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On Stable Ideals

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Introduction

Let A be a d-dimensional Cohen-Macaulay semi-local ring. We say A is equi-dimensional, if $\dim(A_M)=d$ for all maximal ideals M of A, or if A is a Macaulay ring of Nagata [3]. The length of an A-module E will be denoted by $\ell(E)$ or $\ell_A(E)$ to avoid ambiguity.

Sally proved in [5], [6], [7], and [8] that a d-dimensional Cohen-Macaulay local ring A with its maximal ideal M and multiplicity e, has the maximal embedding dimension $e+d-1$, if and only if the Hilbert-Samuel function $\ell(A/M^{n+1})$ of A equals a polynomial

$$P(n)=e\binom{n+d-1}{d}+\binom{n+d-1}{d-1}$$

for all $n\geqq 0$. In fact, more was proved in [8]: For A to have the maximal embedding dimension, it is sufficient that the above $P(n)$ is known to be the Hilbert-Samuel polynomial of A, or $\ell(A/M^{n+1})=P(n)$ for all large n. Our previous work [1] contains an extension of the first assertion: Let I be an open ideal of an equi-dimensional Cohen-Macaulay semi-local ring A of dimension d, then

$$\ell(I/I^2)=e+(d-1)\ell(A/I)\ ,$$

if and only if the Hilbert-Samuel function of I $\ell(A/I^{n+1})$ equals a polynomial

$$Q(n)=e\binom{n+d-1}{d}+\ell(A/I)\binom{n+d-1}{d-1}$$

for all $n\geqq 0$, where e is the multiplicity of I. In this paper, we shall show that the above conditions for I will be satisfied, if we know that

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the polynomial $Q(n)$ is the Hilbert-Samuel polynomial of I.

Let I be an open ideal of a semi-local ring R such that $\dim(R_I)=d$. For integers k $(0\leqq k\leqq d)$, e_k or $e_k(I)$ to avoid confusion, denote the normalised Hilbert-Samuel coefficients of I. This means that

$$P_I(n)=\sum_{k=0}^{d}(-1)^k e_k\binom{n+d-k}{d-k}$$

is the Hilbert-Samuel polynomial of I. We consider, for convenience $e_k=0$ for $k>d$. Throughout the paper, we assume that A is an equi-dimensional Cohen-Macaulay semi-local ring of dimension $d>0$, and I an open ideal of A. Let t be an indeterminate, and $B=A[t]_{IA[t]}$. Then $\ell_A(I^n/I^{n+1})=\ell_B(I^nB/I^{n+1}B)$, and our results in this paper will take effect, even if we prove them considering B and IB for A and I respectively. Accordingly, we may assume if necessary, that the residue fields A/M are infinite for all maximal ideals M of A, and that an open ideal I is an ideal of definition of A.

§ 1. Preliminaries.

In this section, we recall some fundamental facts. Let x be an element of an open ideal I of A, then

$$\ell(A/(I^{n+1}+xA))=\ell(A/I^{n+1})-\ell(A/I^n)+\ell((I^{n+1}: x)/I^n)$$

for all $n\geqq 0$. If x is a superficial element of I, then $\dim(A/xA)=d-1$. If I is an ideal of definition of A and if x is a superficial element of I, then x is regular ([1] Lemma 4 (1)) and A/xA is an equi-dimensional Cohen-Macaulay semi-local ring of dimension $d-1$. If x is a regular element of A, then it is a superficial element of I if and only if $I^{n+1}: x=I^n$ for all large n. Hence, we have the following lemma, which is [1] Lemma 4 (3) with a slight modification.

LEMMA 1. *Let x be a regular superficial element of an open ideal I of A. Then $e_k(I/xA)=e_k(I)$ for all k $(0\leqq k\leqq d-1)$.*

DEFINITION. Let I be an open ideal of a semi-local ring R such that $\dim(R_I)=d$. We call I a *stable* ideal, if

$$\ell(R/I^{n+1})=e_0\binom{n+d}{d}-e_1\binom{n+d-1}{d-1}$$

for all $n\geqq 0$.

By [2] Theorem 1.5 or [1] Corollary 2, the above definition is compatible with that of Lipman [2], which was given in the case $d=1$.

PROPOSITION 2. *Let the dimension of A $d=1$, and I an open ideal of A. Then we have the following assertions.*

(1) $\ell(A/I^n)\geqq ne_0-e_1$ *for all* $n\geqq 0$, *and for* $n\geqq 1$ *equality holds if and only if* I^n *is stable.*

(2) $\ell(I^n/I^{n+1})\leqq e_0$ *for all* $n\geqq 0$, *and for* $n\geqq 1$ *equality holds if and only if* I^n *is stable.*

PROOF. This is [1] Proposition 1, where we sketched a proof making use of the blowing-up A^I of A with center I. Here, we give another proof along Northcott [4]. We may assume that I is an ideal of definition of A and that there exists a superficial element x of I. Then $\dim(A/xA)=0$ and $I^n\subset xA$ for large n. Therefore $e_0(I/xA)=\ell(A/xA)$. Since x is a regular element, $e_0(I/xA)=e_0(I)$ by Lemma 1. Hence $e_0=\ell(A/xA)$. (In fact, x is a transversal element of I and $I^{n+1}=xI^n$ for large n, which will be seen below.) On the other hand,

$$\begin{aligned}\ell(A/I^n)&=\ell(A/x^nA)-\ell(I^n/x^nA)\\&=ne_0-\ell(I^n/x^nA)\end{aligned}$$

and $\ell(I^n/x^nA)=e_1$ for large n. Since

$$\begin{aligned}\ell(I^{n+1}/x^{n+1}A)&=\ell(I^{n+1}/xI^n)+\ell(xI^n/x^{n+1}A)\\&=\ell(I^{n+1}/xI^n)+\ell(I^n/x^nA)\ ,\end{aligned}$$

we have $\ell(I^{n+1}/x^{n+1}A)\geqq\ell(I^n/x^nA)$. If once equality holds for $n=k$, then it holds for all $n\geqq k$. Hence $e_1\geqq\ell(I^n/x^nA)$ and $\ell(A/I^n)\geqq ne_0-e_1$ for all $n\geqq 0$, and if these equalities hold for $n=k$, then they do for all $n\geqq k$. This proves (1), since $e_0(I^n)=ne_0$ and $e_1(I^n)=e_1$ for any $n\geqq 1$. Furthermore

$$\begin{aligned}\ell(I^n/I^{n+1})&=\ell(A/I^{n+1})-\ell(A/I^n)\\&=e_0-\ell(I^{n+1}/x^{n+1}A)+\ell(I^n/x^nA)\\&=e_0-\ell(I^{n+1}/xI^n)\\&\leqq e_0\ ,\end{aligned}$$

and equality holds if and only if $\ell(I^n/x^nA)=\ell(I^{n+1}/x^{n+1}A)=e_1$. This proves (2).

We get immediately the following corollaries, which were stated in [1] Corollary 2 but a little difference.

COROLLARY 3. *Let $d=1$, and I an open ideal of A. Then the* *following conditions are equivalent.*

(1) *I is stable, or $\ell(A/I^n)=ne_0-e_1$ for all $n\geqq 1$.*

(2) *$\ell(A/I)=e_0-e_1$.*

(3) *$\ell(I^n/I^{n+1})=e_0$ for all $n\geqq 1$.*

(4) *$\ell(I/I^2)=e_0$.*

COROLLARY 4. *Let $d=1$, and I an ideal of definition of A, and assume that the residue fields A/M are infinite for all maximal ideals M of A. Then the conditions in the above corollary are equivalent to any one of the following two.*

(5) *There exists an element x of I such that $I^2=xI$.*

(6) *For any superficial element x of I, it holds that $I^2=xI$.*

§ 2. Theorem and its corollaries.

LEMMA 5. *Let the dimension $d\geqq 2$, I an open ideal of A, $e_d=0$, and x a regular superficial element of I such that I/xA is stable. Then $I^{n+1}: x=I^n$ for all $n\geqq 0$, and I is also stable.*

PROOF. The Hilbert-Samuel function of I/xA is equal to $\ell(A/(I^{n+1}+xA))$, $e_0(I/xA)=e_0(I)$, and $e_1(I/xA)=e_1(I)$. We have therefore

$$\ell(A/(I^{n+1}+xA))=e_0(I)\binom{n+d-1}{d-1}-e_1(I)\binom{n+d-2}{d-2}$$

for all $n\geqq 0$. Accordingly,

$$\begin{aligned}\ell(A/I^{n+1})&=\sum_{k=0}^{n}(\ell(A/I^{k+1})-\ell(A/I^k))\\&=\sum_{k=0}^{n}(\ell(A/(I^{k+1}+xA))-\ell((I^{k+1}: x)/I^k))\\&=e_0(I)\binom{n+d}{d}-e_1(I)\binom{n+d-1}{d-1}-\sum_{k=0}^{n}\ell((I^{k+1}: x)/I^k)\end{aligned}$$

for all $n\geqq 0$. As $I^{n+1}: x=I^n$ for large n, we get

$$\sum_{k=0}^{n}\ell((I^{k+1}: x)/I^k)=(-1)^{d-1}e_d(I)$$

for large n. Therefore we get the assertion, by the assumption $e_d=0$.

THEOREM. *Let I be an open ideal of A.*

(1) *If I is stable, then $\ell(A/I)=e_0-e_1$ and $e_k=0$ for $k\geqq 2$.*

(2) *If $\ell(A/I)=e_0-e_1$ and $e_k=0$ for $k\geqq 3$, then I is stable.*

PROOF. (1) is trivial, and we have to prove (2). When $d=1$, the assertion is true by Corollary 3. Now assume $d\geqq 2$, I an ideal of definition, and x a superficial element of I. Then the ideal $J=I/xA$ of $B=A/xA$ satisfies $\ell_B(B/J)=\ell_A(A/I)$ and $e_k(J)=e_k(I)$ for all k $(0\leqq k\leqq d-1)$. Therefore we may assume that J is a stable ideal of B by induction on d. Then I is stable by the preceding lemma, and we have the assertion proved.

Our Theorem gives, in particular, an extension of Sally's result [8] Theorem 3.2.

COROLLARY 6. *An open ideal I of A is stable, if and only if its Hilbert-Samuel polynomial is*

$$e_0\binom{n+d-1}{d}+\ell(A/I)\binom{n+d-1}{d-1}.$$

COROLLARY 7. *Let I be a stable ideal of definition of A, and x a superficial element of I. Then $I^{n+1}: x=I^n$ for all $n\geqq 0$. If $d\geqq 2$, then I/xA is also stable.*

PROOF. When $d=1$, the assertion is valid by Corollary 4. Assume $d\geqq 2$. Since x is a regular element, I/xA is stable by Lemma 1 and Theorem. Therefore $I^{n+1}: x=I^n$ for all $n\geqq 0$, by Lemma 5.

Let I be an ideal of definition of A. We call a system of d elements of I, $x_1, x_2, \cdots, x_d$ a system of superficial parameters of I, if x_k mod. $(x_1, \cdots, x_{k-1})$ is a superficial element of $I/(x_1, \cdots, x_{k-1})$ for any k $(1\leqq k\leqq d)$.

PROPOSITION 8. *Let the residue fields A/M be infinite for all maximal ideals M of A, and I an ideal of definition of A. Then the following conditions are equivalent.*

(1) *I is stable.*

(2) *There exists an ideal X generated by a system of parameters of I, such that $I^2=XI$.*

(3) *Any ideal X generated by a system of superficial parameters of I, satisfies $I^2=XI$.*

PROOF. Equivalence of (1) and (2) was obtained in [1] Theorem. Obviously (3) implies (2). So we are to prove that (1) implies (3). When $d=1$, we have the assertion by Corollary 4. Assume $d\geqq 2$, and $x_1, x_2, \cdots, x_d$ is a system of superficial parameters of I. By the preceding Corollary 7, I/x_1A is stable and $I^{n+1}\cap x_1A=x_1I^n$ for all $n\geqq 0$. Therefore, we may assume that

$$I^2 \subset x_1 A + (x_2, \cdots, x_d) I$$

by induction on d, and we have

$$\begin{aligned} I^2 &= x_1 A \cap I^2 + (x_2, \cdots, x_d) I \\ &= (x_1, x_2, \cdots, x_d) I . \end{aligned}$$

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