R_k} \neq \gamma_\mu^{\pm 1}$ $(1 \le \mu \le m)$ for all $k \in \mathbf{Z}$. Then the values*

$$F_{\mu\xi}(1,\alpha) = \sum_{k \in \mathbf{Z}} \frac{\xi^k \alpha^{R_{2k-1}}}{1 - \gamma_\mu \alpha^{R_{2k-1}}} \quad (1 \le \mu \le m,\ \xi \in \overline{\mathbf{Q}},\ |\xi| > 1)$$

and

$$G_{\mu\eta}(1,\alpha) = \sum_{k \in \mathbf{Z}} \frac{\eta^k \alpha^{R_{2k}}}{1 - \gamma_\mu \alpha^{R_{2k}}} \quad (1 \le \mu \le m,\ \eta \in \overline{\mathbf{Q}},\ |\eta| > 1)$$

are algebraically independent, where $\overline{\mathbf{Q}}$ is the set of algebraic numbers.

REMARK 1. The sequence $\{R_k\}_{k \in \mathbf{Z}}$ is uniquely determined if the consecutive terms, for example, R_0 and R_1 are given. The explicit expression of R_k for any $k \in \mathbf{Z}$ and the convergence of $F_{\mu\xi}(z_1, z_2)$ and $G_{\mu\eta}(z_1, z_2)$ $(1 \le \mu \le m)$ are explained in Section 3 as the first half of the proof of Theorem 2. For any $k \in \mathbf{Z}$, R_k is expressed, for instance, as (8) in Section 3.

COROLLARY 1. *Let $\{R_k\}_{k \in \mathbf{Z}}$ and $\gamma_1, \dots, \gamma_m$ be as in Theorem 2. Define the series*

$$f_{\mu\xi}(z_1, z_2) = \sum_{k \in \mathbf{Z}} \frac{\xi^k z_1^{R_{k+2}} z_2^{R_k}}{1 - \gamma_\mu z_1^{R_{k+2}} z_2^{R_k}} \quad (1 \le \mu \le m),$$

where ξ is an algebraic number with $|\xi| > 1$. Then $f_{\mu\xi}(z_1, z_2)$ $(1 \le \mu \le m)$ are power series which converge in the domain D defined by (2) and satisfy the following two properties:

(i) *Define the transformation Ω as in Theorem 2. Then each of $f_{\mu\xi}(z_1, z_2)$ $(1 \le \mu \le m)$ satisfies the functional equation*

$$f_{\mu\xi}(z_1, z_2) = \xi^2 f_{\mu\xi}(\Omega(z_1, z_2)).$$

(ii) *Let α be an algebraic number with $0 < |\alpha| < 1$ such that $\alpha^{R_k} \neq \gamma_\mu^{\pm 1}$ $(1 \le \mu \le m)$ for all $k \in \mathbf{Z}$. Then the values*

$$f_{\mu\xi}(1,\alpha) = \sum_{k \in \mathbf{Z}} \frac{\xi^k \alpha^{R_k}}{1 - \gamma_\mu \alpha^{R_k}} \quad (1 \le \mu \le m,\ \xi \in \overline{\mathbf{Q}},\ |\xi| > 1)$$

are algebraically independent.

EXAMPLE 1. Let $\{F_k\}_{k \ge 0}$ be the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k \quad (k \ge 0).$$

Then, as is stated in Section 3, F_k is defined for any integer k. Define the function $G_n(x)$ $(n =$

$1, 2, 3, \ldots$) of the complex variable x by

$$G_n(x) = \sum_{k \in \mathbf{Z}} \frac{x^k}{2^{F_k} + n} = \cdots + \frac{x^{-4}}{2^{-3} + n} + \frac{x^{-3}}{2^2 + n} + \frac{x^{-2}}{2^{-1} + n} + \frac{x^{-1}}{2 + n} + \frac{1}{1 + n} + \frac{x}{2 + n} + \frac{x^2}{2 + n} + \frac{x^3}{2^2 + n} + \cdots .$$

Then $G_n(x)$ ($n = 1, 2, 3, \ldots$) converge in $|x| > 1$ and the infinite set of the values

$$\left\{ G_n(\xi) = \sum_{k \in \mathbf{Z}} \frac{\xi^k}{2^{F_k} + n} \;\middle|\; n = 1, 2, 3, \ldots;\ \xi \in \overline{\mathbf{Q}},\ |\xi| > 1 \right\}$$

is algebraically independent by Corollary 1 with $-\gamma_\mu \in \{1, 2, 3, \ldots\}$ and $\alpha = 1/2$.

Taking C as $\overline{\mathbf{Q}}$ in the following theorem, we see that, if the entries of Ω are nonnegative, there are no solutions of the functional equations such as (3) which take transcendental values. Hence it is essential in Theorem 2 that there is a negative entry in the matrix Ω, otherwise the values $F_{\mu\xi}(1, \alpha)$ and $G_{\mu\eta}(1, \alpha)$ in Theorem 2 must be algebraic.

THEOREM 3 (Loxton and van der Poorten [2], see also Theorem 3.1 of Nishioka [6]). *Let C be a field and Ω a nonsingular 2×2 matrix with nonnegative integer entries. Assume that none of the eigenvalues of Ω is a root of unity. Let ξ be an element of C and $F(z_1, z_2)$ a quotient of the elements of $C[[z_1, z_2]]$. If*

$$F(z_1, z_2) = \xi F(\Omega(z_1, z_2)),$$

then $F(z_1, z_2) \in C$.

The property (ii) of Theorem 2 is proved in Section 3 by using the following:

THEOREM 4. *Let $\{R_k^{(i)}\}_{k \geq 0}$ $(i = 1, 2)$ be sequences of positive integers satisfying*

$$R_{k+2}^{(i)} = a R_{k+1}^{(i)} + R_k^{(i)} \quad (k = 0, 1, 2, \ldots),$$

where a is a positive integer. Let α be an algebraic number with $0 < |\alpha| < 1$, $\gamma_1, \ldots, \gamma_m$ nonzero distinct algebraic numbers, and $\delta_1, \ldots, \delta_n$ nonzero distinct algebraic numbers such that $\alpha^{R_k^{(1)}} \neq \gamma_\mu^{-1}$ $(1 \leq \mu \leq m)$ and $\alpha^{R_k^{(2)}} \neq \delta_\nu^{-1}$ $(1 \leq \nu \leq n)$ for all $k \geq 0$. Then the numbers

$$\sum_{k \geq 0} \frac{\xi^k \alpha^{R_k^{(1)}}}{1 - \gamma_\mu \alpha^{R_k^{(1)}}} \quad (1 \leq \mu \leq m,\ \xi \in \overline{\mathbf{Q}},\ |\xi| > 1)$$

and

$$\sum_{k \geq 0} \frac{\eta^k \alpha^{R_k^{(2)}}}{1 - \delta_\nu \alpha^{R_k^{(2)}}} \quad (1 \leq \nu \leq n,\ \eta \in \overline{\mathbf{Q}},\ 0 < |\eta| < 1)$$

are algebraically independent.

2. Proof of Theorem 4

Let $M(z_1, z_2) = z_1^m z_2^n$, where m and n are nonnegative integers not both zero, and let $\overline{C}$ be an algebraically closed field of characteristic 0. In order to prove Theorem 4 we use the following:

LEMMA 1 (A special case of Theorem 1 of [7]). *Let* $\Omega = \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix}$, *where a and b are positive integers. Suppose that the characteristic polynomial $X^2 - aX - b$ of Ω is irreducible over* $\mathbf{Q}$. *Assume that $F(z_1, z_2) \in \overline{C}[[z_1, z_2]]$ satisfies the functional equation of the form*

$$F(z_1, z_2) = \xi F(\Omega^p(z_1, z_2)) + \sum_{k=q}^{p+q-1} Q_k(M(\Omega^k(z_1, z_2))),$$

where $\xi \neq 0$ is an element of $\overline{C}$, $p > 0$, $q \geq 0$ are integers, and $Q_k(X) \in \overline{C}(X)$ $(q \leq k \leq p+q-1)$ are defined at $X = 0$. If $F(z_1, z_2) \in \overline{C}(z_1, z_2)$, then $F(z_1, z_2) \in \overline{C}$ and $Q_k(X) \in \overline{C}$ $(q \leq k \leq p+q-1)$.

PROOF OF THEOREM 4. The numbers in question are the values $f_{\mu\xi}(1, \alpha)$ $(1 \leq \mu \leq m)$ and $g_{\nu\eta}(1, \alpha)$ $(1 \leq \nu \leq n)$ of the functions

$$f_{\mu\xi}(z_1, z_2) = \sum_{k \geq 0} \frac{\xi^k z_1^{R_{k+1}^{(1)}} z_2^{R_k^{(1)}}}{1 - \gamma_\mu z_1^{R_{k+1}^{(1)}} z_2^{R_k^{(1)}}} \quad (1 \leq \mu \leq m)$$

and

$$g_{\nu\eta}(z_1, z_2) = \sum_{k \geq 0} \frac{\eta^k z_1^{R_{k+1}^{(2)}} z_2^{R_k^{(2)}}}{1 - \delta_\nu z_1^{R_{k+1}^{(2)}} z_2^{R_k^{(2)}}} \quad (1 \leq \nu \leq n),$$

which respectively satisfy the functional equations

$$f_{\mu\xi}(z_1, z_2) = \xi f_{\mu\xi}(\Omega(z_1, z_2)) + \frac{M_1(z_1, z_2)}{1 - \gamma_\mu M_1(z_1, z_2)}$$

and

$$g_{\nu\eta}(z_1, z_2) = \eta g_{\nu\eta}(\Omega(z_1, z_2)) + \frac{M_2(z_1, z_2)}{1 - \delta_\nu M_2(z_1, z_2)},$$

where

$$\Omega = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, \quad M_1(z_1, z_2) = z_1^{R_1^{(1)}} z_2^{R_0^{(1)}}, \quad M_2(z_1, z_2) = z_1^{R_1^{(2)}} z_2^{R_0^{(2)}},$$

and

$$M_1(\Omega^k(z_1, z_2)) = z_1^{R_{k+1}^{(1)}} z_2^{R_k^{(1)}}, \quad M_2(\Omega^k(z_1, z_2)) = z_1^{R_{k+1}^{(2)}} z_2^{R_k^{(2)}} \quad (k \geq 0).$$

Suppose on the contrary that there exist distinct algebraic numbers $\xi_1, \ldots, \xi_s$ with $|\xi_i| > 1$ and distinct algebraic numbers $\eta_1, \ldots, \eta_t$ with $0 < |\eta_j| < 1$ such that the values $f_{\mu\xi_i}(1, \alpha)$ $(1 \leq \mu \leq m,\ 1 \leq i \leq s)$ and $g_{\nu\eta_j}(1, \alpha)$ $(1 \leq \nu \leq n,\ 1 \leq j \leq t)$ are algebraically dependent. Noting that the characteristic polynomial $\Phi(X) = X^2 - aX - 1$ of Ω is irreducible over $\mathbf{Q}$ since ± 1 are not the roots of $\Phi(X)$, we see by Theorem 1 that the functions $f_{\mu\xi_i}(z_1, z_2)$ $(1 \leq \mu \leq m,\ 1 \leq i \leq s)$ and $g_{\nu\eta_j}(z_1, z_2)$ $(1 \leq \nu \leq n,\ 1 \leq j \leq t)$ are algebraically dependent over $\mathbf{C}(z_1, z_2)$. Since $\xi_1, \ldots, \xi_s$ and $\eta_1, \ldots, \eta_t$ are distinct, by Loxton and van der Poorten's theorem [2, Theorem 2] or by Kubota's result [1, Corollary 9], there exist complex numbers $c_1, \ldots, c_m$, not all zero, such that

$$F(z_1, z_2) := \sum_{\mu=1}^{m} c_\mu f_{\mu\xi_i}(z_1, z_2) \in \mathbf{C}(z_1, z_2)$$

for some i, or there exist complex numbers $d_1, \ldots, d_n$, not all zero, such that

$$G(z_1, z_2) := \sum_{\nu=1}^{n} d_\nu g_{\nu\eta_j}(z_1, z_2) \in \mathbf{C}(z_1, z_2)$$

for some j. Then $F(z_1, z_2)$ and $G(z_1, z_2)$ satisfy

$$F(z_1, z_2) = \xi_i F(\Omega(z_1, z_2)) + \sum_{\mu=1}^{m} \frac{c_\mu M_1(z_1, z_2)}{1 - \gamma_\mu M_1(z_1, z_2)}$$

and

$$G(z_1, z_2) = \eta_j G(\Omega(z_1, z_2)) + \sum_{\nu=1}^{n} \frac{d_\nu M_2(z_1, z_2)}{1 - \delta_\nu M_2(z_1, z_2)},$$

respectively. By Lemma 1 with $p = 1$ and $q = 0$, we see that

$$\sum_{\mu=1}^{m} \frac{c_\mu X}{1 - \gamma_\mu X} \in \mathbf{C} \qquad \text{or} \qquad \sum_{\nu=1}^{n} \frac{d_\nu X}{1 - \delta_\nu X} \in \mathbf{C},$$

which is a contradiction since these functions of the variable X have some poles. □

3. Proof of Theorem 2

PROOF OF THEOREM 2. Let ρ_1 and ρ_2 be the roots of $X^2 - aX - 1$. Since $a > 0$, we may assume $\rho_1 > 1$ and $-1 < \rho_2 < 0$. For any real numbers c and d we define

$$R_k^* = c\rho_1^k + d\rho_2^k \quad (k \in \mathbf{Z}).$$

Then the sequence $\{R_k^*\}_{k\geq -1}$ satisfies

$$R_{k+2}^* = aR_{k+1}^* + R_k^* \quad (k \geq -1). \tag{4}$$

By $\rho_1\rho_2 = -1$ we have

$$(-1)^k R_{-k}^* = (\rho_1\rho_2)^k (c\rho_1^{-k} + d\rho_2^{-k}) = d\rho_1^k + c\rho_2^k$$

for $k \geq -1$, and hence $\{(-1)^k R_{-k}^*\}_{k\geq -1}$ satisfies

$$(-1)^{k+2} R_{-k-2}^* = a(-1)^{k+1} R_{-k-1}^* + (-1)^k R_{-k}^* \quad (k \geq -1). \tag{5}$$

By (4) and (5) we see that

$$R_{k+2}^* = aR_{k+1}^* + R_k^* \quad (k \in \mathbf{Z}).$$

Hereafter we put

$$c = \frac{R_{2k_0+1} - \rho_2 R_{2k_0}}{\rho_1 - \rho_2}\rho_1^{-2k_0} = \frac{R_{2k_0+1}\rho_1^{-2k_0}}{\rho_1 - \rho_2} \tag{6}$$

and

$$d = \frac{\rho_1 R_{2k_0} - R_{2k_0+1}}{\rho_1 - \rho_2}\rho_2^{-2k_0} = -\frac{R_{2k_0+1}\rho_2^{-2k_0}}{\rho_1 - \rho_2}. \tag{7}$$

Then

$$c > 0 \quad \text{and} \quad d < 0$$

by the condition $R_{2k_0+1} > 0$. Since $R_{2k_0}^* = R_{2k_0}$ and $R_{2k_0+1}^* = R_{2k_0+1}$, we see that $R_k^* = R_k$ $(k \in \mathbf{Z})$ and so

$$R_k = \frac{R_{2k_0+1}}{\rho_1 - \rho_2}\rho_1^{k-2k_0} - \frac{R_{2k_0+1}}{\rho_1 - \rho_2}\rho_2^{k-2k_0} \quad (k \in \mathbf{Z}), \tag{8}$$

which is the explicit expression for the terms of the sequence $\{R_k\}_{k\in\mathbf{Z}}$.

For any $k \in \mathbf{Z}$, define

$$S_k^{(1)} := R_{2k-1} = c\rho_1^{-1}(\rho_1^2)^k + d\rho_2^{-1}(\rho_2^2)^k \quad \text{and} \quad S_k^{(2)} := R_{2k} = c(\rho_1^2)^k + d(\rho_2^2)^k, \tag{9}$$

where c and d are defined by (6) and (7), respectively. Since $\rho_1^2+\rho_2^2=(\rho_1+\rho_2)^2-2\rho_1\rho_2=a^2+2$ and $\rho_1^2\rho_2^2=1$, the sequences $\{S_k^{(i)}\}_{k\in\mathbf{Z}}$ $(i=1,2)$ satisfy

$$S_{k+2}^{(i)}=(a^2+2)S_{k+1}^{(i)}-S_k^{(i)} \quad (k\in\mathbf{Z})\,. \tag{10}$$

Since $c\rho_1^{-1}, d\rho_2^{-1}>0$ and $\rho_1^2>1>\rho_2^2$, we see that $S_k^{(1)}>0$ for any $k\in\mathbf{Z}$ and $S_k^{(1)}$ tends to $+\infty$ as k tends to $\pm\infty$. Hence

$$F_{\mu\xi}(z_1,z_2)=\sum_{k\geq 0}\frac{\xi^k z_1^{S_{k+1}^{(1)}}z_2^{S_k^{(1)}}}{1-\gamma_\mu z_1^{S_{k+1}^{(1)}}z_2^{S_k^{(1)}}}+\sum_{k<0}\frac{\xi^k z_1^{S_{k+1}^{(1)}}z_2^{S_k^{(1)}}}{1-\gamma_\mu z_1^{S_{k+1}^{(1)}}z_2^{S_k^{(1)}}}$$

converges in $\mathbf{C}[[z_1,z_2]]$ with respect to the topology defined by its maximal ideal (z_1,z_2); moreover, $F_{\mu\xi}(z_1,z_2)$ converges in the domain D since $S_{k+1}^{(1)}/S_k^{(1)}$ tends to $\rho_1^2=\lambda$ as k tends to $+\infty$ and $S_k^{(1)}/S_{k+1}^{(1)}$ tends to $1/\rho_2^2=\lambda$ as k tends to $-\infty$ and so the first sum converges if $|z_1|^\lambda|z_2|<1$ and the second sum converges if $|z_1||z_2|^\lambda<1$. By (10) we have

$$\xi F_{\mu\xi}(\Omega(z_1,z_2))=\xi F_{\mu\xi}(z_1^{a^2+2}z_2,z_1^{-1})=\sum_{k\in\mathbf{Z}}\frac{\xi^{k+1}z_1^{S_{k+2}^{(1)}}z_2^{S_{k+1}^{(1)}}}{1-\gamma_\mu z_1^{S_{k+2}^{(1)}}z_2^{S_{k+1}^{(1)}}}\,,$$

which converges in the domain D and which converges to

$$F_{\mu\xi}(z_1,z_2)=\sum_{k\in\mathbf{Z}}\frac{\xi^k z_1^{S_{k+1}^{(1)}}z_2^{S_k^{(1)}}}{1-\gamma_\mu z_1^{S_{k+1}^{(1)}}z_2^{S_k^{(1)}}}$$

in $\mathbf{C}[[z_1,z_2]]$ with respect to the topology.

Since $\rho_1^2>1>\rho_2^2$ and so $S_{k+1}^{(2)}-S_k^{(2)}=c(\rho_1^2-1)\rho_1^{2k}+d(\rho_2^2-1)\rho_2^{2k}>0$, $\{S_k^{(2)}\}_{k\in\mathbf{Z}}$ is a strictly increasing sequence of integers with $S_{k_0}^{(2)}=R_{2k_0}=0$. Hence

$$\begin{aligned}
&G_{\mu\eta}(z_1,z_2)\\
&=\sum_{k\geq k_0}\frac{\eta^k z_1^{S_{k+1}^{(2)}}z_2^{S_k^{(2)}}}{1-\gamma_\mu z_1^{S_{k+1}^{(2)}}z_2^{S_k^{(2)}}}+\gamma_\mu^{-1}\sum_{k<k_0}\eta^k\left(\frac{z_1^{-S_{k+1}^{(2)}}z_2^{-S_k^{(2)}}}{z_1^{-S_{k+1}^{(2)}}z_2^{-S_k^{(2)}}-\gamma_\mu}-1\right)\\
&=\sum_{k\geq k_0}\frac{\eta^k z_1^{S_{k+1}^{(2)}}z_2^{S_k^{(2)}}}{1-\gamma_\mu z_1^{S_{k+1}^{(2)}}z_2^{S_k^{(2)}}}-\gamma_\mu^{-2}\sum_{k<k_0}\frac{\eta^k z_1^{-S_{k+1}^{(2)}}z_2^{-S_k^{(2)}}}{1-\gamma_\mu^{-1}z_1^{-S_{k+1}^{(2)}}z_2^{-S_k^{(2)}}}-\gamma_\mu^{-1}\sum_{k<k_0}(\eta^{-1})^{-k}
\end{aligned}$$

converges in $\mathbf{C}[[z_1,z_2]]$ with respect to the topology defined by its maximal ideal (z_1,z_2); moreover, $G_{\mu\eta}(z_1,z_2)$ converges in the domain D since $|\eta^{-1}|<1$ and since $S_{k+1}^{(2)}/S_k^{(2)}$ tends to λ as k tends to $+\infty$ and $S_k^{(2)}/S_{k+1}^{(2)}$ tends to λ as k tends to $-\infty$. By (10) we have

$$\eta G_{\mu\eta}(\Omega(z_1,z_2)) = \eta G_{\mu\eta}(z_1^{a^2+2}z_2, z_1^{-1}) = \sum_{k\in\mathbf{Z}} \frac{\eta^{k+1} z_1^{S_{k+2}^{(2)}} z_2^{S_{k+1}^{(2)}}}{1-\gamma_\mu z_1^{S_{k+2}^{(2)}} z_2^{S_{k+1}^{(2)}}},$$

which converges in the domain D and which converges to

$$G_{\mu\eta}(z_1,z_2) = \sum_{k\in\mathbf{Z}} \frac{\eta^k z_1^{S_{k+1}^{(2)}} z_2^{S_k^{(2)}}}{1-\gamma_\mu z_1^{S_{k+1}^{(2)}} z_2^{S_k^{(2)}}}$$

in $\mathbf{C}[[z_1,z_2]]$ with respect to the topology. This gives the property (i) of Theorem 2.

Let

$$R_k^{(1)} = R_{k+2k_0+1} \quad (k \ge 0).$$

Then by (9) we have

$$S_{k+k_0+1}^{(1)} = R_{2k+2k_0+1} = R_{2k}^{(1)} \quad (k \ge 0)$$

and

$$S_{k+k_0+1}^{(2)} = R_{2k+2k_0+2} = R_{2k+1}^{(1)} \quad (k \ge 0).$$

Let

$$R_k^{(2)} = (-1)^k R_{2k_0-k-1} \quad (k \ge 0).$$

Then by (9) we have

$$S_{k_0-k}^{(1)} = R_{2k_0-2k-1} = R_{2k}^{(2)} \quad (k \ge 0)$$

and

$$S_{k_0-k-1}^{(2)} = R_{2k_0-2k-2} = -R_{2k+1}^{(2)} \quad (k \ge 0).$$

Since $R_0^{(1)} = R_{2k_0+1} = S_{k_0+1}^{(1)} > 0$, $R_1^{(1)} = R_{2k_0+2} = S_{k_0+1}^{(2)} > S_{k_0}^{(2)} = 0$, $R_0^{(2)} = R_{2k_0-1} = S_{k_0}^{(1)} > 0$, and $R_1^{(2)} = -R_{2k_0-2} = -S_{k_0-1}^{(2)} > -S_{k_0}^{(2)} = 0$, and since

$$R_k^{(1)} = c\rho_1^{2k_0+1}\rho_1^k + d\rho_2^{2k_0+1}\rho_2^k \quad (k \ge 0)$$

and

$$R_k^{(2)} = (-1)^k(c\rho_1^{2k_0-1}\rho_1^{-k} + d\rho_2^{2k_0-1}\rho_2^{-k}) = d\rho_2^{2k_0-1}\rho_1^k + c\rho_1^{2k_0-1}\rho_2^k \quad (k \ge 0),$$

we see by (4) that $\{R_k^{(i)}\}_{k\ge 0}$ $(i = 1, 2)$ are sequences of positive integers satisfying

$$R_{k+2}^{(i)} = aR_{k+1}^{(i)} + R_k^{(i)} \quad (k \ge 0).$$

For $1 \le \mu \le m$, define the functions

$$f_{1\mu}^{+}(x,z) = \sum_{k\ge 0} \frac{(\sqrt{x})^k z^{R_k^{(1)}}}{1-\gamma_\mu z^{R_k^{(1)}}}, \qquad f_{1\mu}^{-}(x,z) = \sum_{k\ge 0} \frac{(-\sqrt{x})^k z^{R_k^{(1)}}}{1-\gamma_\mu z^{R_k^{(1)}}},$$

$$f_{2\mu}^{+}(x,z) = \sum_{k\ge 0} \frac{(\sqrt{x})^k z^{R_k^{(2)}}}{1-\gamma_\mu z^{R_k^{(2)}}}, \qquad f_{2\mu}^{-}(x,z) = \sum_{k\ge 0} \frac{(-\sqrt{x})^k z^{R_k^{(2)}}}{1-\gamma_\mu z^{R_k^{(2)}}},$$

$$f_{3\mu}^{+}(x,z) = \sum_{k\ge 0} \frac{(\sqrt{x})^k z^{R_k^{(2)}}}{1-\gamma_\mu^{-1} z^{R_k^{(2)}}}, \qquad f_{3\mu}^{-}(x,z) = \sum_{k\ge 0} \frac{(-\sqrt{x})^k z^{R_k^{(2)}}}{1-\gamma_\mu^{-1} z^{R_k^{(2)}}}.$$

Noting that

$$\{1,\ 0,\ \xi,\ 0,\ \xi^2,\ 0,\ \xi^3, , \ldots\} = \left\{\frac{(\sqrt{\xi})^k + (-\sqrt{\xi})^k}{2}\right\}_{k\ge 0},$$

we have

$$\begin{aligned}
F_{\mu\xi}(1,\alpha) &= \sum_{k\ge k_0+1} \frac{\xi^k \alpha^{S_k^{(1)}}}{1-\gamma_\mu \alpha^{S_k^{(1)}}} + \sum_{k\le k_0} \frac{\xi^k \alpha^{S_k^{(1)}}}{1-\gamma_\mu \alpha^{S_k^{(1)}}} \\
&= \sum_{k\ge 0} \frac{\xi^{k+k_0+1} \alpha^{S_{k+k_0+1}^{(1)}}}{1-\gamma_\mu \alpha^{S_{k+k_0+1}^{(1)}}} + \sum_{k\ge 0} \frac{\xi^{k_0-k} \alpha^{S_{k_0-k}^{(1)}}}{1-\gamma_\mu \alpha^{S_{k_0-k}^{(1)}}} \\
&= \sum_{k\ge 0} \frac{\xi^{k+k_0+1} \alpha^{R_{2k}^{(1)}}}{1-\gamma_\mu \alpha^{R_{2k}^{(1)}}} + \sum_{k\ge 0} \frac{\xi^{k_0-k} \alpha^{R_{2k}^{(2)}}}{1-\gamma_\mu \alpha^{R_{2k}^{(2)}}} \\
&= \frac{\xi^{k_0+1}}{2} \sum_{k\ge 0} \frac{\left((\sqrt{\xi})^k + (-\sqrt{\xi})^k\right) \alpha^{R_k^{(1)}}}{1-\gamma_\mu \alpha^{R_k^{(1)}}} \\
&\quad + \frac{\xi^{k_0}}{2} \sum_{k\ge 0} \frac{\left((\sqrt{\xi^{-1}})^k + (-\sqrt{\xi^{-1}})^k\right) \alpha^{R_k^{(2)}}}{1-\gamma_\mu \alpha^{R_k^{(2)}}} \\
&= \frac{\xi^{k_0+1}}{2} \left(f_{1\mu}^{+}(\xi,\alpha) + f_{1\mu}^{-}(\xi,\alpha)\right) \\
&\quad + \frac{\xi^{k_0}}{2} \left(f_{2\mu}^{+}(\xi^{-1},\alpha) + f_{2\mu}^{-}(\xi^{-1},\alpha)\right).
\end{aligned} \tag{11}$$

Next we have

$G_{\mu\eta}(1,\alpha)$

$$= \sum_{k\ge k_0+1} \frac{\eta^k \alpha^{S_k^{(2)}}}{1-\gamma_\mu \alpha^{S_k^{(2)}}} + \frac{\eta^{k_0}\alpha^{S_{k_0}^{(2)}}}{1-\gamma_\mu \alpha^{S_{k_0}^{(2)}}} + \gamma_\mu^{-1} \sum_{k\le k_0-1} \eta^k \left(\frac{1}{1-\gamma_\mu \alpha^{S_k^{(2)}}} - 1 \right)$$

$$= \sum_{k\ge 0} \frac{\eta^{k+k_0+1}\alpha^{R_{2k+1}^{(1)}}}{1-\gamma_\mu \alpha^{R_{2k+1}^{(1)}}} + \frac{\eta^{k_0}}{1-\gamma_\mu} + \gamma_\mu^{-1} \sum_{k\ge 0} \eta^{k_0-k-1} \left(\frac{1}{1-\gamma_\mu \alpha^{-R_{2k+1}^{(2)}}} - 1 \right)$$

$$= \eta^{k_0+1} \sum_{k\ge 0} \frac{\eta^k \alpha^{R_{2k+1}^{(1)}}}{1-\gamma_\mu \alpha^{R_{2k+1}^{(1)}}} + \gamma_\mu^{-1}\eta^{k_0-1} \left(-\sum_{k\ge 0} \frac{\eta^{-k}\alpha^{R_{2k+1}^{(2)}}}{\gamma_\mu - \alpha^{R_{2k+1}^{(2)}}} - \sum_{k\ge 0} \eta^{-k} \right) + \frac{\eta^{k_0}}{1-\gamma_\mu}.$$

Noting that

$$\{0,\ 1,\ 0,\ \eta,\ 0,\ \eta^2,\ 0,\ \eta^3, \ldots\} = \left\{ \frac{(\sqrt{\eta})^k - (-\sqrt{\eta})^k}{2\sqrt{\eta}} \right\}_{k\ge 0},$$

we see that

$$\begin{aligned} G_{\mu\eta}(1,\alpha) &= \frac{\eta^{k_0+1}}{2\sqrt{\eta}} \sum_{k\ge 0} \frac{\left((\sqrt{\eta})^k - (-\sqrt{\eta})^k\right)\alpha^{R_k^{(1)}}}{1-\gamma_\mu \alpha^{R_k^{(1)}}} \\ &\quad - \frac{\gamma_\mu^{-2}\eta^{k_0-1}}{2\sqrt{\eta^{-1}}} \sum_{k\ge 0} \frac{\left((\sqrt{\eta^{-1}})^k - (-\sqrt{\eta^{-1}})^k\right)\alpha^{R_k^{(2)}}}{1-\gamma_\mu^{-1}\alpha^{R_k^{(2)}}} - \frac{\gamma_\mu^{-1}\eta^{k_0-1}}{1-\eta^{-1}} + \frac{\eta^{k_0}}{1-\gamma_\mu} \\ &= \frac{\eta^{k_0+1}}{2\sqrt{\eta}} \left(f_{1\mu}^{+}(\eta,\alpha) - f_{1\mu}^{-}(\eta,\alpha) \right) - \frac{\gamma_\mu^{-2}\eta^{k_0-1}}{2\sqrt{\eta^{-1}}} \left(f_{3\mu}^{+}(\eta^{-1},\alpha) - f_{3\mu}^{-}(\eta^{-1},\alpha) \right) \\ &\quad + \frac{\gamma_\mu^{-1}\eta^{k_0}}{1-\eta} + \frac{\eta^{k_0}}{1-\gamma_\mu}. \end{aligned} \tag{12}$$

For proving the theorem, it is enough to show that, for any distinct algebraic numbers $\xi_1, \ldots, \xi_s$ with $|\xi_i| > 1$ and for any distinct algebraic numbers $\eta_1, \ldots, \eta_t$ with $|\eta_j| > 1$, the values $F_{\mu\xi_i}(1,\alpha)$ $(1 \le \mu \le m,\ 1 \le i \le s)$ and $G_{\mu\eta_j}(1,\alpha)$ $(1 \le \mu \le m,\ 1 \le j \le t)$ are algebraically independent. Replacing $\{\xi_1, \ldots, \xi_s\}$ with $\{\xi_1, \ldots, \xi_s\} \cup \{\eta_1, \ldots, \eta_t\}$ if necessary, we may assume that $\{\xi_1, \ldots, \xi_s\} \supset \{\eta_1, \ldots, \eta_t\}$ without loss of generality. Hence it suffices to prove that, for any distinct algebraic numbers $\xi_1, \ldots, \xi_s$ with $|\xi_i| > 1$, the values $F_{\mu\xi_i}(1,\alpha)$ and $G_{\mu\xi_i}(1,\alpha)$ $(1 \le \mu \le m,\ 1 \le i \le s)$ are algebraically independent.

Letting $\{\delta_1, \ldots, \delta_n\} = \{\gamma_1, \ldots, \gamma_m\} \cup \{\gamma_1^{-1}, \ldots, \gamma_m^{-1}\}$ and noting that $\pm\sqrt{\xi_i}$ $(1 \le i \le s)$ are distinct numbers in $\{x \in \overline{\mathbf{Q}} \mid |x| > 1\}$ and $\pm\sqrt{\xi_i^{-1}}$ $(1 \le i \le s)$ are distinct numbers in $\{x \in \overline{\mathbf{Q}} \mid 0 < |x| < 1\}$, we see by Theorem 4 that the numbers

$$f_{1\mu}^{+}(\xi_i,\alpha)=\sum_{k\geq 0}\frac{\sqrt{\xi_i}^{k}\alpha^{R_k^{(1)}}}{1-\gamma_\mu\alpha^{R_k^{(1)}}},\quad f_{1\mu}^{-}(\xi_i,\alpha)=\sum_{k\geq 0}\frac{(-\sqrt{\xi_i})^{k}\alpha^{R_k^{(1)}}}{1-\gamma_\mu\alpha^{R_k^{(1)}}}$$
$$(1\leq\mu\leq m,\ 1\leq i\leq s)$$

and

$$\theta_{\nu i}^{+}:=\sum_{k\geq 0}\frac{\sqrt{\xi_i^{-1}}^{k}\alpha^{R_k^{(2)}}}{1-\delta_\nu\alpha^{R_k^{(2)}}},\quad \theta_{\nu i}^{-}:=\sum_{k\geq 0}\frac{\left(-\sqrt{\xi_i^{-1}}\right)^{k}\alpha^{R_k^{(2)}}}{1-\delta_\nu\alpha^{R_k^{(2)}}}\quad (1\leq\nu\leq n,\ 1\leq i\leq s)$$

are algebraically independent.

Since

$$f_{2\mu}^{+}(\xi_i^{-1},\alpha),\ f_{3\mu}^{+}(\xi_i^{-1},\alpha)\in\{\theta_{\nu i}^{+}\mid 1\leq\nu\leq n,\ 1\leq i\leq s\},$$
$$f_{2\mu}^{-}(\xi_i^{-1},\alpha),\ f_{3\mu}^{-}(\xi_i^{-1},\alpha)\in\{\theta_{\nu i}^{-}\mid 1\leq\nu\leq n,\ 1\leq i\leq s\}$$

for $1\leq\mu\leq m$ and $1\leq i\leq s$, we see by (11) and (12) that there exists a $2ms\times 2(m+n)s$ matrix $\mathcal{M}$ with algebraic entries such that

$$\begin{aligned}&{}^{t}(F_{1\xi_i}(1,\alpha),G_{1\xi_i}(1,\alpha),\dots,F_{m\xi_i}(1,\alpha),G_{m\xi_i}(1,\alpha)\mid 1\leq i\leq s)\\
&=\mathcal{M}\,{}^{t}(f_{11}^{+}(\xi_i,\alpha),f_{11}^{-}(\xi_i,\alpha),\dots,f_{1m}^{+}(\xi_i,\alpha),f_{1m}^{-}(\xi_i,\alpha),\theta_{1i}^{+},\theta_{1i}^{-},\dots,\theta_{ni}^{+},\theta_{ni}^{-}\mid 1\leq i\leq s)\\
&\quad+{}^{t}(0,\gamma_1^{-1}\xi_i^{k_0}/(1-\xi_i)+\xi_i^{k_0}/(1-\gamma_1),\\
&\qquad\dots,0,\gamma_m^{-1}\xi_i^{k_0}/(1-\xi_i)+\xi_i^{k_0}/(1-\gamma_m)\mid 1\leq i\leq s)\,.\end{aligned}$$

Moreover, by (11) and (12), the matrix $\mathcal{M}$ has a minor of order $2ms$ whose value is equal to

$$\prod_{i=1}^{s}\begin{vmatrix}\xi_i^{k_0+1}/2 & \xi_i^{k_0+1}/2\\ \xi_i^{k_0+1}/(2\sqrt{\xi_i}) & -\xi_i^{k_0+1}/(2\sqrt{\xi_i})\end{vmatrix}^{m}\neq 0\,.$$

Hence the $2ms$ values $F_{\mu\xi_i}(1,\alpha)$ and $G_{\mu\xi_i}(1,\alpha)$ $(1\leq\mu\leq m,\ 1\leq i\leq s)$ are expressed as linearly independent linear combinations of 1 and the algebraically independent numbers $f_{1\mu}^{+}(\xi_i,\alpha), f_{1\mu}^{-}(\xi_i,\alpha)$ $(1\leq\mu\leq m,\ 1\leq i\leq s)$, $\theta_{\nu i}^{+},\theta_{\nu i}^{-}$ $(1\leq\nu\leq n,\ 1\leq i\leq s)$ with algebraic coefficients, which gives the property (ii) of Theorem 2. This completes the proof of the theorem. □

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